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THE $\mathfrak{v}$-RADIAL PALEY–WIENER THEOREM FOR THE HELGASON FOURIER TRANSFORM ON DAMEK–RICCI SPACES

BY

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Abstract. We prove the Paley–Wiener theorem for the Helgason Fourier transform of smooth compactly supported $\mathfrak{v}$-radial functions on a Damek–Ricci space $S = NA$.

1. Introduction. Let $S = NA$ be a *Damek–Ricci space*, i.e., the semidirect product of a (connected and simply connected) nilpotent Lie group N of Heisenberg type [17] and the one-dimensional Lie group $A \cong \mathbb{R}^+$ acting on N by anisotropic dilations. When S is equipped with a suitable left-invariant Riemannian metric γ_S, S becomes a (noncompact, simply connected) homogeneous harmonic Riemannian space [9, 10]. Conversely, every such space is a Damek–Ricci space if we exclude $\mathbb{R}^n$ and the "degenerate" case of real hyperbolic spaces (see [14, Corollary 1.2]). We refer to [23] for a nice introduction to the geometry and harmonic analysis on Damek–Ricci spaces.

We use the ball model B of S, namely we identify S with the unit ball B in the Lie algebra $\mathfrak{s}$ via the Cayley transform C (see [7]):

$$S = NA \overset{C}{\cong} B = \{(V, Z, t) \in \mathfrak{s} : |V|^2 + |Z|^2 + t^2 < 1\}.$$

Here $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$, where $\mathfrak{z}$ is the center of $\mathfrak{n}$ and $\mathfrak{v}$ its orthogonal complement in $\mathfrak{n}$. We let $p = \dim \mathfrak{v}$, $q = \dim \mathfrak{z}$, $Q = p/2 + q$, and let S^{p+q} be the unit sphere in $\mathfrak{s}$,

$$S^{p+q} = \partial B = \{\omega = (V, Z, t) \in \mathfrak{s} : |V|^2 + |Z|^2 + t^2 = 1\}.$$

Let $f \in C_0^\infty(B)$. The *Helgason Fourier transform* of f is defined by

$$\widetilde{f}(\lambda, \omega) = \int_B f(b)\mathcal{Q}_\lambda(b, \omega)\, db \qquad (\lambda \in \mathbb{C},\ \omega \in S^{p+q}), \tag{1.1}$$

where $\mathcal{Q}_\lambda(b, \omega)$ is the normalized Poisson kernel with parameter λ on B (see

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[2, 6] and Section 2). The inversion formula is [2, Theorem 4.4]

$$f(b) = \frac{c_{p,q}}{2\pi} \int_0^\infty \int_{S^{p+q}} \mathcal{Q}_{-\lambda}(b,\omega)\widetilde{f}(\lambda,\omega)\, d\omega\, d\mu(\lambda), \tag{1.2}$$

where $d\omega$ is the normalized Euclidean surface measure on S^{p+q}, and $d\mu(\lambda) = |c(\lambda)|^{-2}d\lambda$ with

$$c(\lambda) = \frac{2^{Q-2i\lambda}\Gamma(2i\lambda)\Gamma(d/2)}{\Gamma(i\lambda+Q/2)\Gamma(i\lambda+p/4+1/2)}, \tag{1.3}$$

and

$$c_{p,q} = 2^{q-1}\Gamma(d/2)\pi^{-d/2}, \qquad d = p+q+1.$$

A C^∞ function $\psi(\lambda,\omega)$ on $\mathbb{C}\times\partial B$, holomorphic in λ, is called a *holomorphic function of uniform exponential type* if there exists a constant $R > 0$ such that, for each integer $j \geq 0$,

$$\sup_{(\lambda,\omega)\in\mathbb{C}\times\partial B} e^{-R|\operatorname{Im}\lambda|}\,(1+|\lambda|)^j\,|\psi(\lambda,\omega)| < \infty. \tag{1.4}$$

THEOREM 1.1. *The Fourier transform $f(b) \mapsto \widetilde{f}(\lambda,\omega)$ is a bijection of $C_0^\infty(B)$ onto the set of holomorphic functions $\psi(\lambda,\omega)$ of uniform exponential type satisfying the condition*

$$\int_{\partial B} \mathcal{Q}_{-\lambda}(b,\omega)\psi(\lambda,\omega)\, d\omega = \int_{\partial B} \mathcal{Q}_{\lambda}(b,\omega)\psi(-\lambda,\omega)\, d\omega \tag{1.5}$$

for any $b \in B$ and $\lambda \in \mathbb{C}$. Moreover, $\widetilde{f}$ satisfies (1.4) *if and only if f has support in the closed ball $B_R = \{b \in B : d(b, C(e)) \leq R\}$.*

The direct part of this theorem, asserting that $\operatorname{supp} f \subset B_R$ implies $\widetilde{f}$ holomorphic of uniform exponential type R, was proved in [2, Theorem 4.5] (in the open model). Here we prove the $\mathfrak{v}$-radial case of Theorem 1.1. The converse part, in particular the surjectivity statement, is proved for $\mathfrak{v}$*-radial* functions f on B, i.e., functions that are radial in the variable V and thus depend only on $|V|$, Z, and t. In this case we show that $f \mapsto \widetilde{f}$ is a bijection onto the set of functions $\psi(\lambda,\omega)$ that are holomorphic of uniform exponential type, $\mathfrak{v}$-radial in ω, and satisfy (1.5).

The case of biradial functions $f = f(|V|,|Z|,t)$ on B was treated recently in [5]. Here we extend the results of [5], by generalizing to $\mathfrak{v}$-radial functions on B the well known expansion into K-types of the symmetric case [15].

By working in geodesic polar coordinates $(r,\omega) \in (0,\infty)\times S^{p+q}$ around the origin in B, we expand both functions $\omega \mapsto f(r,\omega)$ and $\omega \mapsto \widetilde{f}(\lambda,\omega)$ in Fourier series with respect to an orthogonal system of $\mathfrak{v}$-radial eigenfunctions of the angular Laplacian $L_{S(r)}$ in $L^2(S^{p+q})$. Here $S(r) \simeq S^{p+q}$ is the geodesic sphere of radius $r > 0$ centered at the origin. The Fourier coefficients are then functions of r and λ, respectively, related by a suitable Jacobi transform. Using well known estimates for Jacobi functions, we prove the result.

Our method should generalize to arbitrary functions on B. The problem in the nonsymmetric case is that there is no analogue of the group K acting transitively by isometries on the geodesic spheres. This makes more difficult the identification of non-$\mathfrak{v}$-radial eigenfunctions of $L_{S(r)}$, as it requires the explicit form of the full angular Laplacian, which is not yet available.

The outline of this paper is as follows. In Section 2 we first obtain a formula for the $\mathfrak{v}$-radial part of the angular Laplacian. It generalizes the formula obtained in [6, 5] in the biradial case. Then, using results of Koornwinder [19], we write down a decomposition of the space $\mathcal{H}^{p+q+1,n}$ of spherical harmonics of degree n on S^{p+q} as an orthogonal direct sum of subspaces invariant and irreducible under the group $\mathrm{SO}(p) \times \mathrm{SO}(q)$. This enables us to identify the $\mathfrak{v}$-radial eigenfunctions of $L_{S(r)}$ in terms of spherical harmonics on S^{p+q}. We then compute the $\mathfrak{v}$-radial eigenfunctions of the full Laplacian L_B on B that separate in geodesic polar coordinates. The radial part of these eigenfunctions is given by associated Jacobi functions. We also obtain a Poisson integral representation for these $\mathfrak{v}$-radial eigenfunctions of L_B.

In Section 3 we prove that $\mathfrak{v}$-radiality is preserved by the Helgason Fourier transform, i.e., f $\mathfrak{v}$-radial on B implies $\omega \mapsto \widetilde{f}(\lambda,\omega)$ $\mathfrak{v}$-radial on S^{p+q}. The proof involves the Radon transform and the method of "descent" to complex hyperbolic spaces [24, Proposition 5.1]. Then we write down the Fourier transform and prove the $\mathfrak{v}$-radial case of Theorem 1.1.

Let us mention some earlier results on the Paley–Wiener theorem for Damek–Ricci spaces. For radial functions on NA, the Helgason Fourier transform reduces to the spherical transform [2]. The Paley–Wiener theorem for the spherical transform follows from the general theory developed in [20] (Jacobi function analysis): see, for instance, [1, pp. 649–650]. For nonradial functions, a partial result that uses the Radon transform and reduction to complex hyperbolic spaces appears in [24]. A Paley–Wiener theorem for nonradial functions on NA supported in a set whose boundary is a horocycle was obtained in [3]. A Paley–Wiener theorem for the inverse Fourier transform on NA was proved in [4].

2. $\mathfrak{v}$-radial eigenfunctions on B

2.1. The Cayley transform and the $\mathfrak{v}$-radial Laplacian. We denote by $\langle\cdot,\cdot\rangle$ and $|\cdot|$ the fixed inner product and associated norm on $\mathfrak{s}$, and by $(V,Z,t) \in \mathfrak{s}$ the element $\exp(V+Z)\exp(tH)$ of S, where $V \in \mathfrak{v}$, $Z \in \mathfrak{z}$, $t \in \mathbb{R}$, and $H \in \mathfrak{a}$ is a unit vector. For each $Z \in \mathfrak{z}$ we have the linear map $J_Z : \mathfrak{v} \to \mathfrak{v}$ defined by $\langle J_Z V, V'\rangle = \langle Z, [V,V']\rangle$ for $V, V' \in \mathfrak{v}$. The *Cayley* *transform* is defined by

$$C : NA \to B, \qquad (V, Z, t) \mapsto (V', Z', t'),$$

where

$$\text{(2.1)} \qquad \begin{cases} V' = \dfrac{\left(1 + e^t + \frac{1}{4}|V|^2\right)V - J_Z V}{\left(1 + e^t + \frac{1}{4}|V|^2\right)^2 + |Z|^2}, \\ Z' = \dfrac{2Z}{\left(1 + e^t + \frac{1}{4}|V|^2\right)^2 + |Z|^2}, \\ t' = \dfrac{-1 + \left(e^t + \frac{1}{4}|V|^2\right)^2 + |Z|^2}{\left(1 + e^t + \frac{1}{4}|V|^2\right)^2 + |Z|^2}, \end{cases}$$

with inverse

$$\text{(2.2)} \qquad \begin{cases} V = 2\dfrac{(1 - t')V' + J_{Z'}V'}{(1 - t')^2 + |Z'|^2}, \\ Z = \dfrac{2Z'}{(1 - t')^2 + |Z'|^2}, \\ e^t = \dfrac{1 - R^2}{(1 - t')^2 + |Z'|^2}, \end{cases}$$

where $R = \sqrt{|V'|^2 + |Z'|^2 + t'^2}$ (see [23, (18), (19), Sect. 4.4]).

We also have a generalized stereographic projection

$$C_0 : N \to S^{p+q} \setminus \{H\} \qquad (H = (0, 0, 1))$$

obtained by letting $a_t = e^t = 0$, i.e., $t = -\infty$, in (2.1):

$$C_0(n) = \lim_{t \to -\infty} C(na_t) \in \partial B$$

(see [23, Section 4.6]). Its inverse C_0^{-1} is given by the first two lines in (2.2).

In the ball model B of S, equipped with the transported metric $\gamma_B = C^{-1*}(\gamma_S)$, we have $C(\operatorname{Exp}_e r\omega) = \operatorname{th}(r/2)\,\omega$ for $r \geq 0$ and $\omega \in S^{p+q}$. Thus the geodesics through the origin are the diameters, and the Riemannian sphere $S(r)$ of radius r (centered at the origin) is just the Euclidean sphere $S(R)$ of radius $R = \operatorname{th}(r/2)$ [23, Thm. 10].

Let $\operatorname{vol}(S^n) = 2\pi^{(n+1)/2}/\Gamma((n+1)/2)$ be the Euclidean surface measure of the n-sphere. In geodesic polar coordinates (r, ω) around the origin $x_0 = C(e) = (0, 0, 0)$ in B, the Riemannian measure is given by

$$\text{(2.3)} \qquad db = 2^{p+q}(\operatorname{sh}(r/2))^{p+q}(\operatorname{ch}(r/2))^q \operatorname{vol}(S^{p+q})drd\omega =: J(r)drd\omega.$$

Let M be the group of orthogonal automorphisms of NA, namely the automorphisms of S that preserve the inner product on the Lie algebra $\mathfrak{s}$. Using the exponential map, we can identify M as the group of orthogonal automorphisms of the H-type Lie algebra $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, i.e., the elements (ψ, ϕ)

in $\mathrm{O}(\mathfrak{v}) \times \mathrm{O}(\mathfrak{z})$ such that

$$[\psi(V), \psi(V')] = \phi([V, V']),$$

or equivalently,

$$\psi(J_Z V) = J_{\phi(Z)}\psi(V), \quad \forall V, V' \in \mathfrak{v}, \forall Z \in \mathfrak{z}.$$

By conjugating with the Cayley map C, we obtain a group of transformations of B that we still denote by M. It is easy to check that this action of M on B is just the action of M on $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$, i.e., $m \cdot (V', Z', t') = (m \cdot V', m \cdot Z', t')$, where M is trivial on $\mathfrak{a}$ and leaves $\mathfrak{v}$ and $\mathfrak{z}$ invariant. It is known that M acts transitively on the unit sphere S^{q-1} in $\mathfrak{z}$ (see [7, Remark 6.3]). However, M may or may not be transitive on the unit sphere S^{p-1} in $\mathfrak{v}$, depending on the Heisenberg-type group N (see [22]).

