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Periodic solutions to evolution equations: existence, conditional stability and admissibility of function spaces

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Abstract. We prove the existence and conditional stability of periodic solutions to semilinear evolution equations of the form $\dot{u} = A(t)u+g(t,u(t))$, where the operator-valued function $t \mapsto A(t)$ is 1-periodic, and the operator $g(t,x)$ is 1-periodic with respect to t for each fixed x and satisfies the φ-Lipschitz condition $\|g(t,x_1) - g(t,x_2)\| \le \varphi(t)\|x_1 - x_2\|$ for $\varphi(t)$ being a real and positive function which belongs to an admissible function space. We then apply the results to study the existence, uniqueness and conditional stability of periodic solutions to the above semilinear equation in the case that the family $(A(t))_{t\ge 0}$ generates an evolution family having an exponential dichotomy. We also prove the existence of a local stable manifold near the periodic solution in that case.

1. Introduction and preliminaries. Consider the abstract semilinear evolution equation

$$\dot{u} = A(t)u + g(t,u(t)), \qquad t \in \mathbb{R}_+, \tag{1.1}$$

where for each $t \in \mathbb{R}_+$, $A(t)$ is a possibly unbounded operator on a Banach space X such that $(A(t))_{t\ge 0}$ generates an evolution family $(U(t,s))_{t\ge s\ge 0}$ on X, and the operator g is locally Lipschitz and acts on some function space of vector-valued functions. One of the important research directions related to the asymptotic behavior of the solutions to the above equation is to find conditions for the existence of a periodic solution. The most popular conditions are the periodicity of the linear part $A(t)$ and that of the nonlinear term $g(t,x)$ with respect to t combined with its local Lipschitz property. However, in some applications related to complicated reaction-diffusion processes, the nonlinear term g describes the source of material (or population) which, in many contexts, depends on time in diversified manners

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(see [8, Chapt. 11], [9], [20]). Therefore, sometimes one may not hope to have the uniform Lipschitz continuity of g. Thus, one tries to extend the conditions on nonlinear parts so that they describe more precisely such reaction-diffusion processes. Hence, in the present paper, we will prove the existence of a periodic solution to the semilinear equation with the nonlinear term g satisfying the φ-Lipschitz condition, i.e., $\|g(t,x)-g(t,y)\| \le \varphi(t)\|x-y\|$ for φ being a positive and periodic function belonging to an admissible function space.

The method of using admissibility of function spaces to handle the φ-Lipschitz nonlinear term was first introduced in [12] to prove the existence of invariant manifolds for semilinear evolution equations. Then it was extended in [13] to prove the existence of a new class of invariant manifolds, namely, the invariant manifold of *admissible class*. We will use this method in the present paper combined with the folklore methodology of Massera [5] for periodic solutions to ordinary differential equations (which roughly says that if an ODE has a bounded solution then it has a periodic one).

Let us briefly explain the reason for choosing such a method: In the present literature, there are several methods to prove the existence of periodic solutions to evolution equations such as Tikhonov's fixed point method [17] or the Lyapunov functionals [21]. The most popular approaches are the use of ultimate boundedness of solutions and the compactness of the Poincaré map realized through some compact embeddings (see [1, 4, 16, 17, 19, 21] and references therein). However, in some concrete applications, e.g., to partial differential equations in unbounded domains or to evolution equations having unbounded solutions, such compact embeddings are no longer valid, and the existence of bounded solutions is not easy to obtain since one has to carefully choose an appropriate initial vector (or condition) to guarantee the boundedness of the solution emanating from that vector.

Therefore, in the present paper, we use the ergodic approach (see [22] for its origin) to overcome such difficulties. Namely, we start with the linear equation

$$\dot{u} = A(t)u + f(t), \qquad t \ge 0, \tag{1.2}$$

and use the Cesàro limit to prove the existence of a periodic solution through the existence of a bounded solution whose sup-norm can be controlled by the norm of the input function f in a relevant admissible function space. We refer the reader to [14] for an extension of such an approach to the case of periodic solutions to Stokes and Navier–Stokes equations around rotating obstacles. We then use the admissibility of function spaces combined with the fixed point argument to prove the existence and uniqueness of a periodic solution of the abstract semilinear evolution equation (1.1) with the φ-Lipschitz nonlinear term.

It is worth noting that our framework fits perfectly to the situation of exponentially dichotomic linear parts, i.e., when the family $(A(t))_{t\geq 0}$ generates an evolution family $(U(t,s))_{t\geq s\geq 0}$ having an exponential dichotomy (see Definition 4.1 below), since in this case we can choose an initial vector from which emanates a bounded solution. We can also prove the conditional stability of periodic solutions in this case. Our main results are contained in Theorems 2.3 and 3.1. The applications of our abstract results to semilinear equations with exponentially dichotomic linear parts are given in Section 4. Moreover, in that section, we also prove the existence of a local stable manifold around the periodic solution.

We now recall some notions for function spaces and refer to Massera and Schäffer [6], Räbiger and Schnaubelt [18], and Nguyen [11] for concrete applications.

Denote by $\mathcal{B}$ the Borel algebra and by λ the Lebesgue measure on $\mathbb{R}_+$. The space $L_{1,\mathrm{loc}}(\mathbb{R}_+)$ of real-valued locally integrable functions on $\mathbb{R}_+$ (modulo λ-nullfunctions) becomes a Fréchet space for the seminorms $p_n(f) := \int_{J_n} |f(t)|\,dt$, where $J_n = [n, n+1]$ for each $n \in \mathbb{N}$ (see [6, Chapt. 2, §20]).

We can now define Banach function spaces as follows.

Definition 1.1. A vector space E of real-valued Borel-measurable functions on $\mathbb{R}_+$ (modulo λ-nullfunctions) is called a *Banach function space* (over $(\mathbb{R}_+, \mathcal{B}, \lambda)$) if

(1) E is a Banach lattice with respect to a norm $\|\cdot\|_E$, i.e., $(E, \|\cdot\|_E)$ is a Banach space, and if $\varphi \in E$ and ψ is a real-valued Borel-measurable function such that $|\psi(\cdot)| \leq |\varphi(\cdot)|$, λ-a.e., then $\psi \in E$ and $\|\psi\|_E \leq \|\varphi\|_E$,
(2) the characteristic functions χ_A belong to E for all $A \in \mathcal{B}$ of finite measure, and $\sup_{t\geq 0} \|\chi_{[t,t+1]}\|_E < \infty$ and $\inf_{t\geq 0} \|\chi_{[t,t+1]}\|_E > 0$,
(3) $E \hookrightarrow L_{1,\mathrm{loc}}(\mathbb{R}_+)$, i.e., for each seminorm p_n of $L_{1,\mathrm{loc}}(\mathbb{R}_+)$ there exists a number $\beta_{p_n} > 0$ such that $p_n(f) \leq \beta_{p_n}\|f\|_E$ for all $f \in E$.

Let now E be a Banach function space and X a Banach space. We set

$$\mathcal{E} := \mathcal{E}(\mathbb{R}_+, X) := \{f : \mathbb{R}_+ \to X : f \text{ is strongly measurable and } \|f(\cdot)\| \in E\}$$

(modulo λ-nullfunctions) endowed with the norm

$$\|f\|_{\mathcal{E}} := \big\|\|f(\cdot)\|\big\|_E.$$

Then $\mathcal{E}$ is a Banach space called the *Banach space corresponding to the Banach function space E*.

Definition 1.2. A Banach function space E is called *admissible* if

(i) there is a constant $C \geq 1$ such that for every compact interval $[a,b] \subset \mathbb{R}_+$ we have

$$\int_a^b |\varphi(t)|\,dt \le \frac{C(b-a)}{\|\chi_{[a,b]}\|_E}\|\varphi\|_E \quad \text{for all } \varphi \in E, \tag{1.3}$$

(ii) for $\varphi \in E$ the function $\Lambda_1\varphi$ defined by $\Lambda_1\varphi(t) := \int_t^{t+1}\varphi(\tau)\,d\tau$ belongs to E,

(iii) E is T_τ^+-invariant and T_τ^--invariant, where

$$\begin{aligned} T_\tau^+\varphi(t) &:= \begin{cases} \varphi(t-\tau) & \text{for } t \ge \tau \ge 0,\\ 0 & \text{for } 0 \le t \le \tau, \end{cases}\\ T_\tau^-\varphi(t) &:= \varphi(t+\tau) \qquad \text{for } t \ge 0. \end{aligned} \tag{1.4}$$

Moreover, there are constants N_1 and N_2 such that $\|T_\tau^+\| \le N_1$ and $\|T_\tau^-\| \le N_2$ for all $\tau \in \mathbb{R}_+$.

