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FACTORIZATION OF VECTOR MEASURES AND THEIR INTEGRATION OPERATORS

BY

JOSÉ RODRÍGUEZ (Murcia)

Abstract. Let X be a Banach space and ν a countably additive X-valued measure defined on a σ-algebra. We discuss some generation properties of the Banach space $L^1(\nu)$ and its connection with uniform Eberlein compacta. In this way, we provide a new proof that $L^1(\nu)$ is weakly compactly generated and embeds isomorphically into a Hilbert generated Banach space. The Davis–Figiel–Johnson–Pełczyński factorization of the integration operator $I_\nu : L^1(\nu) \to X$ is also analyzed. As a result, we prove that if I_ν is both completely continuous and Asplund, then ν has finite variation and $L^1(\nu) = L^1(|\nu|)$ with equivalent norms.

1. Introduction. The factorization method of Davis, Figiel, Johnson and Pełczyński [9] (briefly DFJP) is one of the keystones of Banach space theory. In this paper we apply this technique to study the Banach lattice $L^1(\nu)$ of all real-valued functions which are integrable with respect to a vector measure ν. Such spaces represent (via order isometries) all order continuous Banach lattices having weak unit. It is well known that any order continuous Banach lattice having weak unit, say E, is weakly compactly generated (WCG) [4] (cf. [8]). Therefore, by the DFJP theorem, there exist a reflexive Banach space Y and an operator $T : Y \to E$ with dense range (we say that E is *generated by* Y *via* T). An elementary example is $E = L^1(\mu)$, where μ is a non-negative finite measure, which is generated by the Hilbert space $L^2(\mu)$ via the identity operator from $L^2(\mu)$ to $L^1(\mu)$. A Banach space Z is called *Hilbert generated* if there exist a Hilbert space H and an operator $T : H \to Z$ with dense range. Hilbert generated spaces are a proper subclass of WCG spaces containing all separable ones. Kutzarova and Troyanski [19] proved that every order continuous Banach lattice having weak unit admits an equivalent uniformly Gâteaux smooth norm, a condition which is equivalent to being isomorphic to a *subspace* of a Hilbert generated space (see [15],

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cf. [18, Theorem 6.30]). It seems to be unknown whether such spaces are Hilbert generated in general.

Throughout this paper, (Ω, Σ) is a measurable space, X a Banach space and $\mathrm{ca}(\Sigma, X)$ denotes the set of all (countably additive) X-valued vector measures defined on the σ-algebra Σ. In Section 2 we provide a new insight into the weakly compact generation of $L^1(\nu)$ for $\nu \in \mathrm{ca}(\Sigma, X)$. Namely, we show that $L^1(\nu)$ is generated by a reflexive space of the form $L^2(\tilde{\nu})$, where $\tilde{\nu}$ is a reflexive Banach space-valued measure through which ν factors (Theorem 2.1). We also give another proof that $L^1(\nu)$ is isomorphic to a subspace of a Hilbert generated space by showing that $(B_{L^1(\nu)^*}, w^*)$ is uniform Eberlein (Theorem 2.2). Recall that a compact Hausdorff topological space is said to be *uniform Eberlein compact* (UEC) if it is homeomorphic to a weakly compact subset of a Hilbert space. It is known that a Banach space Z is isomorphic to a subspace of a Hilbert generated space if and only if (B_{Z^*}, w^*) is UEC (see [15], cf. [18, Theorem 6.30]).

In Section 3 we deal with the integration operator

$$I_\nu : L^1(\nu) \to X, \quad I_\nu(f) := \int_\Omega f \, d\nu,$$

associated to $\nu \in \mathrm{ca}(\Sigma, X)$. The operator ideal properties of I_ν have strong connections with the structure of $L^1(\nu)$. For instance, if I_ν is compact, or p-summing ($1 \leq p < \infty$), or completely continuous and X is an Asplund space, then ν has finite variation and $L^1(\nu) = L^1(|\nu|)$ with equivalent norms; see [22] (cf. [7, 24]), [6, 23] and [7], respectively. By applying the DFJP factorization method to the integration operator, we are able to generalize simultaneously these results in the following way: if I_ν is completely continuous and Asplund, then ν has finite variation and $L^1(\nu) = L^1(|\nu|)$ with equivalent norms (Theorem 3.3).