Let f be a $\mathfrak{v}$-radial function on B, i.e., $f(V_1', Z', t') = f(V_2', Z', t')$ if $|V_1'| = |V_2'|$. Then f depends only on the variables $|V'|$, Z' and t', and we write $f = f(|V'|, Z', t')$. We denote by $C_0^\infty(B)^{\mathfrak{v}\text{-rad}}$ the subspace of $\mathfrak{v}$-radial functions in $C_0^\infty(B)$. In geodesic polar coordinates we write

$$f(b) = f(\mathrm{th}(r/2)\omega) = f(r, \omega).$$

For each $r > 0$, the function $\omega = (V, Z, t) \mapsto f(r, \omega)$ is $\mathfrak{v}$-radial on S^{p+q}, i.e., it depends only on Z and t. We use the following notations:

$$\begin{aligned} S^{p-1} &= \{(V, 0, 0) : |V| = 1\} = S^{p+q} \cap \mathfrak{v} && \text{(unit sphere in } \mathfrak{v}\text{)},\\ S^{q} &= \{(0, Z, t) : |Z|^2 + t^2 = 1\} = S^{p+q} \cap \mathfrak{z} \oplus \mathfrak{a} && \text{(unit sphere in } \mathfrak{z} \oplus \mathfrak{a}\text{)},\\ S^{q-1} &= \{(0, Z, 0) : |Z| = 1\} = S^{p+q} \cap \mathfrak{z} && \text{(unit sphere in } \mathfrak{z}\text{)}. \end{aligned}$$

Every $\omega \in S^{p+q}$ can be written as $\omega = \sqrt{1-\rho^2}\,\omega_1 + \rho\tilde{\omega}_2$, where $0 \le \rho \le 1$, $\omega_1 \in S^{p-1}$, and $\tilde{\omega}_2 \in S^q$. By writing $\tilde{\omega}_2$ as $\tilde{\omega}_2 = \cos\phi\, H + \sin\phi\,\omega_2$, where $\omega_2 \in S^{q-1}$, $H = (0, 0, 1)$, and $0 \le \phi \le \pi$, we see that every $\omega = (V, Z, t) \in S^{p+q}$ can be represented in the form

$$\begin{cases} V = \sqrt{1-\rho^2}\,\omega_1, \\ Z = \rho \sin\phi\,\omega_2, \\ t = \rho\cos\phi, \end{cases}$$

where

$$0 \le \rho \le 1, \quad 0 \le \phi \le \pi, \quad \omega_1 \in S^{p-1}, \quad \omega_2 \in S^{q-1}.$$

We write $\omega = (\rho, \phi, \omega_1, \omega_2)$ and refer to this as a system of *bispherical coordinates* on S^{p+q}. The choices of ω_1, ω_2 and ϕ are unique except when $V = 0$, or $Z = 0$, or $(Z, t) = (0, 0)$. The coordinates (ρ, ϕ) can be regarded as polar coordinates in the space $(|Z|, t)$:

$$\begin{cases} t = \rho\cos\phi, \\ |Z| = \rho\sin\phi. \end{cases}$$

We let D_+ be the upper-half unit disk, defined by $0 \le \rho \le 1$ and $0 \le \phi \le \pi$.

A $\mathfrak{v}$-radial function χ on S^{p+q} depends only on ρ, ϕ, ω_2, and we write $\chi = \chi(\rho, \phi, \omega_2)$. A $\mathfrak{v}$-radial function f on B depends only on r, ρ, ϕ, ω_2, and we write $f = f(r, \rho, \phi, \omega_2)$.

The *Laplace–Beltrami operator* on B in geodesic polar coordinates reads

$$L_B = L_{\mathrm{rad}} + L_{S(r)}, \tag{2.4}$$

where L_{rad} is the radial part, given by

$$L_{\mathrm{rad}} = \partial_r^2 + \left(\frac{p}{2}\operatorname{cth}\frac{r}{2} + q\operatorname{cth} r\right)\partial_r \qquad (\partial_r = \partial/\partial r), \tag{2.5}$$

and $L_{S(r)}$ is the angular part, i.e., the Laplacian on the Riemannian sphere $S(r)$ with respect to the induced metric. We identify $S(r)$ with S^{p+q} by the map $C(\mathrm{Exp}_e\, r\omega) \mapsto \omega$, i.e., $\operatorname{th}(r/2)\omega \mapsto \omega$, for any fixed $r > 0$.

Let L_{S^n} denote the round Laplacian on the unit sphere S^n. Then the round Laplacian on S^{p+q} can be written in bispherical coordinates as

$$L_{S^{p+q}} = (1-\rho^2)\partial_\rho^2 + \left(\frac{q}{\rho} - (p+q)\rho\right)\partial_\rho + \frac{1}{\rho^2}L_{S^q} + \frac{1}{1-\rho^2}L_{S^{p-1}} \tag{2.6}$$

(see [6]), where the round Laplacian on S^q is

$$L_{S^q} = \partial_\phi^2 + (q-1)\cot\phi\,\partial_\phi + \frac{1}{\sin^2\phi}L_{S^{q-1}}, \tag{2.7}$$

with ϕ playing the role of "radial" coordinate on S^q.

THEOREM 2.1. *Let $\chi = \chi(\rho, \phi, \omega_2)$ be a $\mathfrak{v}$-radial function on $S^{p+q} \simeq S(r)$. Then the angular Laplacian $L_{S(r)}$ acting on χ is given by*

$$\begin{aligned} L_{S(r)}\chi &= \frac{1}{4\operatorname{sh}^2(r/2)}L_{S^{p+q}}\chi - \frac{1}{4\operatorname{ch}^2(r/2)}L_{S^q}\chi \\ &= \frac{1}{4\operatorname{sh}^2(r/2)}\left\{(1-\rho^2)\partial_\rho^2 + \left(\frac{q}{\rho} - (p+q)\rho\right)\partial_\rho + \frac{1}{\rho^2}L_{S^q}\right\}\chi \\ &\quad - \frac{1}{4\operatorname{ch}^2(r/2)}L_{S^q}\chi. \end{aligned} \tag{2.8}$$

Proof. The idea is to change variables $(V, Z, t) \overset{C}{\mapsto} (V', Z', t')$ directly in the known expression of the Laplacian on $S = NA$. This can be carried out in a rather explicit way, up to some point, which is enough to obtain the $\mathfrak{v}$-radial part.

The Laplace–Beltrami operator on S is given in the usual NA-chart by

$$L_S = e^t \sum_{i=1}^{p} E_i^2 + e^{2t}\sum_{j=1}^{q} Y_j^2 + H^2 - QH,$$

where $\{E_1, \ldots, E_p, Y_1, \ldots, Y_q, H\}$ is an orthonormal basis of $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$ with respect to the Euclidean inner product $\langle\,,\rangle$ on $\mathfrak{s}$. We identify $H = \partial_t$

and regard E_i, Y_j as left-invariant vector fields on the group N, given at (V, Z) by

$$E_i = \partial_{v_i} + \frac{1}{2}\sum_{j=1}^{q} \langle J_{Y_j} V, E_i\rangle \partial_{z_j}, \qquad Y_j = \partial_{z_j},$$

if we write $V = \sum_i v_i E_i$ and $Z = \sum_j z_j Y_j$. Then

$$\begin{aligned} L_S &= e^t \sum_i \Big(\partial_{v_i} + \frac{1}{2}\sum_j \langle J_{Y_j} V, E_i\rangle \partial_{z_j}\Big)\Big(\partial_{v_i} + \frac{1}{2}\sum_k \langle J_{Y_k} V, E_i\rangle \partial_{z_k}\Big) \\ &\quad + e^{2t}\sum_j \partial_{z_j}^2 + \partial_t^2 - Q\,\partial_t \\ &= e^t \sum_i \partial_{v_i}^2 + \frac{1}{2}e^t \sum_{i,k} (\partial_{v_i}\langle J_{Y_k} V, E_i\rangle)\partial_{z_k} \\ &\quad + \frac{1}{2}e^t \sum_{i,k} \langle J_{Y_k} V, E_i\rangle \partial_{v_i}\partial_{z_k} + \frac{1}{2}e^t \sum_{i,j} \langle J_{Y_j} V, E_i\rangle \partial_{z_j}\partial_{v_i} \\ &\quad + \frac{1}{4}e^t \sum_{i,j,k} \langle J_{Y_j} V, E_i\rangle\langle J_{Y_k} V, E_i\rangle \partial_{z_j}\partial_{z_k} + e^{2t}\sum_j \partial_{z_j}^2 + \partial_t^2 - Q\partial_t. \end{aligned}$$

The second term vanishes since

$$\begin{aligned} \sum_{i,k} (\partial_{v_i}\langle J_{Y_k} V, E_i\rangle)\partial_{z_k} &= \sum_{i,k} \Big(\partial_{v_i}\Big\langle Y_k, \Big[\sum_l v_l E_l, E_i\Big]\Big\rangle\Big)\partial_{z_k} \\ &= \sum_{i,k} \langle Y_k, [E_i, E_i]\rangle \partial_{z_k} = 0. \end{aligned}$$

The third and fourth terms are equal, and in the fifth term we have

$$\begin{aligned} \sum_{i,j,k} \langle J_{Y_j} V, E_i\rangle\langle J_{Y_k} V, E_i\rangle \partial_{z_j}\partial_{z_k} &= \sum_{j,k} \langle J_{Y_j} V, J_{Y_k} V\rangle \partial_{z_j}\partial_{z_k} \\ &= \sum_{j,k} |V|^2 \langle Y_j, Y_k\rangle \partial_{z_j}\partial_{z_k} = |V|^2 \sum_j \partial_{z_j}^2. \end{aligned}$$

Here we have used the identity $\langle J_Z V, J_{Z'} V\rangle = \langle Z, Z'\rangle |V|^2$.

We can rewrite L_S as

$$\begin{aligned} L_S = e^t \Big(\sum_i \partial_{v_i}^2 + \frac{1}{4}|V|^2 \sum_j \partial_{z_j}^2\Big) + e^{2t}\sum_j \partial_{z_j}^2 + \partial_t^2 - Q\partial_t \\ + e^t \sum_{i,j} \langle J_{Y_j} V, E_i\rangle \partial_{z_j}\partial_{v_i}. \end{aligned}$$

The operators

$$L_{\mathfrak{v}} = \sum_i \partial_{v_i}^2, \qquad L_{\mathfrak{z}} = \sum_j \partial_{z_j}^2$$

are of course the Euclidean Laplacians on $\mathfrak{v} \simeq \mathbb{R}^p$ and $\mathfrak{z} \simeq \mathbb{R}^q$, respectively. In Euclidean polar coordinates they read

$$L_{\mathfrak{v}} = \partial^2_{|V|} + \frac{p-1}{|V|}\partial_{|V|} + \frac{1}{|V|^2} L_{S^{p-1}},$$
$$L_{\mathfrak{z}} = \partial^2_{|Z|} + \frac{q-1}{|Z|}\partial_{|Z|} + \frac{1}{|Z|^2} L_{S^{q-1}},$$

where $L_{S^{p-1}}$ and $L_{S^{q-1}}$ are the round Laplacians on the unit spheres S^{p-1} and S^{q-1} in $\mathfrak{v}$ and $\mathfrak{z}$, respectively. We denote by L_2 the last term in L_S,

$$L_2 = e^t \sum_{i,j} \langle J_{Y_j} V, E_i \rangle \partial_{z_j} \partial_{v_i},$$

and observe that L_2 gives zero when acting on a $\mathfrak{v}$-radial function on S, i.e., $f = f(|V|, Z, t)$. Indeed in this case we have

$$\partial_{v_i} f = \frac{\partial |V|}{\partial v_i} \partial_{|V|} f = \frac{v_i}{|V|} \partial_{|V|} f,$$

so that

$$L_2 f = e^t \sum_{i,j} \langle J_{Y_j} V, E_i \rangle \frac{v_i}{|V|} \partial_{z_j} \partial_{|V|} f = e^t \sum_j \frac{1}{|V|} \langle J_{Y_j} V, V \rangle \partial_{z_j} \partial_{|V|} f = 0.$$

If we define the structure constants C_{ij}^k by

$$[E_i, E_j] = \sum_k C_{ij}^k \, Y_k,$$

we can rewrite L_2 as

$$L_2 = e^t \sum_{i,j,k} C_{ij}^k v_i \partial_{z_k} \partial_{v_j} = \frac{1}{2} e^t \sum_{i,j,k} C_{ij}^k (v_i \partial_{v_j} - v_j \partial_{v_i}) \partial_{z_k}.$$

Note that $v_i \partial_{v_j} - v_j \partial_{v_i}$ is a well defined differential operator on the unit sphere S^{p-1} for any $i, j = 1, \ldots, p$.