Example 1.3. Besides the spaces $L_p(\mathbb{R}_+)$, $1 \le p \le \infty$, and the space

$$\mathbf{M} = \mathbf{M}(\mathbb{R}_+) := \Big\{ f \in L_{1,\mathrm{loc}}(\mathbb{R}_+) : \sup_{t\ge 0} \int_t^{t+1} |f(\tau)|\,d\tau < \infty \Big\} \tag{1.5}$$

endowed with the norm $\|f\|_{\mathbf{M}} := \sup_{t\ge0}\int_t^{t+1}|f(\tau)|\,d\tau$, many other function spaces occurring in interpolation theory, e.g. the Lorentz spaces $L_{p,q}$, $1 < p, q < \infty$, are admissible.

Remark 1.4. It can be easily seen that if E is an admissible Banach function space, then $E \hookrightarrow \mathbf{M}(\mathbb{R}_+)$.

We now collect some properties of admissible Banach function spaces in the following proposition (see [11, Proposition 2.6]).

Proposition 1.5. *Let E be an admissible Banach function space. Then:*

(a) *Let $\varphi \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ be such that $\varphi \ge 0$ and $\Lambda_1\varphi \in E$, where Λ_1 is as in Definition 1.2(ii). For $\sigma > 0$ define*

$$\Lambda'_\sigma\varphi(t) = \int_0^t e^{-\sigma(t-s)}\varphi(s)\,ds, \qquad \Lambda''_\sigma\varphi(t) = \int_t^\infty e^{-\sigma(s-t)}\varphi(s)\,ds.$$

Then $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ belong to E. In particular, if $\sup_{t\ge0}\int_t^{t+1}|\varphi(\tau)|\,d\tau < \infty$ (this will be satisfied if $\varphi \in E$, see Remark 1.4) then $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ are bounded. Moreover, denoting by $\|\cdot\|_\infty$ the ess sup-norm, we have

$$\|\Lambda'_\sigma\varphi\|_\infty \le \frac{N_1}{1-e^{-\sigma}}\|\Lambda_1 T_1^+\varphi\|_\infty, \qquad \|\Lambda''_\sigma\varphi\|_\infty \le \frac{N_2}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_\infty. \tag{1.6}$$

(b) *E contains the exponentially decaying functions $\psi(t) = e^{-\alpha t}$ for $t \ge 0$ and any fixed constant $\alpha > 0$.*

(c) *E does not contain any exponentially growing function $f(t) = e^{bt}$ for $t \ge 0$ and a constant $b > 0$.*

Next, in the admissible space $\mathbf{M} = \mathbf{M}(\mathbb{R}_+)$ defined in (1.5) we consider the subset

$$\mathbf{P} := \{f \in \mathbf{M} : f \text{ is 1-periodic}\}. \tag{1.7}$$

Let now φ be a positive function belonging to $\mathbf{P}$. Then, for $0 \le t \le 1$,

$$\begin{aligned}(\Lambda_1 T_1^+\varphi)(t) &= \int_t^{t+1} (T_1^+\varphi)(\tau)\,d\tau = \int_1^{t+1} (T_1^+\varphi)(\tau)\,d\tau = \int_1^{t+1} \varphi(\tau-1)\,d\tau \\ &= \int_1^{t+1} \varphi(\tau)\,d\tau \le \int_t^{t+1} \varphi(\tau)\,d\tau = (\Lambda_1\varphi)(t),\end{aligned}$$

while for $t > 1$,

$$(\Lambda_1 T_1^+\varphi)(t) = \int_t^{t+1} (T_1^+\varphi)(\tau)\,d\tau = \int_t^{t+1} \varphi(\tau-1)\,d\tau = \int_t^{t+1} \varphi(\tau)\,d\tau = (\Lambda_1\varphi)(t).$$

Hence, $(\Lambda_1 T_1^+\varphi)(t) \le (\Lambda_1\varphi)(t)$ for all $t \in \mathbb{R}_+$. Therefore, from (1.6) we get

$$\|\Lambda'_\sigma\varphi\|_\infty \le \frac{N_1}{1-e^{-\sigma}}\|\varphi\|_{\mathbf{M}} \quad\text{and}\quad \|\Lambda''_\sigma\varphi\|_\infty \le \frac{N_2}{1-e^{-\sigma}}\|\varphi\|_{\mathbf{M}} \quad \text{for all positive } \varphi \in \mathbf{P}. \tag{1.8}$$

We now recall the cone inequality theorem which will be used to prove the conditional stability of solutions. Firstly, we introduce the following notion. A closed subset $\mathcal{K}$ of a Banach space W is called a *cone* if it has the following properties:

(i) $x_0 \in \mathcal{K}$ implies $\lambda x_0 \in \mathcal{K}$ for all $\lambda \ge 0$,
(ii) $x_1, x_2 \in \mathcal{K}$ implies $x_1 + x_2 \in \mathcal{K}$,
(iii) $\pm x_0 \in \mathcal{K}$ implies $x_0 = 0$.

Suppose $\mathcal{K}$ is a cone in a Banach space W. For $x, y \in W$ we will write $x \le y$ if $y - x \in \mathcal{K}$. If $\mathcal{K}$ is invariant under a linear operator A, then it is easy to see that A preserves the inequality, i.e., $x \le y$ implies $Ax \le Ay$.

The following cone inequality theorem is taken from [2, Theorem I.9.3].

THEOREM 1.6 (Cone inequality). *Let $\mathcal{K}$ be a cone in a Banach space W such that $\mathcal{K}$ is invariant under a bounded linear operator $A \in \mathcal{L}(W)$ having spectral radius $r < 1$. If a vector $x \in W$ satisfies the inequality*

$$x \le Ax + z \quad \textit{for some given } z \in W,$$

then also $x \le y$, where $y \in W$ is the solution of the equation $y = Ay + z$.

We then introduce the notion of local φ-Lipschitz functions.

DEFINITION 1.7 (Local φ-Lipschitz functions). Let E be an admissible Banach function space, φ be a positive function belonging to E, and B_ρ be the ball with radius ρ in X, i.e., $B_\rho := \{x \in X : \|x\| \le \rho\}$. A function

$g : [0,\infty) \times B_\rho \to X$ is said to belong to the class (L, φ, ρ) for some positive constants L, ρ if:

(i) $\|g(t,x)\| \le L\varphi(t)$ for a.e. $t \in \mathbb{R}_+$ and $x \in B_\rho$,
(ii) $\|g(t,x_1) - g(t,x_2)\| \le \varphi(t)\|x_1 - x_2\|$ for a.e. $t \in \mathbb{R}_+$ and all $x_1, x_2 \in B_\rho$.

REMARK 1.8. If $g(t,0) = 0$ then condition (ii) above already implies that g belongs to the class (ρ, φ, ρ).

We also need the following space of bounded and continuous functions:

$$C_b(\mathbb{R}_+, X) := \Big\{ v : \mathbb{R}_+ \to X : v \text{ is continuous and } \sup_{t\in\mathbb{R}_+} \|v(t)\| < \infty \Big\} \tag{1.9}$$

endowed with the norm $\|v\|_{C_b} := \sup_{t\in\mathbb{R}_+} \|v(t)\|$.

2. Bounded and periodic solutions to linear evolution equations. Given a function f taking values in a Banach space X having a separable predual Y (i.e., $X = Y'$ for a separable Banach space Y) we consider the following nonhomogeneous linear problem for the unknown function $u(t)$:

$$\begin{cases} \dfrac{du}{dt} = A(t)u(t) + f(t) \quad \text{for } t > 0, \\ u(0) = u_0 \in X, \end{cases} \tag{2.1}$$

where the family $(A(t))_{t\ge 0}$ of partial differential operators is given such that the homogeneous Cauchy problem

$$\begin{cases} \dfrac{du}{dt} = A(t)u(t) \quad \text{for } t > s \ge 0, \\ u(s) = u_s \in X \end{cases} \tag{2.2}$$

is well-posed. By this we understand that there exists an evolution family $(U(t,s))_{t\ge s\ge 0}$ such that the solution of the Cauchy problem (2.2) is given by $u(t) = U(t,s)u(s)$. For more details on the notion of evolution families, conditions for the existence of such families and applications to partial differential equations we refer the readers to Pazy [15] (see also Nagel and Nickel [10] for a detailed discussion of well-posedness for nonautonomous abstract Cauchy problems on the whole line $\mathbb{R}$). We next give the precise concept of an evolution family.

DEFINITION 2.1. A family $(U(t,s))_{t\ge s\ge 0}$ of bounded linear operators on a Banach space X is a (*strongly continuous*, *exponentially bounded*) *evolution family* if:

(i) $U(t,t) = \mathrm{Id}$ and $U(t,r)U(r,s) = U(t,s)$ for all $t \ge r \ge s \ge 0$,
(ii) the map $(t,s) \mapsto U(t,s)x$ is continuous for every $x \in X$, where $(t,s) \in \{(t,s) \in \mathbb{R}^2 : t \ge s \ge 0\}$,

(iii) there are constants $K, \alpha \geq 0$ such that $\|U(t,s)x\| \leq Ke^{\alpha(t-s)}\|x\|$ for all $t \geq s \geq 0$ and $x \in X$.