Terminology. All our linear spaces are real. By an *operator* we mean a continuous linear map between Banach spaces. By a *subspace* of a Banach space we mean a closed linear subspace. The closed unit ball of a Banach space Z is denoted by B_Z and the (topological) dual of Z is denoted by Z^*. The symbol $\overline{\mathrm{aco}}(C)$ stands for the closed absolutely convex hull of any set $C \subseteq Z$. The weak topology (resp. weak* topology) of Z (resp. Z^*) is denoted by w (resp. w^*). We write $Z \not\supseteq Y$ to denote that no subspace of Z is isomorphic to the Banach space Y.

Let $\nu \in \mathrm{ca}(\Sigma, X)$. We write $x^*\nu \in \mathrm{ca}(\Sigma, \mathbb{R})$ to denote the composition of ν with any $x^* \in X^*$. The *semivariation* of ν is the function $\|\nu\| : \Sigma \to \mathbb{R}$ defined by $\|\nu\|(A) = \sup_{x^* \in B_{X^*}} |x^*\nu|(A)$ for all $A \in \Sigma$ (as usual, $|x^*\nu|$ stands for the variation of $x^*\nu$). The collection of *ν-null sets* is $\mathcal{N}(\nu) := \{A \in \Sigma : \|\nu\|(A) = 0\}$. A *Rybakov control measure* of ν is a non-negative finite measure of the form $\mu = |x_0^* \nu|$ for some $x_0^* \in B_{X^*}$ such that $\mathcal{N}(\nu) = \{A \in \Sigma : \mu(A) = 0\}$. A Σ-measurable function $f : \Omega \to \mathbb{R}$ is said to be *ν-integrable* if it is $|x^*\nu|$-integrable for all $x^* \in X^*$ and, for each $A \in \Sigma$, there is a vector $\int_A f\, d\nu \in X$ such that $x^*(\int_A f\, d\nu) = \int_A f\, d(x^*\nu)$ for every $x^* \in X^*$. By identifying functions which coincide $\|\nu\|$-a.e. we obtain the Banach lattice $L^1(\nu)$ of all (equivalence classes of) ν-integrable functions, equipped with the $\|\nu\|$-a.e. order and the norm

$$\|f\|_{L^1(\nu)} := \sup_{x^* \in B_{X^*}} \int_\Omega |f|\, d|x^*\nu|, \qquad f \in L^1(\nu).$$

For $1 < p < \infty$, we shall also consider the Banach lattice $L^p(\nu)$ made up of all $f \in L^1(\nu)$ for which $|f|^p \in L^1(\nu)$, equipped with the $\|\nu\|$-a.e. order and the norm

$$\|f\|_{L^p(\nu)} := \big(\||f|^p\|_{L^1(\nu)}\big)^{1/p}.$$

The basic properties of the spaces $L^1(\nu)$ and $L^p(\nu)$ can be found in [25, Chapter 3]. Simple functions are dense in these spaces and the identity map $L^p(\nu) \to L^1(\nu)$ is an injective operator. By a *simple function* we mean a finite linear combination of functions of the form 1_A (the characteristic function of A), where $A \in \Sigma$.

2. Generating L^1 of a vector measure. We begin this section with an application of the DFJP factorization method to vector measure theory. Recall first that the range $\nu(\Sigma) = \{\nu(A) : A \in \Sigma\}$ of any $\nu \in \mathrm{ca}(\Sigma, X)$ is relatively weakly compact in X (see e.g. [12, p. 14, Corollary 7]).