The biradial part $\tilde{L}_S$ of L_S is given by

$$\tilde{L}_S = e^t \left(\partial^2_{|V|} + \frac{p-1}{|V|} \partial_{|V|} + \frac{1}{4} |V|^2 \left(\partial^2_{|Z|} + \frac{q-1}{|Z|} \partial_{|Z|} \right) \right) + e^{2t} \left(\partial^2_{|Z|} + \frac{q-1}{|Z|} \partial_{|Z|} \right) + (\partial_t^2 - Q \partial_t), \tag{2.9}$$

and the Laplacian on S becomes

$$L_S = \tilde{L}_S + e^t \left(\frac{1}{|V|^2} L_{S^{p-1}} + \frac{|V|^2}{4|Z|^2} L_{S^{q-1}} \right) + \frac{e^{2t}}{|Z|^2} L_{S^{q-1}} + L_2. \tag{2.10}$$

Consider now the change of variables $(V, Z, t) \overset{C}{\mapsto} (V', Z', t')$ given by the Cayley map in (2.1)–(2.2). It is convenient to separate out the norms of V, V', Z, Z' from their respective angular variables, and to transform to Euclidean polar coordinates $(R,\omega) \in (0,1)\times S^{p+q}$ on B. We get the following transformations:

$$(|V|,|Z|,t,\omega_1,\omega_2) \mapsto (|V'|,|Z'|,t',\omega_1',\omega_2') \mapsto (R,\rho,\phi,\omega_1',\omega_2'),$$

where $V = |V|\omega_1$, $V' = |V'|\omega_1'$ ($\omega_1,\omega_1' \in S^{p-1}$), $Z = |Z|\omega_2$, $Z' = |Z'|\omega_2'$ ($\omega_2,\omega_2' \in S^{q-1}$),

$$\begin{cases} |V'| = \dfrac{|V|}{\left[\left(1+e^t+\frac14|V|^2\right)^2+|Z|^2\right]^{1/2}}, \\ |Z'| = \dfrac{2|Z|}{\left(1+e^t+\frac14|V|^2\right)^2+|Z|^2}, \\ t' = \dfrac{-1+\left(e^t+\frac14|V|^2\right)^2+|Z|^2}{\left(1+e^t+\frac14|V|^2\right)^2+|Z|^2}, \\ \omega_1' = \dfrac{(1+e^t+\frac14|V|^2)\omega_1-|Z|J_{\omega_2}\omega_1}{\left[\left(1+e^t+\frac14|V|^2\right)^2+|Z|^2\right]^{1/2}}, \\ \omega_2' = \omega_2, \end{cases} \tag{2.11}$$

with inverse

$$\begin{cases} |V| = \dfrac{2|V'|}{[(1-t')^2+|Z'|^2]^{1/2}}, \\ |Z| = \dfrac{2|Z'|}{(1-t')^2+|Z'|^2}, \\ e^t = \dfrac{1-R^2}{(1-t')^2+|Z'|^2}, \\ \omega_1 = \dfrac{(1-t')\omega_1'+|Z'|J_{\omega_2'}\omega_1'}{[(1-t')^2+|Z'|^2]^{1/2}}, \\ \omega_2 = \omega_2', \end{cases} \tag{2.12}$$

where $R^2 = |V'|^2+|Z'|^2+t'^2$, and (R,ρ,ϕ) can be regarded as spherical coordinates in the space $(|V'|,|Z'|,t')$:

$$\begin{cases} |V'| = R\sqrt{1-\rho^2}, \\ |Z'| = R\rho\sin\phi, \\ t' = R\rho\cos\phi \end{cases} \qquad (0 \le R,\rho \le 1,\ 0 \le \phi \le \pi).$$

The Jacobian of the change of variables

$$(|V|,|Z|,t,\omega_1,\omega_2) \mapsto (|V'|,|Z'|,t',\omega_1',\omega_2')$$

gives the transformation between the gradients. We write it symbolically as

$$
(2.13)\qquad \begin{cases}
\partial_{|V|} = \dfrac{\partial |V'|}{\partial |V|}\partial_{|V'|} + \dfrac{\partial |Z'|}{\partial |V|}\partial_{|Z'|} + \dfrac{\partial t'}{\partial |V|}\partial_{t'} + \dfrac{\partial \omega_1'}{\partial |V|}\partial_{\omega_1'},\\
\partial_{|Z|} = \dfrac{\partial |V'|}{\partial |Z|}\partial_{|V'|} + \dfrac{\partial |Z'|}{\partial |Z|}\partial_{|Z'|} + \dfrac{\partial t'}{\partial |Z|}\partial_{t'} + \dfrac{\partial \omega_1'}{\partial |Z|}\partial_{\omega_1'},\\
\partial_t = \dfrac{\partial |V'|}{\partial t}\partial_{|V'|} + \dfrac{\partial |Z'|}{\partial t}\partial_{|Z'|} + \dfrac{\partial t'}{\partial t}\partial_{t'} + \dfrac{\partial \omega_1'}{\partial t}\partial_{\omega_1'},\\
\partial_{\omega_1} = \dfrac{\partial \omega_1'}{\partial \omega_1}\partial_{\omega_1'},\\
\partial_{\omega_2} = \dfrac{\partial \omega_1'}{\partial \omega_2}\partial_{\omega_1'} + \partial_{\omega_2'},
\end{cases}
$$

with inverse

$$
\begin{cases}
\partial_{|V'|} = \dfrac{\partial |V|}{\partial |V'|}\partial_{|V|} + \dfrac{\partial t}{\partial |V'|}\partial_t,\\
\partial_{|Z'|} = \dfrac{\partial |V|}{\partial |Z'|}\partial_{|V|} + \dfrac{\partial |Z|}{\partial |Z'|}\partial_{|Z|} + \dfrac{\partial t}{\partial |Z'|}\partial_t + \dfrac{\partial \omega_1}{\partial |Z'|}\partial_{\omega_1},\\
\partial_{t'} = \dfrac{\partial |V|}{\partial t'}\partial_{|V|} + \dfrac{\partial |Z|}{\partial t'}\partial_{|Z|} + \dfrac{\partial t}{\partial t'}\partial_t + \dfrac{\partial \omega_1}{\partial t'}\partial_{\omega_1},\\
\partial_{\omega_1'} = \dfrac{\partial \omega_1}{\partial \omega_1'}\partial_{\omega_1},\\
\partial_{\omega_2'} = \dfrac{\partial \omega_1}{\partial \omega_2'}\partial_{\omega_1} + \partial_{\omega_2},
\end{cases}
$$

where $\frac{\partial |Z|}{\partial |V'|} = 0$, $\frac{\partial |\omega_1|}{\partial |V'|} = 0$. For the change of variables $(|V'|, |Z'|, t') \mapsto (R, \rho, \phi)$ we get

$$
\begin{cases}
\partial_{|V'|} = \sqrt{1-\rho^2}\,\partial_R - \dfrac{1}{R}\rho\sqrt{1-\rho^2}\,\partial_\rho,\\
\partial_{|Z'|} = \rho\sin\phi\,\partial_R + \dfrac{1}{R}(1-\rho^2)\sin\phi\,\partial_\rho + \dfrac{1}{R\rho}\cos\phi\,\partial_\phi,\\
\partial_{t'} = \rho\cos\phi\,\partial_R + \dfrac{1}{R}(1-\rho^2)\cos\phi\,\partial_\rho - \dfrac{1}{R\rho}\sin\phi\,\partial_\phi,
\end{cases}
$$

with inverse

$$
\begin{cases}
\partial_R = \dfrac{1}{R}(t'\partial_{t'} + |Z'|\partial_{|Z'|} + |V'|\partial_{|V'|}),\\
\partial\rho = \dfrac{1}{\rho}(t'\partial_{t'} + |Z'|\partial_{|Z'|}) - \dfrac{R\rho}{\sqrt{1-\rho^2}}\partial_{|V'|},\\
\partial_\phi = t'\partial_{|Z'|} - |Z'|\partial_{t'}.
\end{cases}
$$

Now we observe from (2.11)–(2.12) that f is biradial on B if and only if $f\circ C$ is biradial on S, and more generally, f is $\mathfrak{v}$-radial on B if and only if $f\circ C$ is $\mathfrak{v}$-radial on S.

Consider then the $\mathfrak{v}$-radial part of L_S, given by

$$L_{\mathfrak{v}\text{-rad}} = \tilde{L}_S + \frac{e^t}{|Z|^2}\left(\frac{1}{4}|V|^2 + e^t\right)L_{S^{q-1}}. \tag{2.14}$$

When we transform $L_{\mathfrak{v}\text{-rad}}$ to B we must get the $\mathfrak{v}$-radial part of the Laplacian on B, plus some operator L_1 such that $L_1 f = 0$ for f $\mathfrak{v}$-radial on B. Let us examine the transformation of the two terms in (2.14) separately. When we transform $\tilde{L}_S$ (given by (2.9)) using (2.13), we get the biradial part $\tilde{L}_B$ of the Laplacian on B (which is known, see below) plus an operator L_1' such that $L_1' f = 0$ for f biradial on B. Moreover, $L_1' f = 0$ if f is $\mathfrak{v}$-radial on B, since every term in L_1' will carry derivatives with respect to the angular variable ω_1'. Next, by the transformation

$$\partial_{\omega_2} = \partial_{\omega_2'} + \frac{\partial \omega_1'}{\partial \omega_2}\partial_{\omega_1'},$$

we see that under the Cayley map, $L_{S^{q-1}} \overset{C}{\mapsto} L_{S^{q-1}} + L_1''$, where $L_1'' f = 0$ for $\mathfrak{v}$-radial f, since L_1'' carries derivatives with respect to ω_1' in every term. Transforming $L_{\mathfrak{v}\text{-rad}}$ to B we then get

$$L_{\mathfrak{v}\text{-rad}} \overset{C}{\mapsto} \tilde{L}_B + \frac{1-R^2}{4|Z'|^2}(|V'|^2 + 1 - R^2)L_{S^{q-1}} + L_1, \tag{2.15}$$

where the first two terms give the $\mathfrak{v}$-radial part of the Laplacian on B, and the operator

$$L_1 = L_1' + \frac{1-R^2}{4|Z'|^2}(|V'|^2 + 1 - R^2)L_1''$$

satisfies $L_1 f = 0$ for f $\mathfrak{v}$-radial on B.

The biradial part $\tilde{L}_B$ is known, namely ([6, Theorem 4.1], [5, Theorem 2.1])

$$\tilde{L}_B = L_{\text{rad}} + \frac{1-R^2}{4R^2}D_1 - \frac{1-R^2}{4}D_2,$$

where

$$D_1 = (1-\rho^2)\partial_\rho^2 + \left(\frac{q}{\rho} - (p+q)\rho\right)\partial_\rho + \frac{1}{\rho^2}(\partial_\phi^2 + (q-1)\cot\phi\,\partial_\phi),$$

$$D_2 = \partial_\phi^2 + (q-1)\cot\phi\,\partial_\phi.$$

The coefficient of $L_{S^{q-1}}$ in (2.15) can be rewritten in terms of R, ρ and ϕ as

$$\frac{1-R^2}{4R^2\rho^2\sin^2\phi}(R^2(1-\rho^2) + 1 - R^2) = \frac{1-R^2}{4R^2\rho^2\sin^2\phi} - \frac{1-R^2}{4\sin^2\phi}.$$

Consider now the term $L_{S^{p-1}}$ in (2.10). It is easy to check that $L_{S^{p-1}} \overset{C}{\mapsto} L_{S^{p-1}}$, i.e., the round Laplacian on S^{p-1}, is invariant under the Cayley transformation. For example, for $p = 2$ and $q = 1$ a direct computation shows that if $\omega_1 = e^{i\phi_1}$ and $\omega_1' = e^{i\phi_1'}$, then the angular coordinates on $S^{p-1} = S^1$ are related by

$$\phi_1 = \phi_1' + \arctan \frac{R\rho \sin\phi}{1 - R\rho\cos\phi},$$

so that $\partial_{\phi_1} = \partial_{\phi_1'}$ and $\partial^2_{\phi_1} = \partial^2_{\phi_1'}$. In the general case we observe that the map $\mathcal{R} : \mathfrak{v} \to \mathfrak{v}$ induced by $\omega_1 \to \omega_1'$, namely

$$\tilde{V} \mapsto \mathcal{R}\tilde{V} = \frac{\left(1 + e^t + \frac{1}{4}|V|^2 - |Z| J_{\omega_2}\right)\tilde{V}}{\left[\left(1 + e^t + \frac{1}{4}|V|^2\right)^2 + |Z|^2\right]^{1/2}},$$

is a linear map preserving the Euclidean norm for any $|V|$, $|Z|$, t and ω_2 fixed. Thus $\mathcal{R} \in \mathrm{O}(\mathfrak{v})$ and the round Laplacian $L_{S^{p-1}}$ is invariant under $\mathcal{R}$, as claimed.