The existence of an evolution family $(U(t,s))_{t\geq s\geq 0}$ allows us to define a notion of mild solutions as follows. By a *mild solution* to (2.1) we mean a function u satisfying the integral equation

$$u(t) = U(t,0)u_0 + \int_0^t U(t,\tau)f(\tau)\,d\tau \quad \text{for all } t \geq 0. \tag{2.3}$$

We refer the reader to Pazy [15] for a more detailed treatment of the relations between classical and mild solutions of evolution equations of the form (2.1).

We now state an assumption that will be used in the rest of the paper.

ASSUMPTION 2.2. We assume that $A(t)$ is 1-periodic, i.e., $A(t+1) = A(t)$ for all $t \in \mathbb{R}_+$. Then $(U(t,s))_{t\geq s\geq 0}$ becomes 1-periodic in the sense that

$$U(t+1,s+1) = U(t,s) \quad \text{for all } t \geq s \geq 0. \tag{2.4}$$

We also assume that the space Y considered as a subspace of Y'' (through the canonical embedding) is invariant under the operator $U'(1,0)$ dual to $U(1,0)$.

We now state a Massera type theorem for the existence of a periodic solution.

THEOREM 2.3. *Let X be a Banach space with a separable predual Y. Assume that $f \in \mathbf{M}$ and there exists $u_0 \in X$ such that a mild solution u of* (2.1) *with $u(0) = u_0$ (i.e., $u(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s)\,ds$ for $t \geq 0$) satisfies $u \in C_b(\mathbb{R}_+, X)$ and*

$$\|u\|_{C_b} \leq M_u \|f\|_{\mathbf{M}} \tag{2.5}$$

for some M_u. Then, under Assumption 2.2, *if f is* 1*-periodic, then equation* (2.1) *has a* 1*-periodic mild solution $\hat{u}$ satisfying*

$$\|\hat{u}\|_{C_b} \leq (M_u + 1)Ke^{\alpha}\|f\|_{\mathbf{M}}. \tag{2.6}$$

Furthermore, if the evolution family $U(t,s)_{t\geq s\geq 0}$ satisfies

$$\lim_{t\to\infty} \|U(t,0)x\| = 0 \quad \text{for } x \in X \text{ such that } U(t,0)x \text{ is bounded in } \mathbb{R}_+, \tag{2.7}$$

then the 1*-periodic mild solution of* (2.1) *is unique.*

Proof. The existence of a 1-periodic solution is clearly equivalent to the existence of an $\hat{x} \in X$ such that $\hat{x} = U(1,0)\hat{x} + \int_0^1 U(1,s)f(s)\,ds$. To prove the existence of such an $\hat{x}$ we consider the function $u(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s)\,ds$, which belongs to $C_b(\mathbb{R}_+, X)$ by hypothesis. For a 1-periodic function f and $(U(t,s))_{t\geq s\geq 0}$ satisfying (2.4), it can be easily seen by induction that

$$u(k+1) = U(1,0)u(k) + \int_0^1 U(1,s)f(s)\,ds \quad \text{for all } k \in \mathbb{N}. \tag{2.8}$$

Next, for each $n \in \mathbb{N}$ we define the Cesàro sum x_n by

$$x_n := \frac{1}{n}\sum_{k=1}^{n} u(k). \tag{2.9}$$

Then, the inequality (2.5) implies

$$\sup_{k\in\mathbb{N}} \|u(k)\| \leq M_u \|f\|_{\mathbf{M}}. \tag{2.10}$$

Hence, $\{x_n\}_{n\in\mathbb{N}}$ is also bounded in X, and by (2.10) we have

$$\sup_{n\in\mathbb{N}} \|x_n\| \leq M_u \|f\|_{\mathbf{M}}. \tag{2.11}$$

Since the space $X = Y'$ and Y is separable, by Banach–Alaoglu's Theorem there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\{x_{n_k}\} \xrightarrow{\text{weak}^*} \hat{x} \quad \text{with } \|\hat{x}\| \leq M_u \|f\|_{\mathbf{M}}. \tag{2.12}$$

A straightforward calculation using (2.9) yields

$$U(1,0)x_n + \int_0^1 U(1,s)f(s)\,ds - x_n = \frac{1}{n}\big(u(n+1) - u(1)\big).$$

Since the sequence $\{u(n)\}_{n\in\mathbb{N}}$ is bounded in X, we deduce that

$$\lim_{n\to\infty}\Big(U(1,0)x_n + \int_0^1 U(1,s)f(s)\,ds - x_n\Big) = \lim_{n\to\infty}\frac{1}{n}\big(u(n+1) - u(1)\big) = 0$$

strongly in X. This implies that for the subsequence $\{x_{n_k}\}$ from (2.12) we have

$$U(1,0)x_{n_k} + \int_0^1 U(1,s)f(s)\,ds - x_{n_k} \xrightarrow{\text{weak}^*} 0. \tag{2.13}$$

Combining (2.12) and (2.13) we obtain

$$U(1,0)x_{n_k} + \int_0^1 U(1,s)f(s)\,ds \xrightarrow{\text{weak}^*} \hat{x} \in X. \tag{2.14}$$

We will prove that

$$U(1,0)\hat{x} + \int_0^1 U(1,s)f(s)\,ds = \hat{x}.$$

To do so, denoting by $\langle \cdot, \cdot \rangle$ the duality between Y and Y' and using the fact that $U'(1,0)$ leaves Y invariant (see Assumption 2.2), for all $h \in Y$ we have

$$\begin{aligned}
&\Big\langle U(1,0)x_{n_k} + \int_0^1 U(1,s)f(s)\,ds, h \Big\rangle \\
&\qquad = \langle U(1,0)x_{n_k}, h\rangle + \Big\langle \int_0^1 U(1,s)f(s)\,ds, h \Big\rangle \\
&\qquad = \langle x_{n_k}, U'(1,0)h\rangle + \Big\langle \int_0^1 U(1,s)f(s)\,ds, h \Big\rangle \\
&\qquad \xrightarrow{n_k\to\infty} \langle \hat{x}, U'(1,0)h\rangle + \Big\langle \int_0^1 U(1,s)f(s)\,ds, h \Big\rangle \\
&\qquad = \langle U(1,0)\hat{x}, h\rangle + \Big\langle \int_0^1 U(1,s)f(s)\,ds, h \Big\rangle \\
&\qquad = \Big\langle U(1,0)\hat{x} + \int_0^1 U(1,s)f(s)\,ds, h \Big\rangle.
\end{aligned}$$

This yields

$$U(1,0)x_{n_k} + \int_0^1 U(1,s)f(s)\,ds \xrightarrow{\text{weak}^*} U(1,0)\hat{x} + \int_0^1 U(1,s)f(s)\,ds \in X. \tag{2.15}$$

It now follows from (2.14) and (2.15) that

$$U(1,0)\hat{x} + \int_0^1 U(1,s)f(s)\,ds = \hat{x}. \tag{2.16}$$

The solution $\hat{u} \in C_b(\mathbb{R}_+, X)$ of (2.3) with $\hat{u}(0) = \hat{x}$ is clearly 1-periodic. Therefore, $\hat{u}(t)$ is a 1-periodic mild solution to (2.1).

The inequality (2.6) now follows from (2.5) and (2.12).

We now prove that if $(U(t,s))_{t\ge s\ge 0}$ satisfies (2.7), then the periodic mild solution is unique. Indeed, let $\hat{u}_1$, $\hat{u}_2$ be two 1-periodic mild solutions to (2.1). Then, setting $v = \hat{u}_1 - \hat{u}_2$, we see that v is 1-periodic and, by (2.3),

$$v(t) = U(t,0)(\hat{u}_1(0) - \hat{u}_2(0)) \quad \text{for } t \ge 0. \tag{2.17}$$

Since $v(\cdot)$ is bounded on $\mathbb{R}_+$, (2.7) implies that

$$\lim_{t\to\infty} \|v(t)\| = 0. \tag{2.18}$$

This fact, together with the periodicity of v, shows that $v(t) = 0$ for all $t \ge 0$, so $\hat{u}_1 = \hat{u}_2$. ∎

3. Bounded and periodic solutions to semilinear problems. For a Banach space X with a separable predual Y as in the previous section, we now consider the semilinear evolution equation

$$(3.1)\qquad \begin{cases} \dfrac{du}{dt} = A(t)u(t) + g(t,u(t)), \\ u(0) = u_0 \in X, \end{cases}$$

where the linear operators $A(t)$, $t \geq 0$, act on X and satisfy the hypotheses of Theorem 2.3, and the nonlinear term $g : [0,\infty) \times X \to X$ satisfies:

$$(3.2)\qquad \begin{array}{l} (1)\ g \text{ belongs to the class } (L,\varphi,\rho) \text{ for some } L,\rho > 0 \text{ and } 0 < \varphi \in \mathbf{P}, \\ (2)\ g(t,x) \text{ is 1-periodic with respect to } t \text{ for each fixed } x \in X. \end{array}$$

Furthermore, by a *mild solution* to (3.1) we mean a function u satisfying

$$(3.3)\qquad u(t) = U(t,0)u_0 + \int_0^t U(t,\tau)g(\tau,u(\tau))\,d\tau \quad \text{for all } t \geq 0.$$

We then come to our next result on the existence and uniqueness of a periodic mild solution to (3.1).

