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CHARACTERIZATION OF REALCOMPACTNESS AND HEREDITARY REALCOMPACTNESS IN THE CLASS OF NORMAL NODEC (SUBMAXIMAL) SPACES

BY

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Abstract. Is it true in ZFC that every normal submaximal space of non-measurable cardinality is hereditarily realcompact? This question (posed by O. T. Alas et al. (2002)) is given a complete affirmative answer, for a wider class of spaces. In fact, this answer is a part of a bi-conditional statement: A normal nodec space X is hereditarily realcompact if and only if it is realcompact if and only if every closed discrete (or nowhere dense) subset of X has non-measurable cardinality.

1. Introduction and preliminaries. A topological space X is called *crowded* or *dense in itself* if no point of X is an isolated point. Throughout this paper, X stands for a crowded Tychonoff space. The notations βX and υX denote the Stone–Čech compactification and Hewitt realcompactification of X, respectively. A subspace $A \subseteq X$ is *C-embedded* if every continuous real function on A is continuously extendable on X. It is well known that if A is C-embedded in X, then $\mathrm{cl}_{\nu X} A = \nu A$.

Definition 1.1. A topological space Y is called *submaximal* if every dense subset of Y is open.

When N. Bourbaki [B] introduced submaximal spaces, he called them MI-spaces and defined as spaces in which every subset is locally closed, i.e., every subset is open in its closure. In [H], it is pointed out that every subset of a space is locally closed if and only if every dense subset is open. K. Padmavally [P] proved that no locally connected Hausdorff space could be a submaximal space. A space X is *maximal* if it is dense in itself and any strictly stronger topology on X has isolated points. A T_1 space is maximal if and only if it is submaximal and extremally disconnected. In 1995, a countable maximal space was constructed by E. van Douwen [D]. This example was of crucial significance to the submaximal spaces' life, since it guaranteed the existence of crowded regular (Tychonoff, even perfectly normal) submaximal spaces. Some authors used this example to construct new submaximal spaces in order

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to answer some related questions (for instance see [AP] or [LP]). From another point of view, Arhangel'skiĭ and Collin's article [AC] has played the most effective role in investigations of submaximal spaces. In this comprehensive study various necessary and sufficient conditions for a space to be submaximal have been presented. Since then, the notion of submaximality has been of importance in the realm of general topology. Readers can find further useful information and results on submaximal spaces in [TK] and [HH].

Definition 1.2. A topological space Y is called a *nodec space* if every nowhere dense subset of it is closed, or equivalently, every nowhere dense subset of Y is closed discrete.

Initially, nodec spaces appeared as a part of the aforementioned van Douwen's example [D]. The condition of being nodec is weaker than being submaximal, but they are close. Some authors consider nodec spaces mostly due to their relation to submaximals; for example [AC]. More recently, they are considered indenpendently of submaximals or in relation to some other spaces (see for example [HMW] or [BD]).

The article [AS] contains some interesting results about both irresolvable and submaximal spaces; it also contains partial answers to some questions posed in [AC]. Alas et al. posed ten questions about submaximal spaces in the last section of [AS].

Here, the main purpose is to answer Problem 4.7 of [AS]: "Is it true in ZFC that every normal submaximal space of non-measurable cardinality is hereditarily realcompact?" A result close to the answer of this question was asserted in [HH]. F. Hernández-Hernández et al. [HH] proved that every normal submaximal space of non-measurable cardinality is realcompact. A similar result, yet for nodec spaces, was obtained by Aliabad et al. [ABK]. This question is given the affirmative answer in the following section.

It is important to mention that without the assumption of normality, the result is not valid even for the realcompact case. Problem 4.8 of [AS]: "Let X be a submaximal space of non-measurable cardinality. Is it necessarily realcompact?" has a negative answer. As mentioned in [HH, Corollary 6]: "There is a separable submaximal not realcompact space of cardinality ω_1."

Here are some necessary preliminaries.

A point $p \in \beta X \setminus X$ is said to be a *remote point* if $p \notin \mathrm{cl}_{\beta X} D$ for every nowhere dense subset D of X. The following theorem is the main result of [T].

Theorem 1.3. *For a space X, if its cardinality $c(X)$ is non-measurable, then $\upsilon X \setminus X$ contains no remote point of X.*

The following theorem [GJ, 12.2] is well known.

Theorem 1.4. *A discrete space is realcompact if and only if its cardinality is non-measurable.*

2. Main result. In this section we obtain a necessary and sufficient condition for a normal nodec space to be realcompact (or hereditarily realcompact).

Clearly, for a space X and $A \subseteq X$, a set $D \subseteq A$ is discrete in A if and only if D is discrete in X. And if X is crowded and $D \subset A$ is discrete in X, then D is nowhere dense in X (the converse is true when X is nodec).

LEMMA 2.1. *If X is a crowded nodec space and for some $A \subseteq X$, $D \subseteq A$ is discrete in A, then D is discrete closed in X.*

LEMMA 2.2. *If X is a nodec space and $A \subseteq X$, then the subspace A is also a nodec space.*

Proof. Suppose $D \subseteq X$ is nowhere dense in A; we will show that D is also nowhere dense in X. This follows from the fact that if $V \subseteq \operatorname{cl}_X D$ is open and non-void, then $V \cap A \subseteq \operatorname{cl}_A D$ is open non-void in A. This contradiction shows that D must be nowhere dense in X. Now, D is discrete in X, so it is discrete in A. ∎

THEOREM 2.3. *The following statements are equivalent for a normal nodec space X:*

(i) *X is hereditarily realcompact;*
(ii) *X is realcompact;*
(iii) *every closed discrete (nowhere dense) subset of X has non-measurable cardinality.*

Proof. (ii)⇒(iii). Suppose that, on the contrary, $D \subseteq X$ is a closed discrete (or nowhere dense) subset, and it has measurable cardinality. So, by Theorem 1.4, D is not realcompact. Since D is closed in the normal space X, we have $\operatorname{cl}_{\upsilon X} D = \upsilon D$. Thus, $\emptyset \neq \upsilon D \setminus D = \operatorname{cl}_{\upsilon X} D \setminus D \subseteq \upsilon X \setminus X$. This contradicts realcompactness of X.

(iii)⇒(i). Assume that $A \subseteq X$. First, we prove that $c(X)$ is non-measurable. By Lemmas 2.1 and 2.2, this also holds for $c(A)$. Let $\{U_i\}_{i \in I}$ be a family of pairwise disjoint open subsets of X. For every $i \in I$, select $a_i \in U_i$ and set $D = \{a_i : i \in I\}$. Clearly, D is nowhere dense, and since X is a nodec space, it is also closed discrete. Therefore, by hypothesis, $|D|$ is non-measurable. Since $|D| = |I|$ and non-measurability is closed under taking supremum, it follows that $c(X)$ is non-measurable. Now, if $p \in \upsilon A \setminus A$, then by Theorem 1.3, p is not a remote point. By definition, $p \in \operatorname{cl}_{\beta A} D$ for some nowhere dense $D \subseteq A$. By Corollary 2.2, D is closed and discrete in X. Therefore, $p \in \upsilon D \setminus D$, and by hypothesis, D is non-measurable, which contradicts Theorem 1.4. Hence, $\upsilon A \setminus A = \emptyset$. ∎

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