By transforming L_S in (2.10), we then obtain the Laplacian on B in the form

$$\begin{aligned} L_B = L_{\mathrm{rad}} &+ \frac{1-R^2}{4R^2}\left(D_1 + \frac{1}{\rho^2 \sin^2\phi} L_{S^{q-1}} + \frac{1}{1-\rho^2} L_{S^{p-1}}\right) \\ &- \frac{1-R^2}{4}\left(D_2 + \frac{1}{\sin^2\phi} L_{S^{q-1}}\right) + L_1 + L_2, \end{aligned} \tag{2.16}$$

where we write L_2 for the image of L_2 under the Cayley transform. Note that the operators in the round brackets of (2.16) are precisely the round Laplacians $L_{S^{p+q}}$ and L_{S^q} (cf. (2.6), (2.7)). Defining L_3 by

$$L_1 + L_2 = -\frac{1-R^2}{4} L_3 = -\frac{1}{4\,\mathrm{ch}^2(r/2)} L_3,$$

and recalling the relationship $R = \mathrm{th}(r/2)$ between the Euclidean and Riemannian distance in B, we can rewrite the Laplacian on B in geodesic polar coordinates as

$$\begin{aligned} L_B &= L_{\mathrm{rad}} + \frac{1-R^2}{4R^2} L_{S^{p+q}} - \frac{1-R^2}{4}(L_{S^q} + L_3) \\ &= L_{\mathrm{rad}} + \frac{1}{4\,\mathrm{sh}^2(r/2)} L_{S^{p+q}} - \frac{1}{4\,\mathrm{ch}^2(r/2)}(L_{S^q} + L_3). \end{aligned}$$

The angular Laplacian $L_{S(r)}$ is identified as

$$L_{S(r)} = \frac{1}{4\,\mathrm{sh}^2(r/2)} L_{S^{p+q}} - \frac{1}{4\,\mathrm{ch}^2(r/2)}(L_{S^q} + L_3). \tag{2.17}$$

Now $L_3 f = 0$ for f $\mathfrak{v}$-radial on B, since both L_1 and L_2 have this property, so the result follows. ■

REMARK 2.2. The unknown part in $L_{S(r)}$ is the operator L_3. It will be some expression in the derivatives ∂_ϕ, $\partial_{\omega_1'}$, $\partial_{\omega_2'}$. (The derivative ∂_R must obviously cancel out in L_3. The derivative ∂_ρ cancels out in L_3 since the

ρ-coordinate decouples from the remaining coordinates, as we know from the explicit form of the induced metric $\gamma_{S(r)}$ [6]. Thus the derivatives with respect to ρ only occur in the term $(4\,\mathrm{sh}^2(r/2))^{-1}L_{S^{p+q}}$, corresponding to the constant curvature part of the induced metric, namely $4\,\mathrm{sh}^2(r/2)\gamma_{S^{p+q}}$; see [6, Theorem 3.1].) Since $L_3 f = 0$ for f $\mathfrak{v}$-radial, every term of L_3 will contain derivatives with respect to the angular variable ω_1'. Symbolically, L_3 carries the derivatives $\partial^2_{\omega_1'}$, $\partial_{\omega_1'}$, $\partial_{\omega_1'}\partial_{\omega_2'}$, $\partial_{\omega_1'}\partial_\phi$.

REMARK 2.3. In the symmetric case, i.e., when S is a rank-1 symmetric space G/K, L_3 is r-independent and the operator $L' = L_{S^q} + L_3$ in (2.17) is the "vertical" Laplacian acting along the fibers of the Hopf fibration of S^{p+q}. For example for $p = 2$ and $q = 1$ we have

$$v_1\partial_{v_2} - v_2\partial_{v_1} = \partial_{\phi_1} = \partial_{\phi_1'} = v_1'\partial_{v_2'} - v_2'\partial_{v_1'}$$

and

$$L_2 = e^t(v_1\partial_{v_2} - v_2\partial_{v_1})\partial_z = e^t\partial_{\phi_1}\partial_z = e^t\partial_{\phi_1'}\partial_z,$$

where $\partial_z = a\partial_R + b\partial_\rho + c\partial_\phi + d\partial_{\phi_1'}$, with a, b, c, d suitable functions of R, ρ and ϕ. Adding on the contribution from L_1, we see that all terms with the derivatives $\partial_R, \partial_\rho$ cancel out, and we get

$$L_3 = \partial^2_{\phi_1'} + 2\partial_\phi\partial_{\phi_1'} \;\Rightarrow\; L' = L_{S^q} + L_3 = \partial^2_\phi + L_3 = (\partial_\phi + \partial_{\phi_1'})^2 = \partial^2_\theta.$$

Here $\partial_\theta = \partial_\phi + \partial_{\phi_1'} = t'\partial_{z'} - z'\partial_{t'} + v_1'\partial_{v_2'} - v_2'\partial_{v_1'}$ is the Hopf vector field, generating the Hopf action along the fibers isomorphic to S^1 at each point of $S^{p+q} = S^3$.

In the nonsymmetric case, S^{p+q} is no longer a fibration with fiber S^q, and there does not seem to be a natural interpretation of the operator $L' = L_{S^q} + L_3$ in (2.17). Moreover, L_3 will generally depend on r, since the term L' is due to the "perturbed" part of the induced metric (denoted $4\,\mathrm{sh}^4(r/2)h_{\mathrm{th}(r/2)}$ in [6, Theorem 3.1]), which is a complicated differential expression on S^{p+q} explicitly depending on r.

2.2. Spherical harmonics on S^{p+q}. We recall some results of Koornwinder [19]. Let $\mathcal{H}^{p+q+1,n}$ be the space of spherical harmonics of degree n on S^{p+q}. Recall that every $\omega \in S^{p+q}$ can be written as $\omega = \sqrt{1-\rho^2}\omega_1 + \rho\tilde{\omega}_2$, with $0 \le \rho \le 1$, $\omega_1 \in S^{p-1}$, and $\tilde{\omega}_2 \in S^q$. By [19, Theorem 4.2] (with $q \mapsto q+1$, $\cos\theta = \rho$, $m \mapsto n$, $k \mapsto r$, $l \mapsto s$) we have the decomposition

$$\mathcal{H}^{p+q+1,n} = \sum_{\substack{0\le r,s\le n\\ n-r-s\text{ even}\ge 0}} \mathcal{H}^{p+q+1,n}_{r,s},$$

where $\mathcal{H}^{p+q+1,n}_{r,s}$ is the vector space which is spanned by the functions

$$S(\omega) = \rho^s(1-\rho^2)^{r/2}R^{(p/2-1+r,(q-1)/2+s)}_{(n-r-s)/2}(2\rho^2-1)\,S^{(1)}_r(\omega_1)S^{(2)}_s(\tilde{\omega}_2)$$

with

$$S_r^{(1)} \in \mathcal{H}^{p,r} = \text{spherical harmonics of degree } r \text{ on } S^{p-1},$$

$$S_s^{(2)} \in \mathcal{H}^{q+1,s} = \text{spherical harmonics of degree } s \text{ on } S^q,$$

and $R_m^{(a,b)}(x)$ is a Jacobi polynomial normalized so that $R_m^{(a,b)}(1) = 1$. The spaces $\mathcal{H}_{r,s}^{p+q+1,n}$ are mutually orthogonal and they are invariant and irreducible under $\mathrm{SO}(p) \times \mathrm{SO}(q+1)$.

We now refine this decomposition by adapting it to the bispherical coordinate chart $(\rho, \phi, \omega_1, \omega_2)$ of S^{p+q}. As before, we write $\tilde{\omega}_2 \in S^q$ as $\tilde{\omega}_2 = \cos\phi\, H + \sin\phi\, \omega_2$, with $\omega_2 \in S^{q-1}$, $H = (0, 0, 1)$, and $0 \leq \phi \leq \pi$. Then by [19, Theorem 2.4] (with $q \mapsto q+1$) we have the decomposition

$$\mathcal{H}^{q+1,s} = \sum_{j=0}^{s} \mathcal{H}_j^{q+1,s}, \tag{2.18}$$

where $\mathcal{H}_j^{q+1,s}$ is the linear span of the functions

$$S(\tilde{\omega}_2) = (\sin\phi)^j R_{s-j}^{(q/2-1+j, q/2-1+j)}(\cos\phi) S_j^{(2)}(\omega_2)$$

with

$$S_j^{(2)} \in \mathcal{H}^{q,j} = \text{spherical harmonics of degree } j \text{ on } S^{q-1}.$$

The spaces $\mathcal{H}_j^{q+1,s}$ are mutually orthogonal and they are invariant and irreducible under $\mathrm{SO}(q)$. Using this decomposition for the spherical harmonics $S_s^{(2)}(\tilde{\omega}_2)$ in $\mathcal{H}_{r,s}^{p+q+1,n}$ above, we obtain the following decompositions of the spaces $\mathcal{H}_{r,s}^{p+q+1,n}$ and $\mathcal{H}^{p+q+1,n}$:

$$\mathcal{H}_{r,s}^{p+q+1,n} = \sum_{j=0}^{s} \mathcal{H}_{r,s,j}^{p+q+1,n}, \tag{2.19}$$

$$\mathcal{H}^{p+q+1,n} = \sum_{\substack{0 \leq r,s \leq n \\ n-r-s \text{ even} \geq 0}} \sum_{j=0}^{s} \mathcal{H}_{r,s,j}^{p+q+1,n}, \tag{2.20}$$

where $\mathcal{H}_{r,s,j}^{p+q+1,n}$ is the linear span of the functions

$$\begin{aligned} S(\omega) = {} & \rho^s (1-\rho^2)^{r/2} R_{(n-r-s)/2}^{(p/2-1+r, (q-1)/2+s)}(2\rho^2 - 1) \\ & \times (\sin\phi)^j R_{s-j}^{(q/2-1+j, q/2-1+j)}(\cos\phi) S_r^{(1)}(\omega_1) S_j^{(2)}(\omega_2) \end{aligned}$$

with $S_r^{(1)} \in \mathcal{H}^{p,r}$ and $S_j^{(2)} \in \mathcal{H}^{q,j}$. The spaces $\mathcal{H}_{r,s,j}^{p+q+1,n}$ are mutually orthogonal and they are invariant and irreducible under $\mathrm{SO}(p) \times \mathrm{SO}(q)$.

Remark 2.4. For $q = 1$ the decompositions (2.18)–(2.20) must be modified as follows. The index j must be restricted to take the values $0 \leq j \leq \min(s, 1)$ and $S_j^{(2)}(\omega_2) \equiv 1$. The remaining formulas correctly reproduce the decomposition of spherical harmonics of degree s on $S^q = S^1$. For example, if $s \geq 1$, then (2.18) reads $\mathcal{H}^{2,s} = \mathcal{H}_0^{2,s} \oplus \mathcal{H}_1^{2,s}$, where $\mathcal{H}_0^{2,s}$ and $\mathcal{H}_1^{2,s}$ are the linear spans of the functions $R_s^{(-1/2,-1/2)}(\cos\phi) = \cos(s\phi)$ and $(\sin\phi)R_{s-1}^{(1/2,1/2)}(\cos\phi) = s^{-1}\sin(s\phi)$, respectively.