Theorem 3.1. *Assume that there exists a constant M such that for each $f \in \mathbf{M}$ there is a mild solution u of* (2.1) *satisfying $u \in C_b(\mathbb{R}_+, X)$ and*

$$\|u\|_{C_b} \leq M\|f\|_{\mathbf{M}},$$

and that the evolution family $U(t,s)_{t\geq s\geq 0}$ satisfies

$$\lim_{t\to\infty} \|U(t,0)x\| = 0 \quad \textit{for } x \in X \textit{ such that } U(t,0)x \textit{ is bounded in } \mathbb{R}_+.$$

Let g satisfy (3.2). *Then, if $\gamma := \|\varphi\|_{\mathbf{M}}$ is small enough, equation* (3.1) *has a unique* 1*-periodic mild solution $\hat{u}$ in $C_b(\mathbb{R}_+, X)$.*

Proof. Consider the closed set $\mathcal{B}^1_\rho \subset C_b(\mathbb{R}_+, X)$ defined by

$$(3.4)\qquad \mathcal{B}^1_\rho := \{v \in C_b(\mathbb{R}_+, X) : v \text{ is 1-periodic and } \|v\|_{C_b} \leq \rho\}.$$

We then define a transformation Φ as follows: Consider the equation

$$(3.5)\qquad u(t) = U(t,0)u(0) + \int_0^t U(t,\tau)g(\tau,v(\tau))\,d\tau \quad \text{for } t \geq 0.$$

Then for $v \in \mathcal{B}^1_\rho$ we set $\Phi(v) := u$ where $u \in C_b(\mathbb{R}_+, X)$ is the unique 1-periodic solution to (3.5) (guaranteed by Theorem 2.3). We will prove that if γ is small enough, then Φ acts from $\mathcal{B}^1_\rho$ into itself and is a contraction. To do so, fixing any $v \in \mathcal{B}^1_\rho$, since g belongs to the class (L,φ,ρ) with $\varphi \in \mathbf{P}$ we have

$$\|g(\cdot,v(\cdot))\|_{\mathbf{M}} = \sup_{t\ge 0}\int_t^{t+1} \|g(\tau,v(\tau))\|\,d\tau \le L\sup_{t\ge 0}\int_t^{t+1} \|\varphi(\tau)\|\,d\tau \le L\|\varphi\|_{\mathbf{M}} := L\gamma. \tag{3.6}$$

Applying Theorem 2.3 for the right-hand side $g(\cdot,v(\cdot))$ instead of $f(\cdot)$ we deduce that for $v\in\mathcal{B}^1_\rho$ there exists a unique 1-periodic solution u to (3.5) satisfying

$$\|u\|_{C_b} \le (M+1)Ke^{\alpha}\|g(\cdot,v(\cdot))\|_{\mathbf{M}} \le (M+1)KL\gamma e^{\alpha}. \tag{3.7}$$

Therefore, if γ is small enough, then Φ acts from $\mathcal{B}^1_\rho$ into itself.

Now, by (3.5) we have

$$\Phi(v)(t) = U(t,0)u(0) + \int_0^t U(t,\tau)g(\tau,v(\tau))\,d\tau \quad \text{for } \Phi(v)=u. \tag{3.8}$$

Furthermore, for $v_1,v_2\in\mathcal{B}^1_\rho$ and $u_1=\Phi(v_1)$ and $u_2=\Phi(v_2)$, by the representation (3.8) we find that $u=\Phi(v_1)-\Phi(v_2)$ is the unique 1-periodic mild solution to

$$u(t) = U(t,0)u(0) + \int_0^t U(t,\tau)\big(g(\tau,v_1(\tau)) - g(\tau,v_2(\tau))\big)\,d\tau \quad \text{for all } t\ge 0.$$

Thus, from Theorem 2.3 and the fact that g belongs to the class (L,φ,ρ) we get

$$\begin{aligned}\|\Phi(v_1)-\Phi(v_2)\|_{C_b} &\le (M+1)Ke^{\alpha}\sup_{t\ge 0}\int_t^{t+1}\|g(\tau,v_1(\tau))-g(\tau,v_2(\tau))\|\,d\tau\\ &\le (M+1)Ke^{\alpha}\sup_{t\ge 0}\int_t^{t+1}\|\varphi(\tau)\|\,d\tau\,\|v_1-v_2\|_{C_b}\\ &\le (M+1)Ke^{\alpha}\|\varphi\|_{\mathbf{M}}\|v_1-v_2\|_{C_b}\\ &= (M+1)K\gamma e^{\alpha}\|v_1-v_2\|_{C_b}.\end{aligned} \tag{3.9}$$

We thus conclude that if $\gamma:=\|\varphi\|_{\mathbf{M}}$ is small enough, then $\Phi:\mathcal{B}^1_\rho\to\mathcal{B}^1_\rho$ is a contraction. Therefore, for this value of γ there exists a unique fixed point $\hat{u}$ of Φ in $\mathcal{B}^1_\rho$, and by definition of Φ, $\hat{u}$ is the unique 1-periodic mild solution to (3.1). ∎

4. Periodic solutions in the case of dichotomic evolution families

4.1. Existence, uniqueness and conditional stability. In this subsection, we will consider equations (2.3) and (3.3) in the case that the evolution family $(U(t,s))_{t\geq s\geq 0}$ has an exponential dichotomy. This assumption is convenient to prove the existence of bounded solutions to (2.3) (i.e., bounded mild solutions to (2.1)). Therefore, the existence and uniqueness of periodic solutions to (2.3) and hence to (3.3) easily follow. Moreover, using the cone inequality (Theorem 1.6), we will show the conditional stability of such periodic solutions.

We start with the definitions of exponential dichotomy and stability of an evolution family.

Definition 4.1. Let $\mathcal{U} := (U(t,s))_{t\geq s\geq 0}$ be an evolution family on a Banach space X.

(1) $\mathcal{U}$ is said to have an *exponential dichotomy* on $[0,\infty)$ if there exist bounded linear projections $P(t)$, $t\geq 0$, on X and positive constants N, ν such that

(a) $U(t,s)P(s) = P(t)U(t,s)$, $t \geq s \geq 0$,

(b) the restriction $U(t,s)_| : \operatorname{Ker} P(s) \to \operatorname{Ker} P(t)$, $t\geq s\geq 0$, is an isomorphism, with inverse $U(s,t)_| := (U(t,s)_|)^{-1}$, $0\leq s\leq t$,

(c) $\|U(t,s)x\| \leq Ne^{-\nu(t-s)}\|x\|$ for $x\in P(s)X$, $t\geq s\geq 0$,

(d) $\|U(s,t)_| x\| \leq Ne^{-\nu(t-s)}\|x\|$ for $x\in \operatorname{Ker} P(t)$, $t\geq s\geq 0$.

The projections $P(t)$, $t\geq 0$, are called the *dichotomy projections*, and the constants N,ν the *dichotomy constants.*

(2) $\mathcal{U}$ is called *exponentially stable* if it has an exponential dichotomy with the dichotomy projections $P(t) = \mathrm{Id}$ for all $t\geq 0$. In other words, $\mathcal{U}$ is exponentially stable if there exist positive constants N and ν such that

$$\|U(t,s)\| \leq Ne^{-\nu(t-s)} \quad \text{for all } t\geq s\geq 0. \tag{4.1}$$

We remark that properties (a)–(d) of the dichotomy projections $P(t)$ already imply that

(1) $H := \sup_{t\geq 0}\|P(t)\| < \infty$,

(2) $t\mapsto P(t)$ is strongly continuous

(see [7, Lemma 4.2]). We refer the reader to [11] for characterizations of exponential dichotomies of evolution families in general admissible spaces.

If $(U(t,s))_{t\geq s\geq 0}$ has an exponential dichotomy with dichotomy projections $(P(t))_{t\geq 0}$ and constants $N,\nu>0$, then we can define the *Green function* on a half-line as follows:

$$\mathcal{G}(t,\tau) := \begin{cases} P(t)U(t,\tau) & \text{for } t>\tau\geq 0,\\ -U(t,\tau)_|(I-P(\tau)) & \text{for } 0\leq t<\tau. \end{cases} \tag{4.2}$$

Then

$$\|\mathcal{G}(t,\tau)\| \leq (1+H)Ne^{-\nu|t-\tau|} \quad \text{for } t\neq\tau\geq 0. \tag{4.3}$$

The following lemma gives the form of bounded solutions of (2.3), (3.3).