2.3. Separation of variables. The $\mathfrak{v}$-radial eigenfunctions of the angular Laplacian $L_{S(r)}$ are those that are independent of ω_1 in the bispherical coordinate chart $(\rho, \phi, \omega_1, \omega_2)$ of $S^{p+q} \simeq S(r)$. It follows from (2.8) that the $\mathfrak{v}$-radial eigenfunctions of $L_{S(r)}$ in $\mathcal{H}^{p+q+1,n}$ are the elements of

$$\mathcal{H}_{0,s}^{p+q+1,n} = \sum_{j=0}^{s} \mathcal{H}_{0,s,j}^{p+q+1,n} \qquad (0 \leq s \leq n,\ n-s \text{ even } \geq 0),$$

namely

$$Y \in \mathcal{H}_{0,s}^{p+q+1,n} \ \Rightarrow\ L_{S(r)}Y = \left(-\frac{n(n+p+q-1)}{4\operatorname{sh}^2(r/2)} + \frac{s(s+q-1)}{4\operatorname{ch}^2(r/2)}\right)Y,$$

and conversely, if $Y \in \mathcal{H}^{p+q+1,n}$ is a $\mathfrak{v}$-radial eigenfunctions of $L_{S(r)}$, then $Y \in \mathcal{H}_{0,s}^{p+q+1,n}$ for some s with $0 \leq s \leq n$ and s of the same parity of n. Letting $n = k+l$ and $s = k-l$, we find that the $\mathfrak{v}$-radial eigenfunctions of $L_{S(r)}$ with the eigenvalue

$$\lambda_{k,l} = -\frac{(k+l)(k+l+p+q-1)}{4\operatorname{sh}^2(r/2)} + \frac{(k-l)(k-l+q-1)}{4\operatorname{ch}^2(r/2)} \tag{2.21}$$

are the elements of $\mathcal{H}_{0,k-l}^{p+q+1,k+l}$. They belong to subspaces invariant and irreducible under $\mathrm{SO}(p)\times\mathrm{SO}(q+1)$. The $\mathfrak{v}$-radial eigenfunctions with the eigenvalue $\lambda_{k,l}$ that belong to subspaces invariant and irreducible under $\mathrm{SO}(p)\times\mathrm{SO}(q)$ are the elements of $\mathcal{H}_{0,k-l,j}^{p+q+1,k+l}$, with $0 \leq j \leq k-l$. A basis of $\mathcal{H}_{0,k-l,j}^{p+q+1,k+l}$ is given by the functions

$$\begin{aligned} \chi_{k,l,j,i}(\rho,\phi,\omega_2) = \rho^{k-l} & R_l^{(p/2-1,(q-1)/2+k-l)}(2\rho^2-1) \\ & \times (\sin\phi)^j R_{k-l-j}^{(q/2-1+j,q/2-1+j)}(\cos\phi) S_{j,i}^{(2)}(\omega_2), \end{aligned} \tag{2.22}$$

where $S_{j,i}^{(2)}$ $(i = 1,\dots,\dim\mathcal{H}^{q,j})$ is a basis of $\mathcal{H}^{q,j}$. For k and l fixed, the only eigenfunctions that are invariant under $\mathrm{SO}(p)\times\mathrm{SO}(q)$ are those with $j=0$, namely the biradial eigenfunctions (cf. [6, 5])

$$\begin{aligned} \chi_{k,l}(\rho,\phi) &= \rho^{k-l} R_l^{(p/2-1,(q-1)/2+k-l)}(2\rho^2-1) R_{k-l}^{(q/2-1,q/2-1)}(\cos\phi) \\ &\in \mathcal{H}_{0,k-l,0}^{p+q+1,k+l} \subset \mathcal{H}_{0,k-l}^{p+q+1,k+l}. \end{aligned}$$

The degeneracy of $\lambda_{k,l}$ is then at least $\dim\mathcal{H}_{0,k-l}^{p+q+1,k+l} = \sum_{j=0}^{k-l}\dim\mathcal{H}^{q,j}$. It will actually be bigger than this, since this number only depends on $k-l$ but not on $k+l$.

REMARK 2.5. In [18, pp. 27–28] it is observed that, for any noncompact harmonic space X, the number $(f'/f)'(r)$, where f is the density function, is an eigenvalue of $L_{S(r)}$ with degeneracy $\geq \dim X$. For a Damek–Ricci space we have $f(r) \propto (\operatorname{sh}(r/2))^{p+q}(\operatorname{ch}(r/2))^q$, so that $(f'/f)'(r)$ is precisely the first nonzero $\mathfrak{v}$-radial eigenvalue $\lambda_{1,0} = \lambda_{1,0}(r)$ in (2.21). Thus the degeneracy of $\lambda_{1,0}$ is at least $p+q+1$, whereas $\dim \mathcal{H}_{0,1}^{p+q+1,1} = q+1$. Note that $p+q+1 = \dim \mathcal{H}_{0,1}^{p+q+1,1} + \dim \mathcal{H}_{1,0}^{p+q+1,1}$.

The normalized Euclidean measure $d\omega$ on S^{p+q} can be written in bi-spherical coordinates as

$$\begin{aligned} d\omega &= \frac{\operatorname{vol}(S^{p-1}) \operatorname{vol}(S^{q-1})}{\operatorname{vol}(S^{p+q})} \rho^q (1-\rho^2)^{p/2-1} (\sin\phi)^{q-1} d\rho d\phi d\omega_1 d\omega_2 \\ &:= dm(\rho,\phi) d\omega_1 d\omega_2, \end{aligned}$$

where $d\omega_1$ and $d\omega_2$ are the normalized Euclidean measures on S^{p-1} and S^{q-1}, respectively.

For a $\mathfrak{v}$-radial function χ on S^{p+q} we get (writing $\chi(\omega) = \chi(\rho,\phi,\omega_2)$)

$$\int_{S^{p+q}} \chi(\omega)\, d\omega = \int_0^1 \int_0^\pi \int_{S^{q-1}} \chi(\rho,\phi,\omega_2)\, dm(\rho,\phi)\, d\omega_2.$$

Suppose the basis $\{S_{j,i}^{(2)}\}$ of $\mathcal{H}^{q,j}$ is orthonormal in $L^2(S^{q-1}, d\omega_2)$. Then the system $\{\chi_{k,l,j,i}\}$ is orthogonal on $D_+ \times S^{q-1}$ with respect to the measure $d\mu = dm(\rho,\phi) d\omega_2$,

$$\begin{aligned} \int_{S^{p+q}} \chi_{k,l,j,i}(\omega) \chi_{k',l',j',i'}(\omega)\, d\omega &= \int_{D_+ \times S^{q-1}} \chi_{k,l,j,i}\, \chi_{k',l',j',i'}\, d\mu \\ &= \|\chi_{k,l,j,i}\|^2 \delta_{kk'} \delta_{ll'} \delta_{jj'} \delta_{ii'}. \end{aligned}$$

The squared L^2-norm $\|\chi_{k,l,j,i}\|^2$ is computed to be $(\pi_{k,l,j})^{-1}$ with

$$\pi_{k,l,j} = \frac{(2k-2l+2\beta)(k+l+\alpha)(\alpha-\beta)_l (2\beta+1)_{k-l} (\alpha+1)_k (k-l+2\beta)_j}{2^{2j}(k-l+2\beta)(k+\alpha)\, l!\, (k-l-j)! (\beta+1)_k (\beta+1/2)_j^2},$$

where $\alpha = (p+q-1)/2$, $\beta = (q-1)/2$, and $(a)_k$ is defined by $(a)_0 = 1$ and

$$(a)_n = \Gamma(a+n)/\Gamma(a) = a(a+1)\cdots(a+n-1).$$

A smooth $\mathfrak{v}$-radial function χ on S^{p+q} can then be expanded as

$$\chi = \sum_{k=0}^{\infty} \sum_{l=0}^{k} \sum_{j=0}^{k-l} \sum_{i=1}^{\dim \mathcal{H}^{q,j}} \pi_{k,l,j} a_{k,l,j,i} \chi_{k,l,j,i},$$

where

$$a_{k,l,j,i} = \int_{S^{p+q}} \chi(\omega) \chi_{k,l,j,i}(\omega)\, d\omega.$$

Consider now the eigenvalue equation for the Laplacian on B:

$$L_B f = -(\lambda^2 + Q^2/4) f \qquad (\lambda \in \mathbb{C},\, Q = p/2 + q). \tag{2.23}$$

Here we can separate variables in geodesic polar coordinates, by looking for the $\mathfrak{v}$-radial solutions of the form $f(r,\omega) = \phi(r)\chi(\omega)$.

Recall that for $\lambda \in \mathbb{C}$, $t \in \mathbb{R}$, $\alpha > \beta > -1/2$, and $k, l \in \mathbb{Z}$, $k \geq l \geq 0$, one defines the *associated Jacobi functions* by (see [20])

$$\phi^{(\alpha,\beta)}_{\lambda,k,l}(t) = c(2\,\mathrm{sh}\, t)^{k+l}(2\,\mathrm{ch}\, t)^{k-l}\phi^{(\alpha+k+l,\beta+k-l)}_{\lambda}(t),$$

where c is a normalization constant and $\phi^{(\alpha,\beta)}_{\lambda}$ is a Jacobi function:

$$\phi^{(\alpha,\beta)}_{\lambda}(t) = F\left(\frac{\alpha+\beta+1-i\lambda}{2}, \frac{\alpha+\beta+1+i\lambda}{2}, \alpha+1, -\mathrm{sh}^2\, t\right)$$

($F(a,b,c,z)$ is the hypergeometric function). The functions $\phi = \phi^{(\alpha,\beta)}_{\lambda,k,l}$ are the unique solutions (up to normalization) of the following equation that are regular at $t = 0$:

$$\begin{aligned}\Big\{\partial_t^2 &+ \big((2\alpha+1)\,\mathrm{cth}\, t + (2\beta+1)\,\mathrm{th}, t\big)\partial_t \\ &- \frac{(k+l)(k+l+2\alpha)}{\mathrm{sh}^2\, t} + \frac{(k-l)(k-l+2\beta)}{\mathrm{ch}^2\, t}\Big\}\phi = -(\lambda^2 + (\alpha+\beta+1)^2)\phi.\end{aligned} \tag{2.24}$$

Putting together (2.4), (2.5), (2.8), (2.21), (2.22) and (2.24), we obtain the following result.

Theorem 2.6. *Let $S = NA \overset{C}{\cong} B$ be a Damek–Ricci space. The $\mathfrak{v}$-radial eigenfunctions of the Laplacian, solutions of* (2.23) *that separate in geodesic polar coordinates and belong to subspaces invariant and irreducible under* $\mathrm{SO}(p)\times\mathrm{SO}(q)$*, are given by*

$$f_{\lambda,k,l,j,i}(b) = f_{\lambda,k,l,j,i}(\mathrm{th}(r/2)\omega) = \phi_{\lambda,k,l}(r)\chi_{k,l,j,i}(\omega), \tag{2.25}$$

where the $\phi_{\lambda,k,l}$ are the associated Jacobi functions

$$\begin{aligned}\phi_{\lambda,k,l}(r) &= \phi^{(\alpha,\beta)}_{2\lambda,k,l}(r/2) \\ &= q_{k,l}(\lambda)(2\,\mathrm{sh}(r/2))^{k+l}(2\,\mathrm{ch}(r/2))^{k-l}\phi^{(\alpha+k+l,\beta+k-l)}_{2\lambda}(r/2).\end{aligned} \tag{2.26}$$

Here $q_{k,l}(\lambda)$ is a normalization constant, the functions $\chi_{k,l,j,i}$ are given by (2.22)*, and the indices are as follows:*

$$k,l \in \mathbb{Z}, \quad k \geq l \geq 0, \quad 0 \leq j \leq k-l, \quad 1 \leq i \leq \dim \mathcal{H}^{q,j},$$
$$\alpha = (p+q-1)/2, \quad \beta = (q-1)/2.$$

The functions $f_{\lambda,k,l,j,i}$ are biradial if and only if $j = 0$, in which case they reduce to the biradial eigenfunctions

$$f_{\lambda,k,l}(b) = f_{\lambda,k,l}(\mathrm{th}(r/2)\omega) = \phi_{\lambda,k,l}(r)\chi_{k,l}(\omega)$$

(*cf.* [6, *Theorem* 5.1]). *They are radial if and only if* $k = l = j = 0$, *in which case they reduce to the spherical functions* $\phi_\lambda(r) = \phi_{2\lambda}^{(\alpha,\beta)}(r/2)$.