LEMMA 4.2. *Let the evolution family $(U(t,s))_{t\geq s\geq 0}$ have an exponential dichotomy with dichotomy projections $(P(t))_{t\geq 0}$ and dichotomy constants $N,\nu>0$. Let $f\in\mathbf{M}$ and let g satisfy* (3.2). *Then:*

(a) *Equation* (2.3) *has bounded solutions in $C_b(\mathbb{R}_+,X)$. Every bounded solution v of* (2.3) *can be written in the form*

$$v(t)=U(t,0)\zeta_0+\int_0^\infty \mathcal{G}(t,\tau)f(\tau)\,d\tau,\qquad t\geq 0, \tag{4.4}$$

for some $\zeta_0\in X_0:=P(0)X$ where $\mathcal{G}(t,\tau)$ is defined by (4.2).

(b) *Let $u\in C_b(\mathbb{R}_+,X)$ be a solution to* (3.3) *such that* $\sup_{t\geq 0}\|u(t)\|\leq\rho$ *for a fixed $\rho>0$. Then, for $t\geq 0$,*

$$u(t)=U(t,0)v_0+\int_0^\infty \mathcal{G}(t,\tau)g(\tau,u(\tau))\,d\tau\quad \text{for some } v_0\in X_0, \tag{4.5}$$

where $\mathcal{G}$ and X_0 are as in (a).

Proof. (a) Set $y(t):=\int_0^\infty \mathcal{G}(t,\tau)f(\tau)\,d\tau$ for $t\geq 0$. Since $f\in\mathbf{M}$, using (4.3) and (1.6) we obtain

$$\begin{aligned}\|y(t)\|&\leq (1+H)N\int_0^\infty e^{-\nu|t-\tau|}\|f(\tau)\|\,d\tau\\&\leq \frac{N(1+H)(N_1\|\Lambda_1T_1^+f\|_\infty+N_2\|\Lambda_1 f\|_\infty)}{1-e^{-\nu}}\qquad\text{for all } t\geq 0.\end{aligned}$$

Moreover, it is straightforward that

$$y(t)=U(t,0)y(0)+\int_0^t U(t,\tau)f(\tau)\,d\tau\quad\text{for } t\geq 0.$$

Since $v(t)$ is a solution of (2.3), we obtain

$$v(t)-y(t)=U(t,0)(v(0)-y(0))\quad\text{for } t\geq 0.$$

Let now $\zeta_0=v(0)-y(0)$. The boundedness of $v(\cdot)$ and $y(\cdot)$ on $[0,\infty)$ implies that $\zeta_0\in X_0$. Finally, since $v(t)=U(t,0)\zeta_0+y(t)$ for $t\geq 0$, the equality (4.4) follows.

(b) Similarly to (a) we set $y(t):=\int_0^\infty \mathcal{G}(t,\tau)g(\tau,u(\tau))\,d\tau$ for $t\geq 0$. Since g satisfies the conditions in (3.2), using estimates (4.3) and (1.8) we obtain

$$\begin{aligned}\|y(t)\|&\leq (1+H)N\int_0^\infty e^{-\nu|t-\tau|}\|g(\tau,u(\tau))\|\,d\tau\leq (1+H)NL\int_0^\infty e^{-\nu|t-\tau|}\varphi(\tau)\,d\tau\\&\leq \frac{(1+H)NL(N_1+N_2)}{1-e^{-\nu}}\|\varphi\|_{\mathbf{M}}\quad\text{for } t\geq 0.\end{aligned}$$

Also, it is straightforward that

$$y(t) = U(t,0)y(0) + \int_0^t U(t,\tau)g(\tau,u(\tau))\,d\tau \quad \text{ for } t \geq 0.$$

Since $u(t)$ is a solution of (3.3) we obtain $u(t) - y(t) = U(t,0)(u(0) - y(0))$ for $t \geq 0$. Let now $v_0 = u(0) - y(0)$. The boundedness of $u(\cdot)$ and $y(\cdot)$ on $\mathbb{R}_+$ implies that $v_0 \in X_0$. Finally, the relation $u(t) = U(t,0)v_0 + y(t)$ for $t \geq 0$ yields (4.5). ■

Remark 4.3. By straightforward computations we can prove that the converses of (a) and (b) are also true, i.e., a solution of (4.4) satisfies (2.3) for $t \geq 0$, and that of (4.5) satisfies (3.3) for $t \geq 0$.

We next prove the existence of bounded solutions to (2.3) and (3.3) (i.e., bounded mild solutions to (2.1) and (3.1)) and hence of periodic solutions.

Theorem 4.4. *Consider equations* (2.3) *and* (3.3). *Let the evolution family* $(U(t,s))_{t\geq s\geq 0}$ *satisfy* (2.4) *and have an exponential dichotomy with dichotomy projections* $P(t)$, $t \geq 0$, *and constants* N,ν. *Let* $f \in \mathbf{P}$ *and suppose that* g *satisfies* (3.2) *with positive constants* ρ, L *and a function* $\varphi \in \mathbf{P}$. *Then:*

- (a) *Equation* (2.3) *has a unique* 1*-periodic solution in* $C_b(\mathbb{R}_+, X)$.
- (b) *If* $\|\varphi\|_{\mathbf{M}}$ *is sufficiently small, then* (3.3) *has a unique* 1*-periodic solution in* $C_b(\mathbb{R}_+, X)$.

Proof. (a) For a given $f \in \mathbf{P}$, taking $\zeta_0 = 0 \in X_0$ in (4.4) we see that (2.3) has a bounded solution

$$u(t) = \int_0^\infty \mathcal{G}(t,\tau)f(\tau)\,d\tau, \tag{4.6}$$

and by (4.3) and (1.8),

$$\begin{aligned}\|u\|_{C_b} &\leq (1+H)N\int_0^\infty e^{-\nu|t-\tau|}\|f(\tau)\|\,d\tau\\ &\leq \frac{(1+H)N(N_1+N_2)}{1-e^{-\nu}}\|f\|_{\mathbf{M}} \quad \text{ for all } t \geq 0.\end{aligned}$$

Applying Theorem 2.3 we see that for the 1-periodic function $f \in \mathbf{P}$ there exists a 1-periodic solution $\hat{u}$ of (2.3) satisfying

$$\|\hat{u}\|_{C_b} \leq \left(\frac{(1+H)N(N_1+N_2)}{1-e^{-\nu}} + 1\right)Ke^{\alpha}\|f\|_{\mathbf{M}}. \tag{4.7}$$

The uniqueness of the 1-periodic solution follows from the fact that for two 1-periodic and continuous (hence bounded on $\mathbb{R}_+$) solutions $\hat{u}$ and $\hat{v}$ we deduce from (4.4) that $\|\hat{u}(t) - \hat{v}(t)\| = \|U(t,0)(u_0 - v_0)\| \leq Ne^{-\nu t}\|u_0 - v_0\| \to 0$ as

$t \to \infty$ since $u_0, v_0 \in X_0$. This, together with periodicity, implies $\hat{u}(t) = \hat{v}(t)$ for all $t \geq 0$, finishing the proof of (a).

(b) By (a), for each 1-periodic f, the linear problem (2.3) has a unique 1-periodic solution $\hat{u} \in C_b(\mathbb{R}_+, X)$ satisfying (4.7). Therefore, (b) follows from Theorem 3.1. ■

We now prove the conditional stability of periodic solutions to (3.3).

THEOREM 4.5. *Let the assumptions of Theorem* 4.4 *hold, and let* $\hat{u}$ *be the* 1*-periodic solution of* (3.3) *obtained in Theorem* 4.4(b). *Denote by* $B_r(x)$ (*resp.* $\mathcal{B}_r(v)$) *the ball in* X (*resp. in* $C_b(\mathbb{R}_+, X)$) *centered at* x (*resp. at* v) *with radius* r. *Let* $\mathcal{B}_\rho(0)$ *be the ball containing* $\hat{u}$ *as in Theorem* 4.4(b). *Suppose further that there exists a positive* $\varphi_1 \in \mathbf{P}$ *such that*

$$\|g(t, v_1(t)) - g(t, v_2(t))\| \leq \varphi_1(t)\|v_1 - v_2\| \quad \text{for all } v_1, v_2 \in \mathcal{B}_{2\rho}(0).$$

Then, if $\|\varphi_1\|_{\mathbf{M}}$ *is small enough, to each* $v_0 \in B_{\rho/(2N)}(P(0)\hat{u}(0)) \cap P(0)X$ *there corresponds a unique solution* $u(t)$ *of* (3.3) *on* $\mathbb{R}_+$ *with* $P(0)u(0) = v_0$ *and* $u \in \mathcal{B}_\rho(\hat{u})$. *Moreover,*

$$\|u(t) - \hat{u}(t)\| \leq C_\mu e^{-\mu t}\|P(0)u(0) - P(0)\hat{u}(0)\| \quad \text{for } t \geq 0, \tag{4.8}$$

for some positive constants C_μ *and* μ *independent of* u *and* $\hat{u}$.

Proof. For $v_0 \in B_{\rho/(2N)}(P(0)\hat{u}(0)) \cap P(0)X$ we will prove that the transformation F defined by

$$(Fw)(t) = U(t,0)v_0 + \int_0^\infty \mathcal{G}(t,\tau)(g(\tau, w(\tau))\, d\tau \quad \text{for } t \geq 0$$

acts from $\mathcal{B}_\rho(\hat{u})$ into itself and is a contraction. In fact, for $w(\cdot) \in \mathcal{B}_\rho(\hat{u})$,

$$\|w\|_{C_b} \leq \|w - \hat{u}\|_{C_b} + \|\hat{u}\|_{C_b} \leq 2\rho \tag{4.9}$$

and $\|g(t,w) - g(t,\hat{u})\| \leq \varphi_1(t)\|w - \hat{u}\|_{C_b} \leq \rho\varphi_1(t)$. Therefore, setting

$$y(t) := (Fw)(t) = U(t,0)v_0 + \int_0^\infty \mathcal{G}(t,\tau)(g(\tau, w(\tau))\, d\tau \quad \text{for } t \geq 0$$

we obtain

$$\begin{aligned}
\|y(t) - \hat{u}(t)\| &\leq Ne^{-\nu t}\|v_0 - P(0)\hat{u}(0)\| \\
&\qquad + (1+H)N \int_0^\infty e^{-\nu|t-\tau|}\|g(\tau, w) - g(\tau,\hat{u})\|\, d\tau \\
&\leq Ne^{-\nu t}\|v_0 - P(0)\hat{u}(0)\| + (1+H)N \int_0^\infty e^{-\nu|t-\tau|}\rho\varphi_1(\tau)\, d\tau \\
&\leq N\|v_0 - P(0)\hat{u}(0)\| + \frac{(1+H)N\rho(N_1+N_2)\|\varphi_1\|_{\mathbf{M}}}{1-e^{-\nu}}
\end{aligned}$$

for all $t \geq 0$. Hence,

$$\|Fw - \hat{u}\|_{C_b} \leq N\|v_0 - P(0)\hat{u}(0)\| + \frac{(1+H)N\rho(N_1+N_2)\|\varphi_1\|_{\mathbf{M}}}{1-e^{-\nu}}.$$

Using now the fact that $\|v_0 - P(0)\hat{u}(0)\| \leq \rho/(2N)$ we find that if $\|\varphi_1\|_{\mathbf{M}}$ is small enough, then F acts from $\mathcal{B}_\rho(\hat{u})$ into $\mathcal{B}_\rho(\hat{u})$.

Now, for $x, z \in \mathcal{B}_\rho(\hat{u})$ (thus, as in (4.9), $\|x\|_{C_b}, \|z\|_{C_b} \leq 2\rho$) we estimate

$$\begin{aligned}\|(Fx)(t) - (Fz)(t)\| &\leq \int_0^\infty \|\mathcal{G}(t,\tau)\| \, \|g(\tau,x) - g(\tau,z))\| \, d\tau \\ &\leq (1+H)N \int_0^\infty e^{-\nu|t-\tau|}\|g(\tau,x) - g(\tau,z)\| \, d\tau \\ &\leq \frac{(1+H)N(N_1+N_2)\|\varphi_1\|_{\mathbf{M}}}{1-e^{-\nu}}\|x-z\|_{C_b} \quad \text{for all } t \geq 0.\end{aligned}$$

Therefore,

$$\|Fx - Fz\|_{C_b} \leq \frac{(1+H)N(N_1+N_2)\|\varphi_1\|_{\mathbf{M}}}{1-e^{-\nu}}\|x-z\|_{C_b}.$$

Since $\frac{(1+H)N(N_1+N_2)\|\varphi_1\|_{\mathbf{M}}}{1-e^{-\nu}} < 1$ we deduce that $F : \mathcal{B}_\rho(\hat{u}) \to \mathcal{B}_\rho(\hat{u})$ is a contraction. Thus, there exists a unique $u \in \mathcal{B}_\rho(\hat{u})$ such that $Fu = u$. By definition of F we see that u is the unique solution in $\mathcal{B}_\rho(\hat{u})$ of (4.5) for $t \geq 0$. By Lemma 4.2 and Remark 4.3, u is the unique solution of (3.3) in $\mathcal{B}_\rho(\hat{u})$.

Finally, we prove (4.8). To do so, since both $\hat{u}$ and u are bounded on $\mathbb{R}_+$, we can use (4.5) to write

$$\begin{aligned}u(t) - \hat{u}(t) &= U(t,0)(P(0)u(0) - P(0)\hat{u}(0)) \\ &\quad + \int_0^\infty \mathcal{G}(t,\tau)\big(g(\tau,u(\tau)) - g(\tau,\hat{u}(\tau))\big) \, d\tau.\end{aligned}$$

Therefore,

$$\begin{aligned}\|u(t) - \hat{u}(t)\| &\leq Ne^{-\nu t}\|P(0)u(0) - P(0)\hat{u}(0)\| \\ &\quad + (1+H)N\int_0^\infty e^{-\nu|t-\tau|}\|g(\tau,u(\tau)) - g(\tau,\hat{u}(\tau))\| \, d\tau \\ &\leq Ne^{-\nu t}\|P(0)u(0) - P(0)\hat{u}(0)\| \\ &\quad + (1+H)N\int_0^\infty e^{-\nu|t-\tau|}\varphi_1(\tau)\|u(\tau) - \hat{u}(\tau)\| \, d\tau \quad \text{for } t \geq 0.\end{aligned}$$

Set $\phi(t) = \|u(t) - \hat{u}(t)\|$. Then $\operatorname{ess\,sup}_{t\geq 0} \phi(t) < \infty$ and

$$\phi(t) \leq Ne^{-\nu t}\|P(0)u(0) - P(0)\hat{u}(0)\| + (1+H)N \int_0^\infty e^{-\nu|t-\tau|}\varphi_1(\tau)\phi(\tau)\,d\tau \quad \text{for } t \geq 0. \tag{4.10}$$

We will apply the cone inequality (Theorem 1.6) to the Banach space $W := C_b(\mathbb{R}_+, \mathbb{R})$ with $\mathcal{K}$ being the set of all nonnegative functions. We then consider the linear operator B defined for $u \in W$ by

$$(Bu)(t) = (1+H)N \int_0^\infty e^{-\nu|t-\tau|}\varphi_1(\tau)u(\tau)\,d\tau \quad \text{for } t \geq 0.$$

By (1.8) we have

$$\begin{aligned}\sup_{t\geq 0}(Bu)(t) &= \sup_{t\geq 0}(1+H)N \int_0^\infty e^{-\nu|t-\tau|}\varphi_1(\tau)u(\tau)\,d\tau \\ &\leq \frac{(1+H)N}{1-e^{-\nu}}(N_1+N_2)\|\varphi_1\|_{\mathbf{M}}\|u\|_\infty.\end{aligned}$$

Therefore, $B \in \mathcal{L}(W)$ and $\|B\| \leq \frac{(1+H)N}{1-e^{-\nu}}(N_1+N_2)\|\varphi_1\|_{\mathbf{M}} < 1$. Obviously, B leaves the cone $\mathcal{K}$ invariant. Now (4.10) can be rewritten as

$$\phi \leq B\phi + z \quad \text{for } z(t) = Ne^{-\nu t}\|P(0)u(0) - P(0)\hat{u}(0)\|, \quad t \geq 0.$$

Hence, by Theorem 1.6 we obtain $\phi \leq \psi$, where ψ is a solution in W of the equation $\psi = B\psi + z$ which can be rewritten as

$$\psi(t) = Ne^{-\nu t}\|P(0)u(0) - P(0)\hat{u}(0)\| + (1+H)N \int_0^\infty e^{-\nu|t-\tau|}\varphi_1(\tau)\psi(\tau)\,d\tau \quad \text{for } t \geq 0. \tag{4.11}$$

We now estimate ψ. To that end, for

$$0 < \mu < \nu + \ln\bigl(1 - (1+H)N(N_1+N_2)\|\varphi_1\|_{\mathbf{M}}\bigr)$$

we set $w(t) = e^{\mu t}\psi(t)$ for $t \geq 0$. Then, by (4.11) we obtain

$$w(t) = Ne^{-(\nu-\mu)t}\|P(0)u(0) - P(0)\hat{u}(0)\| + (1+H)N \int_0^\infty e^{-\nu|t-\tau|+\mu(t-\tau)}\varphi_1(\tau)w(\tau)\,d\tau \quad \text{for } t \geq 0. \tag{4.12}$$

We next consider the linear operator D defined for $u \in W$ by

$$(Du)(t) = (1+H)N \int_0^\infty e^{-\nu|t-\tau|+\mu(t-\tau)}\varphi_1(\tau)u(\tau)\,d\tau \quad \text{for } t \geq 0.$$

By (1.8) we have

$$\begin{aligned}\sup_{t\ge 0}(Du)(t) &= \sup_{t\ge 0}(1+H)N\int_0^\infty e^{-\nu|t-\tau|+\mu(t-\tau)}\varphi_1(\tau)u(\tau)\,d\tau\\ &\le \sup_{t\ge 0}(1+H)N\int_0^\infty e^{-(\nu-\mu)|t-\tau|}\varphi_1(\tau)u(\tau)\,d\tau\\ &\le \frac{(1+H)N}{1-e^{-(\nu-\mu)}}(N_1+N_2)\|\varphi_1\|_{\mathbf{M}}\|u\|_\infty.\end{aligned}$$

Therefore, $D\in\mathcal{L}(W)$ and $\|D\|\le \frac{(1+H)N}{1-e^{-(\nu-\mu)}}(N_1+N_2)\|\varphi_1\|_{\mathbf{M}}$.