2.4. Poisson integral representation. Let $\mathcal{P}(x,n)$ be the Poisson kernel on NA given by (see [8])

$$\mathcal{P}(a_t,n) = c_{p,q}\left(\frac{e^t}{\left(e^t + \frac{1}{4}|V|^2\right)^2 + |Z|^2}\right)^Q \tag{2.27}$$

for $x = a_t = \exp(tH) \in A$, and by

$$\mathcal{P}(na_t,n') = \mathcal{P}(a_t,n^{-1}n') \qquad (n,n' \in N)$$

for general $x = na_t \in S$. Define the normalized Poisson kernel with parameter $\lambda \in \mathbb{C}$ on NA as the following function on $NA \times N$ (cf. [2]):

$$\mathcal{Q}_\lambda(x,n) = \frac{\mathcal{P}_\lambda(x,n)}{\mathcal{P}_\lambda(e,n)} \qquad (x \in NA,\, n \in N), \tag{2.28}$$

where

$$\mathcal{P}_\lambda(x,n) = (\mathcal{P}(x,n))^{1/2-i\lambda/Q}. \tag{2.29}$$

We define a kernel $\tilde{\mathcal{Q}}_\lambda$ on B by

$$\tilde{\mathcal{Q}}_\lambda(C(x),C_0(n)) = \mathcal{Q}_\lambda(x,n) \qquad (x \in NA,\, n \in N),$$

that is,

$$\tilde{\mathcal{Q}}_\lambda(b,\omega) = \mathcal{Q}_\lambda(C^{-1}(b),C_0^{-1}(\omega)) \qquad (b \in B,\, \omega \in \partial B \setminus \{H\}).$$

From now on we write $\mathcal{Q}_\lambda(b,\omega)$ in place of $\tilde{\mathcal{Q}}_\lambda(b,\omega)$. The kernel $\mathcal{Q}_\lambda(b,\omega)$ extends to a smooth kernel on $B\times\partial B$. For example for $b = C(a_t) = \operatorname{th}(t/2)H$ and $\omega = (\rho,\phi,\omega_1,\omega_2)$, we have

$$\mathcal{Q}_\lambda(C(a_t),\omega) = |\operatorname{ch}(t/2) - \rho e^{i\phi}\operatorname{sh}(t/2)|^{2i\lambda-Q}$$

(cf. [6, (5.20)]). In particular, for $\omega = H = (0,0,1)$,

$$\mathcal{Q}_\lambda(C(a_t),H) = e^{t(Q/2-i\lambda)}.$$

For a suitable choice of the constant $q_{k,l}(\lambda)$ in (2.26) (see below for details), one has the following Poisson integral representation of the associated Jacobi functions (cf. [6, Theorem 5.2]):

$$\phi_{\lambda,k,l}(t) = f_{\lambda,k,l}(\operatorname{th}(t/2)H) = \int_{S^{p+q}} \mathcal{Q}_\lambda(\operatorname{th}(t/2)H,\omega)\chi_{k,l}(\omega)\,d\omega.$$

This formula extends to the biradial eigenfunctions $f_{\lambda,k,l}(b)$ at arbitrary points, namely for any b in B,

$$f_{\lambda,k,l}(b) = \int_{S^{p+q}} \mathcal{Q}_\lambda(b,\omega)\chi_{k,l}(\omega)\,d\omega. \tag{2.30}$$

See [5, Theorem 2.2] for a proof of this result involving the Radon transform and the method of "descent" to complex hyperbolic spaces, which was used also in [11, 21] for the radial case ($k = l = 0$). A different proof in the radial case appears in [1, pp. 654–655].

The expression of $q_{k,l}(\lambda)$ is (see [5])

$$q_{k,l}(\lambda) = \frac{(-i\lambda + Q/2)_k\,(-i\lambda + p/4 + 1/2)_l}{(d/2)_{k+l}}. \tag{2.31}$$

This can also be written as a ratio of c-functions, namely

$$q_{k,l}(\lambda) = \frac{c_{\alpha,\beta}(-2\lambda)}{c_{\alpha+k+l,\beta+k-l}(-2\lambda)}, \tag{2.32}$$

where

$$c_{\alpha,\beta}(\lambda) = \frac{2^{\alpha+\beta+1-i\lambda}\Gamma(i\lambda)\Gamma(\alpha+1)}{\Gamma\big(\frac{i\lambda+\alpha+\beta+1}{2}\big)\Gamma\big(\frac{i\lambda+\alpha-\beta+1}{2}\big)}. \tag{2.33}$$

(See [20, (2.18), (4.15), (8.5), and (8.7)]. For the symmetric case, see [20, (8.13) and the last part of Section 8.1]. See also [13, Theorem 7 and Remark on p. 277].)

Observe that the c-function $c(\lambda)$ in (1.3) is precisely $c_{\alpha,\beta}(2\lambda)$ if $\alpha = (p+q-1)/2$ and $\beta = (q-1)/2$, as we continue to assume.

We now have a result similar to (2.30) for the $\mathfrak{v}$-radial eigenfunctions $f_{\lambda,k,l,j,i}$ in (2.25). The proof can be given along the same lines of [5, Theorem 2.2] for the biradial case ($j = 0$). Since the proof is quite involved already in the biradial case, we shall omit it altogether.

Theorem 2.7. *Let $q_{k,l}(\lambda)$ in* (2.26) *be given by* (2.31). *For all $b \in B$ we have*

$$f_{\lambda,k,l,j,i}(b) = \int_{S^{p+q}} \mathcal{Q}_\lambda(b,\omega)\chi_{k,l,j,i}(\omega)\,d\omega. \tag{2.34}$$

Equivalently, if we define $f_{\lambda,k,l,j,i}(b)$ by (2.34), *then* (2.25) *holds, i.e.,*

$$\int_{S^{p+q}} \mathcal{Q}_\lambda(\mathrm{th}(r/2)\omega,\omega')\chi_{k,l,j,i}(\omega')\,d\omega' = \phi_{\lambda,k,l}(r)\chi_{k,l,j,i}(\omega), \qquad \forall r \geq 0,\ \forall\omega \in S^{p+q}. \tag{2.35}$$

Remark 2.8. In the symmetric case, the functions $\chi_{k,l,j,i}$ on $S^{p+q} \simeq K/M$ can be identified with suitable matrix coefficients of a K-type $\delta_{k,l}$ containing an M-fixed vector. The result (2.34)–(2.35) then follows for this general class of matrix coefficients (not only the $\mathfrak{v}$-radial ones) by an easy change-of-variable argument, rewriting the integral over K/M as an integral over K (see [16, Lemma 4.2]). This can be interpreted by saying that for a symmetric space G/K of rank one, the subspace $\mathcal{E}_{\lambda,\delta_{k,l}}$ of K-finite functions of type $\delta_{k,l}$ in $\mathcal{E}_\lambda$ (the smooth eigenfunctions of L_B satisfying (2.23)) is essentially determined by the single function $\phi_{\lambda,k,l}$ (up to angular functions). Formulas (2.34)–(2.35) generalize this to any Damek–Ricci space, but only for the class of $\mathfrak{v}$-radial functions.

3. The Helgason Fourier transform in the $\mathfrak{v}$-radial case

3.1. Fourier series expansion of f and $\widetilde{f}$. Let f be a $\mathfrak{v}$-radial function in $C_0^\infty(B)^{\mathfrak{v}\text{-rad}}$, with $\operatorname{supp} f \subset B_R$. The function $\omega \mapsto f(r,\omega) = f(\operatorname{th}(r/2)\omega)$ can be expanded in the Fourier series

$$f(r,\omega) = \sum_{k\geq l}\sum_{j}\sum_{i} \pi_{k,l,j} a_{k,l,j,i}(r)\chi_{k,l,j,i}(\omega), \tag{3.1}$$

where the Fourier coefficients

$$a_{k,l,j,i}(r) = \int_{S^{p+q}} f(r,\omega)\chi_{k,l,j,i}(\omega)\, d\omega \tag{3.2}$$

are smooth functions of the geodesic distance r supported in $[0,R]$.

Let $\widetilde{f}(\lambda,\omega)$ be the Fourier transform of f given by (1.1). By the direct part of Theorem 1.1 (that was proved in [2, Theorem 4.5]), the function $\widetilde{f}(\lambda,\omega)$ is holomorphic of uniform exponential type with constant R.

Lemma 3.1. *Let $f \in C_0^\infty(B)^{\mathfrak{v}\text{-rad}}$. Then, for each $\lambda\in\mathbb{C}$, the function $\omega\mapsto\widetilde{f}(\lambda,\omega)$ is $\mathfrak{v}$-radial on S^{p+q}.*

Proof. Consider the normalized Helgason Fourier transform of $f\circ C$ in $S = NA$ given by

$$\widetilde{f\circ C}(\lambda,n) = \int_S (f\circ C)(x)\mathcal{Q}_\lambda(x,n)\,dx \qquad (\lambda\in\mathbb{C},\ n\in N).$$

Then

$$\widetilde{f}(\lambda, C_0(n)) = \widetilde{f\circ C}(\lambda,n),$$

and we need to prove that $n\mapsto \widetilde{f\circ C}(\lambda,n)$ is $\mathfrak{v}$-radial. For simplicity, we write f in place of $f\circ C$ in the following. Let

$$\widehat{f}(\lambda,n) = \int_S f(x)\mathcal{P}_\lambda(x,n)\,dx$$

be the unnormalized Helgason Fourier transform of f, so that (cf. (2.28))

$$\widehat{f}(\lambda,n) = \mathcal{P}_\lambda(e,n)\widetilde{f}(\lambda,n).$$

Since $n\mapsto\mathcal{P}_\lambda(e,n)$ is biradial (cf. (2.27) and (2.29)), it is enough to prove that $n\mapsto\widehat{f}(\lambda,n)$ is $\mathfrak{v}$-radial. We use the Radon transform to reduce the problem to the case $q=1$ of complex hyperbolic spaces.

Fix $\omega\in\mathfrak{z}$ with $|\omega|=1$, let $\mathfrak{z}_o=\mathbb{R}\omega$, and consider the subspaces $\mathfrak{n}_o=\mathfrak{v}\oplus\mathfrak{z}_o$ and $\mathfrak{s}_o=\mathfrak{n}_o\oplus\mathfrak{a}$ of $\mathfrak{s}$, with the scalar product induced from that on $\mathfrak{s}$. Then $\mathfrak{n}_o$ is a Heisenberg-type Lie algebra if one defines the commutator

$[V, V']_o = \pi_o([V, V'])$, where π_o is the orthogonal projection of $\mathfrak{z}$ onto $\mathfrak{z}_o$. The associated Lie group N_o is the classical Heisenberg group of dimension $p+1$. We recall that p is even, so we let $p = 2(n-1)$ $(n \geq 2)$. Then the Damek–Ricci space $S_o = N_oA$ can be identified with the complex hyperbolic space $H^n(\mathbb{C}) \simeq G_o/K_o$, where $G_o = \mathrm{SU}(n,1) \simeq N_oAK_o$ and $K_o = \mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(1))$.

The centralizer M_o of A in K_o is connected and acts trivially on the center $\mathfrak{z}_o$. In matrix form we have

$$M_o = \left\{ m = \begin{pmatrix} u & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & u \end{pmatrix} : U \in \mathrm{U}(n-1),\, u \in \mathrm{U}(1),\, u^2 \det U = 1 \right\}.$$

The map $m \mapsto U$ is a $2:1$ homomorphism of M_o onto $\mathrm{U}(n-1)$. The action of M_o on $\mathfrak{v} \simeq \mathbb{R}^{2(n-1)} \simeq \mathbb{C}^{n-1}$ is given by $m \cdot V = u^{-1}UV$, and M_o acts transitively on the unit sphere $S^{p-1} = S^{2n-3}$ in $\mathfrak{v}$.

The group $\widetilde{M_o}$ of orthogonal automorphisms of N_oA, i.e., of the H-type Lie algebra $\mathfrak{n}_o = \mathfrak{v} \oplus \mathfrak{z}_o$, is given by

$$\widetilde{M_o} = M_o \cup (\sigma M_o),$$

where σ is the automorphism of $\mathfrak{v} \oplus \mathfrak{z}_o$ defined by $\sigma = (\psi, -\mathrm{Id})$, with ψ any orthogonal isomorphism of $\mathfrak{v}$ that anticommutes with J_ω. For example if $\mathfrak{v} = \mathrm{span}\{X_i, Y_i\}_{i=1}^{n-1}$ with $[X_i, Y_j] = \delta_{ij}\omega$, we can take $\psi(X_i, Y_i) = (-Y_i, -X_i)$.

We denote by $\mathcal{P}^{(o)}(x_o, n_o)$ the Poisson kernel on S_o, and define

$$\mathcal{P}^{(o)}_\lambda(x_o, n_o) = (\mathcal{P}^{(o)}(x_o, n_o))^{1/2 - i\lambda/n} \qquad (\lambda \in \mathbb{C},\, x_o \in S_o,\, n_o \in N_o).$$

Given $g \in C_0^\infty(S)$ and $\omega \in S^{q-1}$ $(q > 1)$, we define the Radon transform of g by

$$\mathcal{R}_\omega g(V, \eta, t) = e^{(1-q)t/2} \int_{\omega^\perp} g(V, \eta\omega + \tilde{Z}, t)\, d\tilde{Z},$$

where $\eta, t \in \mathbb{R}$ and $\omega^\perp$ is the orthogonal complement of ω in $\mathfrak{z}$, with Lebesgue measure $d\tilde{Z}$. The function $\mathcal{R}_\omega g$ is in $C_0^\infty(S_o)$. Note that $\mathcal{R}_\omega g = 0$ for all $\omega \in S^{q-1}$ implies $g = 0$. (This corrects a wrong statement in [5, p. 440] about the injectivity of the maps $\mathcal{R}_\omega$ for ω fixed.) We keep the same notation $\mathcal{R}_\omega g$ for the Radon transform applied to functions g on N. In this case the variable t is absent and the factor $e^{(1-q)t/2}$ is omitted. We observe that g is $\mathfrak{v}$-radial in S (resp. N) if and only if $\mathcal{R}_\omega g$ is $\mathfrak{v}$-radial in S_o (resp. N_o) for every $\omega \in \mathfrak{z} \cap S^{p+q}$.