Equation (4.12) can now be rewritten as

$$w = Dw + z \quad \text{ for } z(t) = Ne^{-(\nu-\mu)t}\|P(0)u(0)-P(0)\hat{u}(0)\|, \quad t\ge 0.$$

Since $\mu<\nu+\ln(1-(1+H)N(N_1+N_2)\|\varphi_1\|_{\mathbf{M}})$, we obtain

$$\|D\|\le \frac{(1+H)N}{1-e^{-(\nu-\mu)}}(N_1+N_2)\|\varphi_1\|_{\mathbf{M}}<1.$$

Therefore, the equation $w=Dw+z$ is uniquely solvable in W, and its solution is $w=(I-D)^{-1}z$. Hence,

$$\begin{aligned}\|w\|_\infty &= \|(I-D)^{-1}z\|_\infty \le \|(I-D)^{-1}\|\,\|z\|_\infty \le \frac{\|z\|_\infty}{1-\|D\|}\\ &\le \frac{N}{1-\frac{(1+H)N}{1-e^{-(\nu-\mu)}}(N_1+N_2)\|\varphi_1\|_{\mathbf{M}}}\|P(0)u(0)-P(0)\hat{u}(0)\|.\end{aligned}$$

Therefore,

$$\|w\|_\infty\le C_\mu\|P(0)u(0)-P(0)\hat{u}(0)\| \text{ for } C_\mu := \frac{N}{1-\frac{(1+H)N}{1-e^{-(\nu-\mu)}}(N_1+N_2)\|\varphi_1\|_{\mathbf{M}}}.$$

This yields

$$w(t)\le C_\mu\|P(0)u(0)-P(0)\hat{u}(0)\| \quad \text{ for } t\ge 0.$$

Hence, $\psi(t)=e^{-\mu t}w(t)\le C_\mu e^{-\mu t}\|P(0)u(0)-P(0)\hat{u}(0)\|$. Since $\|u(t)-\hat{u}(t)\| = \phi(t)\le\psi(t)$, we obtain

$$\|u(t)-\hat{u}(t)\|\le C_\mu e^{-\mu t}\|P(0)u(0)-P(0)\hat{u}(0)\|,$$

finishing the proof of Theorem 4.5. ■

Remark 4.6. The assertion of the above theorem gives the *conditional stability* of the periodic solution $\hat{u}$ in the sense that for any other solution u such that $P(0)u(0)\in B_{\rho/(2N)}(P(0)\hat{u}(0))\cap P(0)X$ and u being in a small ball $\mathcal{B}_\rho(\hat{u})$ we have $\|u(t)-\hat{u}(t)\|\to 0$ exponentially as $t\to\infty$ (see (4.8)).

For an exponentially stable evolution family (see Definition 4.1(2)) we have the following direct consequence of Theorem 4.5.

Corollary 4.7. *Let the assumptions of Theorem* 4.4 *hold, and let* $\hat{u}$ *be the periodic solution of* (3.3) *obtained in Theorem* 4.4(b). *Let further the evolution family* $(U(t,s))_{t\geq s\geq 0}$ *be exponentially stable. Then the periodic solution* $\hat{u}$ *is exponentially stable in the sense that for any other solution* $u \in C_b(\mathbb{R}_+, X)$ *of* (3.3) *such that* $\|u(0)-\hat{u}(0)\|$ *is small enough we have*

$$\|u(t)-\hat{u}(t)\| \leq Ce^{-\mu t}\|u(0)-\hat{u}(0)\| \quad \text{for all } t \geq 0, \tag{4.13}$$

for some positive constants C *and* μ *independent of* u *and* $\hat{u}$.

Proof. Just apply Theorem 4.5 for $P(t) = \mathrm{Id}$ for all $t \geq 0$. ■

4.2. Local stable manifold near the periodic solution. In this subsection, under the same hypotheses as in §4.1, we will prove the existence of a local stable manifold for equation (3.3) near its periodic solution. As previously, we denote by $B_r(x)$ the ball in X centered at x with radius r. We now give the definition of a local stable manifold for (3.3) near its periodic solution.

Definition 4.8. Let $\hat{u}$ be a continuous and 1-periodic solution to (3.3). A set $\mathbf{S} \subset \mathbb{R}_+ \times X$ is said to be a *local stable manifold for* (3.3) *near* $\hat{u}$ if for every $t \in \mathbb{R}_+$ the phase space X splits into a direct sum $X = X_0(t) \oplus X_1(t)$ such that

$$\inf_{t\in\mathbb{R}_+} \mathrm{Sn}(X_0(t), X_1(t)) := \inf_{t\in\mathbb{R}_+} \inf_{i=0,1} \{\|x_0+x_1\| : x_i \in X_i(t),\ \|x_i\| = 1\} > 0$$

and if there exist positive constants ρ, ρ_0, ρ_1 and a family of Lipschitz continuous mappings

$$h_t : B_{\rho_0}(\hat{u}(t)) \cap X_0(t) \to B_{\rho_1}(\hat{u}(t)) \cap X_1(t), \qquad t \in \mathbb{R}_+,$$

with the Lipschitz constants being independent of t, such that:

(i) $\mathbf{S} = \{(t, x+h_t(x)) \in \mathbb{R}_+ \times (X_0(t)\oplus X_1(t)) : t \in \mathbb{R}_+,\ x \in B_{\rho_0}(\hat{u}(t)) \cap X_0(t)\}$, and we define $\mathbf{S}_t := \{x+h_t(x) : (t, x+h_t(x)) \in \mathbf{S}\}$, $t \geq 0$,

(ii) $\mathbf{S}_t$ is homeomorphic to
$$B_{\rho_0}(\hat{u}(t)) \cap X_0(t) := \{x \in X_0(t) : \|x-\hat{u}(t)\| \leq \rho_0\}$$
for all $t \geq 0$,

(iii) to each $x_0 \in \mathbf{S}_{t_0}$ there corresponds a unique solution $u(t)$ of (3.3) on $[t_0, \infty)$ with $u(t_0) = x_0$ and $\operatorname{ess\,sup}_{t\geq t_0} \|u(t)\| \leq \rho$.

Note that, if we identify $X_0(t) \oplus X_1(t)$ with $X_0(t) \times X_1(t)$, then we can write $\mathbf{S}_t = \mathrm{graph}(h_t)$.

We now prove our last result on the existence of a stable manifold for solutions to (3.3).

Theorem 4.9. *Let the assumptions of Theorems* 4.4 *and* 4.5 *hold with positive functions* φ *and* φ_1. *Let* $\hat{u}$ *be the* 1*-periodic solution of* (3.3) *obtained in Theorem* 4.4 *thanks to the sufficient smallness of* $\|\varphi\|_{\mathbf{M}}$. *Then, if* $\|\varphi_1\|_{\mathbf{M}}$

is sufficiently small, there exists a local stable manifold $\mathbf{S}$ *near the solution* $\hat{u}$. *Moreover, every solution* $u(t)$ *on* $\mathbf{S}$ *is exponentially attracted to* $\hat{u}(t)$ *in the sense that there exist positive constants* μ *and* C_μ, *independent of* $t_0 \geq 0$, *such that*

$$\|u(t) - \hat{u}(t)\| \leq C_\mu e^{-\mu(t-t_0)} \|P(t_0)u(t_0) - P(t_0)\hat{u}(t_0))\| \text{ for all } t \geq t_0. \tag{4.14}$$

Proof. We will apply [12, Theorem 3.8]. To this end, let u be a solution to (3.3) and set $w = u - \hat{u}$. Then u satisfies (3.3) if and only if

$$\begin{aligned} w(t) &= U(t,0)w(0) \\ &\quad + \int_0^t U(t,\tau)[g(\tau, w(\tau) + \hat{u}(\tau)) - g(\tau, \hat{u}(\tau))]\,d\tau \qquad \text{for } t \geq 0. \end{aligned} \tag{4.15}$$

Setting now $F(t,w) = g(t, w + \hat{u}) - g(t, \hat{u})$, we find that $F(t,0) = 0$ and F belongs to the class $(2\rho, \varphi_1, 2\rho)$ since g satisfies the assumption of Theorem 4.5. Therefore, by [12, Theorem 3.8], if $\|\varphi_1\|_{\mathbf{M}}$ is small enough, then there exists a local stable manifold $\mathbf{S}$ (near 0) for (4.15). Returning to the solution u of (3.3), by replacing w with $u - \hat{u}$, we see that $\mathbf{S}$ is the local stable manifold for (3.3) near $\hat{u}$. Finally, (4.14) follows from (4.8). ∎

We finally illustrate our results by the following example.