Consider now the Radon transform in N of the function $n \mapsto \widehat{f}(\lambda, n)$. It can be shown that this is well defined and that it is related to the Helgason Fourier transform in $S_o = N_oA$ of the function $\mathcal{R}_\omega f$.

Indeed by [24, Proposition 5.1] we have, for all $\omega \in S^{q-1}$,

$$(3.3)\qquad \begin{aligned}(\mathcal{R}_\omega \widehat{f}(\lambda,\cdot))(n_o) &= c_q B_{p,q}(\lambda)\mathcal{P}_\lambda^{(o)}(e,n_o)(\mathcal{R}_\omega f)\tilde{}(\lambda,n_o)\\ &= c_q B_{p,q}(\lambda)(\mathcal{R}_\omega f)\widehat{}(\lambda,n_o) \quad (n_o\in N_o),\end{aligned}$$

where c_q is a constant depending only on q, and $B_{p,q}(\lambda)$ is the meromorphic function

$$B_{p,q}(\lambda) = \frac{\Gamma((q-1)/2)\Gamma(p/4+1/2-i\lambda)}{\Gamma(Q/2-i\lambda)}.$$

We can describe (3.3) as follows: the Radon transform in N of the (unnormalized) Helgason Fourier transform of f in NA is proportional to the (unnormalized) Helgason Fourier transform in N_oA of the Radon transform of f in NA.

Since f is $\mathfrak{v}$-radial in S, the function $g = \mathcal{R}_\omega f$ is $\mathfrak{v}$-radial in S_o, and we need to prove that $n_o \mapsto \widehat{g}(\lambda, n_o)$ is $\mathfrak{v}$-radial in N_o. Since M_o is transitive on the unit sphere S^{p-1} in $\mathfrak{v}_o = \mathfrak{v}$, it is enough to show that

$$\widehat{g}(\lambda,(m\cdot V,\eta)) = \widehat{g}(\lambda,(V,\eta)), \qquad \forall m\in M_o,\ V\in\mathfrak{v},\ \eta\in\mathbb{R}.$$

Now

$$\widehat{g}(\lambda,(V,\eta)) = \int_{\mathfrak{v}\times\mathbb{R}\times\mathbb{R}} g(\tilde V,\tilde\eta,\tilde t)\mathcal{P}_\lambda^{(o)}(\tilde V,\tilde\eta,\tilde t,(V,\eta))\,d\mu(\tilde V,\tilde\eta,\tilde t),$$

where $d\mu(\tilde V,\tilde\eta,\tilde t) = e^{-n\tilde t}d\tilde V d\tilde\eta d\tilde t$ and

$$\mathcal{P}_\lambda^{(o)}(\tilde V,\tilde\eta,\tilde t,(V,\eta)) = \Big(\frac{(n-1)!}{\pi^n}\Big)^{1/2-i\lambda/n}\Big(\frac{e^{\tilde t}}{\big(e^{\tilde t}+\frac14|V-\tilde V|^2\big)^2+\big|(\eta-\tilde\eta)\omega-\frac12[V,\tilde V]\big|^2}\Big)^{n/2-i\lambda}.$$

Since g is $\mathfrak{v}$-radial, we have, for all $m\in M_o$,

$$\begin{aligned}\widehat{g}(\lambda,(m\cdot V,\eta)) &= \int_{\mathfrak{v}\times\mathbb{R}\times\mathbb{R}} g(\tilde V,\tilde\eta,\tilde t)\mathcal{P}_\lambda^{(o)}(\tilde V,\tilde\eta,\tilde t,(m\cdot V,\eta))\,d\mu(\tilde V,\tilde\eta,\tilde t)\\ &= \int_{\mathfrak{v}\times\mathbb{R}\times\mathbb{R}} g(m^{-1}\cdot\tilde V,\tilde\eta,\tilde t)\mathcal{P}_\lambda^{(o)}(\tilde V,\tilde\eta,\tilde t,(m\cdot V,\eta))\,d\mu(\tilde V,\tilde\eta,\tilde t)\\ &= \int_{\mathfrak{v}\times\mathbb{R}\times\mathbb{R}} g(\tilde V,\tilde\eta,\tilde t)\mathcal{P}_\lambda^{(o)}(m\cdot\tilde V,\tilde\eta,\tilde t,(m\cdot V,\eta))\,d\mu(\tilde V,\tilde\eta,\tilde t).\end{aligned}$$

But

$$\begin{aligned}&\mathcal{P}_\lambda^{(o)}(m\cdot\tilde V,\tilde\eta,\tilde t,(m\cdot V,\eta))\\ &\quad = \Big(\frac{(n-1)!}{\pi^n}\Big)^{1/2-i\lambda/n}\Big(\frac{e^{\tilde t}}{\big(e^{\tilde t}+\frac14|V-\tilde V|^2\big)^2+\big|(\eta-\tilde\eta)\omega-\frac12 m\cdot[V,\tilde V]\big|^2}\Big)^{n/2-i\lambda}\\ &\quad = \mathcal{P}_\lambda^{(o)}(\tilde V,\tilde\eta,\tilde t,(V,\eta)),\end{aligned}$$

since M_o is trivial on $\mathfrak{z}_o$. It follows that $\widehat{g}(\lambda,\cdot) = (\mathcal{R}_\omega f)\widehat{\ }(\lambda,\cdot)$ is $\mathfrak{v}$-radial in N_o, so $(\mathcal{R}_\omega \widehat{f}(\lambda,\cdot))$ is $\mathfrak{v}$-radial in N_o for all $\omega \in S^{q-1}$, and finally $\widehat{f}(\lambda,\cdot)$ is $\mathfrak{v}$-radial in N. ■

By this lemma, we have the Fourier expansion

$$\widetilde{f}(\lambda,\omega) = \sum_{k\geq l}\sum_j\sum_i \pi_{k,l,j}\, b_{k,l,j,i}(\lambda)\chi_{k,l,j,i}(\omega), \tag{3.4}$$

where the coefficients

$$b_{k,l,j,i}(\lambda) = \int\limits_{S^{p+q}} \widetilde{f}(\lambda,\omega)\chi_{k,l,j,i}(\omega)\,d\omega$$

are holomorphic functions of λ of exponential type R.

The functions $a_{k,lj,i}(r)$ and $b_{k,l,j,i}(\lambda)$ are related as follows. Let $\alpha' = \alpha + k + l$ and $\beta' = \beta + k - l$. Define $c_{\alpha',\beta'}(\lambda)$ by (2.33). Let $\mathcal{J}^{(\alpha',\beta')}(g)$ be the Jacobi transform of $g \in C_R^\infty(\mathbb{R})_{\mathrm{even}}$, defined by (see [20])

$$\mathcal{J}^{(\alpha',\beta')}(g)(\lambda) = \int\limits_0^\infty g(t)\phi_\lambda^{(\alpha',\beta')}(t)(2\operatorname{sh} t)^{2\alpha'+1}(2\operatorname{ch} t)^{2\beta'+1}\,dt,$$

with inverse (see [20, Theorem 2.3])

$$(\mathcal{J}^{(\alpha',\beta')})^{-1}(h)(t) = \frac{1}{2\pi}\int\limits_0^\infty h(\lambda)\phi_\lambda^{(\alpha',\beta')}(t)|c_{\alpha',\beta'}(\lambda)|^{-2}\,d\lambda.$$

Proposition 3.2. *We have*

$$\begin{aligned} b_{k,l,j,i}(\lambda) &= \frac{\operatorname{vol}(S^{p+q})}{2^q} q_{k,l}(\lambda) \\ &\quad\times \int\limits_0^\infty a_{k,l,j,i}(r)\,\phi_{2\lambda}^{(\alpha',\beta')}\left(\frac{r}{2}\right)\left(2\operatorname{sh}\frac{r}{2}\right)^{p+q+k+l}\left(2\operatorname{ch}\frac{r}{2}\right)^{q+k-l} dr \end{aligned} \tag{3.5}$$

$$= \frac{\operatorname{vol}(S^{p+q})}{2^{q-1}} q_{k,l}(\lambda)\mathcal{J}^{(\alpha',\beta')}\left(\frac{a_{k,l,j,i}(2t)}{(2\operatorname{sh} t)^{k+l}(2\operatorname{ch} t)^{k-l}}\right)(2\lambda), \tag{3.6}$$

where $q_{k,l}(\lambda)$ is given by (2.31), *and conversely*

$$\begin{aligned} a_{k,l,j,i}(r) &= \frac{c_{p,q}}{2\pi}\left(2\operatorname{sh}\frac{r}{2}\right)^{k+l}\left(2\operatorname{ch}\frac{r}{2}\right)^{k-l}\int\limits_0^\infty q_{k,l}(-\lambda)\phi_{2\lambda}^{(\alpha',\beta')}\left(\frac{r}{2}\right)b_{k,l,j,i}(\lambda)\,d\mu(\lambda) \end{aligned} \tag{3.7}$$

$$= \frac{2^{q-1}}{\operatorname{vol}(S^{p+q})}\left(2\operatorname{sh}\frac{r}{2}\right)^{k+l}\left(2\operatorname{ch}\frac{r}{2}\right)^{k-l}(\mathcal{J}^{(\alpha',\beta')})^{-1}\left(\frac{b_{k,l,j,i}(\lambda'/2)}{q_{k,l}(\lambda'/2)}\right)\left(\frac{r}{2}\right) \tag{3.8}$$

$$= \frac{2^q}{\mathrm{vol}(S^{p+q})}\left(2\,\mathrm{sh}\,\frac{r}{2}\right)^{k+l}\left(2\,\mathrm{ch}\,\frac{r}{2}\right)^{k-l} \tag{3.9}$$
$$\times \frac{1}{2\pi}\int_0^\infty \frac{b_{k,l,j,i}(\lambda)}{q_{k,l}(\lambda)}\phi_{2\lambda}^{(\alpha',\beta')}\left(\frac{r}{2}\right)|c_{\alpha',\beta'}(2\lambda)|^{-2}\,d\lambda.$$

Proof. The proof uses (2.3), (2.25), (2.26), (2.32) and (2.34), and it is entirely analogous to [5, Proposition 3.2]. ■

REMARK 3.3. Since the function $r \mapsto a_{k,l,j,i}(r)$ is in $C_R^\infty([0,\infty))$ and $\lambda \mapsto \phi_{2\lambda}^{(\alpha+k+l,\beta+k-l)}(r/2)$ is entire, the integral in (3.5) is an entire function of λ. Since $\lambda \mapsto q_{k,l}(\lambda)$ is a polynomial (cf. (2.31)), both functions

$$\lambda \mapsto b_{k,l,j,i}(\lambda) \quad \text{and} \quad \lambda \mapsto b_{k,l,j,i}(\lambda)/q_{k,l}(\lambda)$$

are entire of exponential type R (see [15, Lemma 5.13, p. 288]), the second one being even.

We also observe from (3.6)–(3.8) and the Paley–Wiener theorem for the Jacobi transform [20, Theorem 2.1] that, for all k, l, j, i, the map $a_{k,l,j,i}(r) \mapsto b_{k,l,j,i}(\lambda)$ is a bijection from the space of smooth functions $a_{k,l,j,i}$ on $[0,\infty)$ compactly supported in $[0,R]$ and such that the function

$$r \mapsto \frac{a_{k,l,j,i}(r)}{(\mathrm{sh}(r/2))^{k+l}(\mathrm{ch}(r/2))^{k-l}}$$

extends to $C_R^\infty(\mathbb{R})_{\mathrm{even}}$, i.e., $a_{k,l,j,i} \in (\mathrm{sh}(r/2))^{k+l}(\mathrm{ch}(r/2))^{k-l}C_R^\infty(\mathbb{R})_{\mathrm{even}}$, onto the space of holomorphic functions $b_{k,l,j,i}$ on $\mathbb{C}$ such that the function $\lambda \mapsto b_{k,l,j,i}(\lambda)/q_{k,l}(\lambda)$ is in $\mathrm{PW}_R(\mathbb{C})_{\mathrm{even}}$, i.e., $b_{k,l,j,i} \in q_{k,l}(\cdot)\,\mathrm{PW}_R(\mathbb{C})_{\mathrm{even}}$. Here $\mathrm{PW}_R(\mathbb{C})_{\mathrm{even}}$ is the space of even entire functions on $\mathbb{C}$ of exponential type R. The proof of the converse part of Theorem 1.1 in the $\mathfrak{v}$-radial case (see below) implies that the Fourier coefficients $b_{k,l,j,i}(\lambda)$ of $\psi(\lambda,\omega)$ satisfying (1.4), (1.5) and $\mathfrak{v}$-radial in ω are indeed in $q_{k,l}(\cdot)\,\mathrm{PW}_R(\mathbb{C})_{\mathrm{even}}$.