4.3. An example. We consider the problem

$$\begin{cases} w_t(x,t) = a(t)[w_{xx}(x,t) + \delta w(x,t)] \\ \qquad\qquad + \psi(t)[|w|^{k-1} w(x,t) + h(x,t)] \quad \text{for } 0 < x < \pi,\ t \geq 0, \\ w(0,t) = w(\pi,t) = 0, \quad t \geq 0. \end{cases} \tag{4.16}$$

Here, $\delta \in \mathbb{R}$ and $\delta \neq n^2$ for all $n \in \mathbb{N}$; the function $a(\cdot) \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ is 1-periodic and satisfies $0 < \gamma_0 \leq a(t) \leq \gamma_1$ for fixed γ_0, γ_1; the exponent $k > 1$ is an integer; the function $h : [0,\pi] \times \mathbb{R}_+ \to \mathbb{R}$ is continuous on $[0,\pi] \times \mathbb{R}_+$ and 1-periodic with respect to t.

We next set $X := L_2[0,\pi]$, and let $A : X \supset D(A) \to X$ be defined by $Ay = y'' + \delta y$, with the domain

$$\begin{aligned} D(A) = \{y \in X : y \text{ and } y' &\text{ are absolutely continuous,} \\ &y'' \in X,\ y(0) = y(\pi) = 0\}. \end{aligned}$$

It can be seen [3] that A is the generator of an analytic semigroup $(\mathbb{T}(t))_{t\geq 0.}$ Since $\sigma(A) = \{-n^2 + \delta : n = 1, 2, \ldots\}$, applying the spectral mapping theorem for analytic semigroups we get

$$\begin{aligned} \sigma(\mathbb{T}(t)) = e^{t\sigma(A)} = \{e^{t(-n^2+\delta)} : n = 1, 2, \ldots\}, \\ \text{and hence} \quad \sigma(\mathbb{T}(t)) \cap \Gamma = \emptyset \quad \text{for all } t > 0, \end{aligned} \tag{4.17}$$

where $\Gamma := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

Setting now $A(t) := a(t)A$ we see that $A(t)$ is 1-periodic, and the family $(A(t))_{t\geq 0}$ generates a 1-periodic (in the sense of Assumption 2.2) evolution family $U(t,s)_{t\geq s\geq 0}$ which is defined by $U(t,s) = \mathbb{T}(\int_s^t a(\tau)\,d\tau)$.

By (4.17) we see that the analytic semigroup $(\mathbb{T}(t))_{t\geq 0}$ is hyperbolic (or has an exponential dichotomy) with the projection P satisfying

(1) $\|\mathbb{T}(t)x\| \leq Ne^{-\beta t}\|x\|$ for $x \in PX$, $t \geq 0$,
(2) $\|\mathbb{T}(-t)_{|}x\| = \|(\mathbb{T}(t)_{|})^{-1}x\| \leq Ne^{-\beta t}\|x\|$ for $x \in \operatorname{Ker} P$, $t \geq 0$, where the invertible operator $\mathbb{T}(t)_{|}$ is the restriction of $T(t)$ to Ker$P$, and N, β are positive constants.

Using the hyperbolicity of $(\mathbb{T}(t))_{t\geq 0}$ it is straightforward to check that the evolution family $U(t,s)_{t\geq s\geq 0}$ has an exponential dichotomy with the projection $P(t) = P$ for all $t \geq 0$ and the dichotomy constants N and $\nu := \beta\gamma_0$ by the following estimates:

$$\|U(t,s)x\| \leq Ne^{-\nu(t-s)}\|x\| \quad \text{for } x \in PX,\ t \geq s \geq 0,$$
$$\|U(s,t)_{|}x\| \leq Ne^{-\nu(t-s)}\|x\| \quad \text{for } x \in \operatorname{Ker} P,\ t \geq s \geq 0.$$

We then define a function $g : \mathbb{R}_+ \times X \to X$ by $g(t,w) = \psi(t)[|w|^{k-1}w + H(t)]$ for $w \in X$ and $H(t) = h(\cdot,t)$ being an X-valued function, where the real function $\psi(t)$ is defined for fixed constants $b > 0$, $c > 1$ by

$$\psi(t) = \begin{cases} b(t-j) & \text{if } t \in \left[j+\frac{1}{2}-\frac{1}{2^c}, j+\frac{1}{2}+\frac{1}{2^c}\right] \text{ for } j = 0,1,2,\ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{4.18}$$

Equation (4.16) can now be rewritten as

$$\frac{du}{dt} = A(t)u(t) + g(t,u(t)) \quad \text{for } u(t) = w(\cdot,t).$$

Since $\psi(t)$ and $H(t) = h(\cdot,t)$ are 1-periodic, it follows that $g(t,w)$ is 1-periodic with respect to t for each fixed $w \in X$. Moreover, $\|g(t,0)\| = \psi(t)\|H(t)\| \leq \gamma\psi(t)$ for $\gamma := \sup_{t\in[0,1]}(\int_0^\pi |h(x,t)|^2\,dx)^{1/2}$, and we have

$$\begin{aligned}
\|g(t,v_1) - g(t,v_2)\| &= \psi(t)\big\||v_1|^{k-1}v_1 - |v_2|^{k-1}v_2\big\| \\
&= \psi(t)\big\||v_1|^{k-1}v_1 - |v_1|^{k-1}v_2 + |v_1|^{k-1}v_2 \\
&\qquad - |v_1|^{k-2}|v_2|v_2 + \cdots + |v_1|\,|v_2|^{k-2}v_2 - |v_2|^{k-1}v_2\big\| \\
&\leq \psi(t)\sum_{j=0}^{k-1}\big\||v_1 - v_2|\,|v_1|^j|v_2|^{k-1-j}\big\| \\
&\leq \psi(t)k\|v_1 - v_2\|r^{k-1} \quad \text{for all } v_1, v_2 \in B_r(0)
\end{aligned}$$

and

$$\|g(t,v)\| = \|g(t,v) - g(t,0)\| + \|g(t,0)\| \leq \psi(t)(kr^k + \gamma) \quad \text{for all } v \in B_r(0).$$

Setting

$$L := r + \frac{\gamma}{kr^{k-1}} \quad \text{and} \quad \varphi(t) := kr^{k-1}\psi(t)$$

we obtain

$$\begin{aligned}\sup_{t\geq 0}\int_t^{t+1} |\varphi(\tau)|\,d\tau &\leq \sup_{j\in\mathbb{N}} \int_{j+1/2-1/2^c}^{j+1/2+1/2^c} kr^{k-1}b(t-j)\,dt \\ &\leq \sup_{j\in\mathbb{N}} \int_{j+1/2-1/2^c}^{j+1/2+1/2^c} kr^{k-1}b\,dt = \frac{bkr^{k-1}}{2^{c-1}}.\end{aligned}$$

Hence, $\varphi \in \mathbf{M}(\mathbb{R}_+)$ and $\|\varphi\|_{\mathbf{M}} \leq bkr^{k-1}/2^{c-1}$. Therefore, for any fixed constants b, k and r, the norm $\|\varphi\|_{\mathbf{M}}$ can be made sufficiently small if we choose c large enough.

Therefore, g satisfies the hypotheses of Theorems 4.4 and 4.5 with $\rho = r$, $\varphi(t) = k\rho^{k-1}\psi(t)$ and $\varphi_1(t) = k(2\rho)^{k-1}\psi(t)$. These theorems imply that if c is large enough (consequently, $\|\varphi\|_{\mathbf{M}}$ and $\|\varphi_1\|_{\mathbf{M}}$ are small enough), then equation (4.16) has one and only one 1-periodic mild solution $\hat{u} \in \mathcal{B}_\rho(0)$, and this solution $\hat{u}$ is conditionally stable in the sense of Remark 4.6. Moreover, by Theorem 4.9, there exists a local stable manifold for mild solutions to (4.16) near the periodic solution $\hat{u}$.

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