3.2. The $\mathfrak{v}$-radial Paley–Wiener theorem. We now prove the $\mathfrak{v}$-radial case of Theorem 1.1.

THEOREM 3.4. *The Fourier transform $f(b) \mapsto \widetilde{f}(\lambda,\omega)$ is a bijection from $C_0^\infty(B)^{\mathfrak{v}\text{-rad}}$ onto the set of holomorphic functions $\psi(\lambda,\omega)$ of uniform exponential type, $\mathfrak{v}$-radial in ω, and satisfying the condition*

$$\int_{\partial B} \mathcal{Q}_{-\lambda}(b,\omega)\psi(\lambda,\omega)\,d\omega = \int_{\partial B} \mathcal{Q}_{\lambda}(b,\omega)\psi(-\lambda,\omega)\,d\omega \tag{3.10}$$

for any $b \in B$ and $\lambda \in \mathbb{C}$. Moreover, $\widetilde{f}$ satisfies (1.4) *if and only if f has support in the closed ball $B_R = \{b \in B : d(b, C(e)) \le R\}$.*

Proof. In view of Lemma 3.1 and [2, Theorem 4.5], we only need to prove the converse part, in particular the onto statement. We proceed as in [5, Theorem 3.3] for the biradial case.

Let $\psi(\lambda,\omega)$ be a holomorphic function of uniform exponential type such that (1.4) and (3.10) hold, and such that the map $\omega \mapsto \psi(\lambda,\omega)$ is $\mathfrak{v}$-radial on S^{p+q} for all $\lambda \in \mathbb{C}$. Define $b_{k,l,j,i}(\lambda) = \int_{S^{p+q}} \psi(\lambda,\omega)\chi_{k,l,j,i}(\omega)\,d\omega$, so that (cf. (3.4))

$$\psi(\lambda,\omega) = \sum_{k\geq l}\sum_{j}\sum_{i} \pi_{k,l,j} b_{k,l,j,i}(\lambda)\chi_{k,l,j,i}(\omega). \tag{3.11}$$

Then $\lambda \mapsto b_{k,l,j,i}(\lambda)$ is holomorphic of exponential type R. Using (3.11) in the integral $\int_{\partial B} \mathcal{Q}_{-\lambda}(b,\omega)\psi(\lambda,\omega)\,d\omega$ we get, by (2.34),

$$\int_{\partial B} \mathcal{Q}_{-\lambda}(b,\omega)\psi(\lambda,\omega)\,d\omega = \sum_{k\geq l}\sum_{j}\sum_{i} \pi_{k,l,j} b_{k,l,j,i}(\lambda) f_{-\lambda,k,l,j,i}(b). \tag{3.12}$$

From (3.10), (2.25)–(2.26) and (3.12), it follows that the function $\lambda \mapsto b_{k,l,j,i}(\lambda)q_{k,l}(-\lambda)$ is even. Thus so is

$$\lambda \mapsto \frac{b_{k,l,j,i}(\lambda)}{q_{k,l}(\lambda)} = \frac{b_{k,l,j,i}(\lambda)\,q_{k,l}(-\lambda)}{q_{k,l}(\lambda)\,q_{k,l}(-\lambda)}.$$

Define f by the inversion formula (1.2):

$$f(b) = \frac{c_{p,q}}{2\pi}\int_0^\infty \int_{S^{p+q}} \mathcal{Q}_{-\lambda}(b,\omega)\,\psi(\lambda,\omega)\,d\omega\,d\mu(\lambda).$$

Then f is smooth and $\mathfrak{v}$-radial on B (by (3.12)). Define $a_{k,l,j,i}(r)$ by (3.1)–(3.2). Then we get again (3.7) and (3.9).

By (2.31) we see that the function $\lambda \mapsto b_{k,l,j,i}(\lambda)/q_{k,l}(\lambda)$ in (3.9) has no poles for $\operatorname{Im}\lambda \geq 0$. Then, using the exponential type conditions for the functions $b_{k,l,j,i}(\lambda)$ and the well known asymptotic estimates for the functions $\phi_{2\lambda}^{(\alpha',\beta')}(r/2)$ in (3.9), we can prove that $a_{k,l,j,i}(r) = 0$ for $r > R$.

In more detail, we use

$$\phi_{2\lambda}^{(\alpha',\beta')}(r/2) = c_{\alpha',\beta'}(2\lambda)\Phi_{2\lambda}^{(\alpha',\beta')}(r/2) + c_{\alpha',\beta'}(-2\lambda)\Phi_{-2\lambda}^{(\alpha',\beta')}(r/2),$$

where the function $\lambda \mapsto \Phi_{\lambda}^{(\alpha',\beta')}(t)$ is holomorphic in $\mathbb{C}\setminus\{-i\mathbb{N}\}$ for each $t > 0$ (cf. [12, Proposition 1]), to rewrite the integral in (3.9) for $r > 0$ as

$$F(r) = \int_{-\infty}^{\infty} \frac{b_{k,l,j,i}(\lambda)}{q_{k,l}(\lambda)}\,\frac{\Phi_{2\lambda}^{(\alpha',\beta')}(r/2)}{c_{\alpha',\beta'}(-2\lambda)}\,d\lambda.$$

Now $(c_{\alpha',\beta'}(-2\lambda))^{-1}$ has no poles for $\operatorname{Im}\lambda \geq 0$, and the integrand is holomorphic for $\operatorname{Im}\lambda \geq 0$. Thus we obtain, by Cauchy's theorem,

$$F(r) = \int_{-\infty}^{\infty} \frac{b_{k,l,j,i}(\xi+i\eta)}{q_{k,l}(\xi+i\eta)}\,\frac{\Phi_{2(\xi+i\eta)}^{(\alpha',\beta')}(r/2)}{c_{\alpha',\beta'}(-2(\xi+i\eta))}\,d\xi$$

for any $\eta \geq 0$.

We now use the estimates for $\Phi_\lambda^{(\alpha',\beta')}(t)$ and $c_{\alpha',\beta'}(\lambda)$ given in [12, Theorem 2] (see also [20, (6.4) and (6.5)]), namely for any $c > 0$ there exists $K_1 > 0$ such that for all $t \geq c$ and all $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda \geq 0$,

$$|\Phi_\lambda^{(\alpha',\beta')}(t)| \leq K_1 e^{-(\operatorname{Im}\lambda+\alpha'+\beta'+1)t}.$$

Moreover, there exists $K_2 > 0$ such that for all $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda \geq 0$,

$$|c_{\alpha',\beta'}(-\lambda)|^{-1} \leq K_2(1+|\lambda|)^{\alpha'+1/2}.$$

Using the exponential type conditions for $b_{k,l,j,i}(\lambda)$ and the inequality

$$\left|\frac{1}{q_{k,l}(\xi+i\eta)}\right| \leq \frac{(d/2)_{k+l}}{(Q/2)_k\,(p/4+1/2)_l} \qquad (\forall\xi,\ \forall\eta \geq 0),$$

which is easily proved from (2.31), we find (as in [12, p. 157])

$$\begin{aligned}|F(r)| &\leq Ke^{-(2\eta+Q+2k)r/2}\int_{-\infty}^{\infty}|b_{k,l,j,i}(\xi+i\eta)|(1+2|\xi+i\eta|)^{(p+q)/2+k+l}\,d\xi\\ &\leq K'e^{-(2\eta+Q+2k)r/2}e^{\eta R} \leq K'e^{\eta(R-r)}\end{aligned}$$

for suitable constants K, K'. Since this holds for all $\eta \geq 0$, we get $F(r) = 0$ for $r > R$, as claimed. It follows from (3.1) that $f(b) = f(r,\omega)$ has support in B_R. The proof is completed by showing that the Fourier transform of f is just $\widetilde{f}(\lambda,\omega) = \psi(\lambda,\omega)$. In fact, the Fourier coefficients of $\omega \mapsto \widetilde{f}(\lambda,\omega)$ are just the $b_{k,l,j,i}(\lambda)$. ■

REFERENCES

[1] J.-P. Anker, E. Damek and C. Yacoub, *Spherical analysis on harmonic AN groups*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 23 (1996), 643–679.

[2] F. Astengo, R. Camporesi and B. Di Blasio, *The Helgason Fourier transform on a class of nonsymmetric harmonic spaces*, Bull. Austral. Math. Soc. 55 (1997), 405–424.

[3] F. Astengo and B. Di Blasio, *A Paley–Wiener theorem on NA harmonic spaces*, Colloq. Math. 80 (1999), 211–233.

[4] F. Astengo and B. Di Blasio, *Huygens' principle and a Paley–Wiener type theorem on Damek–Ricci spaces*, Ann. Math. Blaise Pascal 17 (2010), 327–340.

[5] R. Camporesi, *The biradial Paley–Wiener theorem for the Helgason Fourier transform on Damek–Ricci spaces*, J. Funct. Anal. 267 (2014), 428–451.

[6] R. Camporesi, *Geodesic spheres and non radial eigenfunctions on Damek–Ricci spaces*, Indag. Math. 24 (2013), 313–345.

[7] M. G. Cowling, A. H. Dooley, A. Korányi and F. Ricci, *An approach to symmetric spaces of rank one via groups of Heisenberg type*, J. Geom. Anal. 8 (1998), 199–237.

[8] E. Damek, *A Poisson kernel on Heisenberg type nilpotent groups*, Colloq. Math. 53 (1987), 239–247.

[9] E. Damek and F. Ricci, *A class of nonsymmetric harmonic Riemannian spaces*, Bull. Amer. Math. Soc. 27 (1992), 139–142.

[10] E. Damek and F. Ricci, *Harmonic analysis on solvable extensions of H-type groups*, J. Geom. Anal. 2 (1992), 213–248.
[11] B. Di Blasio, *An extension of the theory of Gelfand pairs to radial functions on Lie groups*, Boll. Un. Mat. Ital. 11 (1997), 623–642.
[12] M. Flensted-Jensen, *Paley–Wiener type theorems for a differential operator connected with symmetric spaces*, Ark. Mat. 10 (1972), 143–162.
[13] M. Flensted-Jensen and T. Koornwinder, *Positive definite spherical functions on a non-compact, rank one symmetric space*, in: Analyse Harmonique sur les Groupes de Lie II, Lecture Notes in Math. 739, Springer, Berlin, 1979, 249–282.
[14] J. Heber, *On harmonic and asymptotically harmonic homogeneous spaces*, Geom. Funct. Anal. 16 (2006), 869–890.
[15] S. Helgason, *Geometric Analysis on Symmetric Spaces*, Math. Surveys Monogr. 39, Amer. Math. Soc., Providence, RI, 1994.
[16] S. Helgason, *Eigenspaces of the Laplacian; integral representations and irreducibility*, J. Funct. Anal. 17 (1974), 328–353.
[17] A. Kaplan, *Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms*, Trans. Amer. Math. Soc. 258 (1980), 147–153.
[18] G. Knieper and N. Peyerimhoff, *Noncompact harmonic manifolds*, arXiv:1302.3841 (2013).
[19] T. Koornwinder, *The addition formula for Jacobi polynomials and spherical harmonics*, SIAM J. Appl. Math. 25 (1973), 236–246.
[20] T. Koornwinder, *Jacobi functions and analysis on noncompact semisimple Lie groups*, in: Special Functions: Group Theoretical Aspects and Applications, Reidel, Dordrecht, 1984, 1–85.
[21] F. Ricci, *The spherical transform on harmonic extensions of H-type groups*, Rend. Sem. Mat. Univ. Politec. Torino 50 (1992), 381–392.
[22] C. Riehm, *The automorphism group of a composition of quadratic forms*, Trans. Amer. Math. Soc. 269 (1982), 403–414.
[23] F. Rouvière, *Espaces de Damek–Ricci, géométrie et analyse*, in: Sémin. Congr. 7, Soc. Math. France, Paris, 2003, 45–100.
[24] S. Thangavelu, *On Paley–Wiener and Hardy theorems for NA groups*, Math. Z. 245 (2003), 483–502.

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