THEOREM 2.1. *Let $\nu \in \mathrm{ca}(\Sigma, X)$. Then:*

(i) *There exist a reflexive Banach space Y, an injective operator $T : Y \to X$ and $\tilde{\nu} \in \mathrm{ca}(\Sigma, Y)$ such that $T \circ \tilde{\nu} = \nu$.*
(ii) *$L^2(\tilde{\nu})$ is reflexive and the identity map $j : L^2(\tilde{\nu}) \to L^1(\nu)$ is a well-defined injective operator with dense range. In particular, $L^1(\nu)$ is WCG.*

Proof. (i) Since $\nu(\Sigma)$ is relatively weakly compact, $\overline{\mathrm{aco}}(\nu(\Sigma))$ is weakly compact (by the Krein–Šmulian theorem, see e.g. [12, p. 51, Theorem 11]). We can apply the DFJP theorem (see e.g. [1, Theorem 5.37]) to find a reflexive Banach space Y and an injective operator $T : Y \to X$ such that $T(B_Y) \supseteq \overline{\mathrm{aco}}(\nu(\Sigma))$. Define $\tilde{\nu} : \Sigma \to Y$ such that $T \circ \tilde{\nu} = \nu$. Note that $(x^* \circ T) \circ \tilde{\nu} = x^* \circ \nu$ is countably additive for all $x^* \in X^*$. Since $\{x^* \circ T : x^* \in X^*\} \subseteq Y^*$ separates the points of Y (because T is injective) and $Y \not\supseteq \ell^\infty$, we conclude that $\tilde{\nu}$ is countably additive (see [10], cf. [12, p. 23, Corollary 7]).

(ii) The space $L^2(\tilde{\nu})$ is reflexive because Y is weakly sequentially complete (see [16, Corollary 3.10]). On the other hand, the equality $T \circ \tilde{\nu} = \nu$ implies that the identity map $L^1(\tilde{\nu}) \to L^1(\nu)$ is a well-defined injective operator (see e.g. [25, Lemma 3.27]), and so is j. Since simple functions are dense in $L^1(\nu)$, we deduce that $j(L^2(\tilde{\nu}))$ is dense in $L^1(\nu)$. ∎

The proof of the following theorem uses the fact that, *for any non-negative finite measure μ, every weakly compact subset of $L^1(\mu)$ is UEC* (see [2], cf. [18, Corollary 6.47]). Part (i) extends that result to the vector measure setting, see also [5, Proposition 2.4] for a slightly more general statement.

THEOREM 2.2. *Let $\nu \in \mathrm{ca}(\Sigma, X)$. Then:*

(i) *Every weakly compact subset of $L^1(\nu)$ is UEC.*
(ii) *$(B_{L^1(\nu)^*}, w^*)$ is UEC. Equivalently, $L^1(\nu)$ is isomorphic to a subspace of a Hilbert generated space.*

Proof. Let μ be a Rybakov control measure of ν.

(i) According to the comments preceding the theorem, every weakly compact subset of $L^1(\mu)$ is UEC. Since the identity operator $i : L^1(\nu) \to L^1(\mu)$ is injective and w-w-continuous, any weakly compact set $K \subseteq L^1(\nu)$ is homeomorphic to the weakly compact set $i(K) \subseteq L^1(\mu)$, and so K is UEC.

(ii) Identify $L^1(\nu)^*$ with $L^1(\nu)^\times$ (the Köthe dual of $L^1(\nu)$ as a Banach function space over μ) in the usual way. Namely,

$$L^1(\nu)^\times = \{h \in L^1(\mu) : fh \in L^1(\mu) \text{ for all } f \in L^1(\nu)\}$$

and $L^1(\nu)^* = \{\varphi_h : h \in L^1(\nu)^\times\}$, where for each $h \in L^1(\nu)^\times$ the functional $\varphi_h \in L^1(\nu)^*$ is given by $\varphi_h(f) := \int_\Omega fh\,d\mu$ for all $f \in L^1(\nu)$.

Since every weakly compact subset of $L^1(\mu)$ is UEC, in order to prove that $(B_{L^1(\nu)^*}, w^*)$ is UEC it suffices to check that the injective map

$$B_{L^1(\nu)^*} \to L^1(\mu), \qquad \varphi_h \mapsto h,$$

is w^*-w-continuous. To this end, let (φ_{h_α}) be a net in $B_{L^1(\nu)^*}$ which is w^*-convergent to $\varphi_h \in B_{L^1(\nu)^*}$. In particular,

$$\varphi_{h_\alpha}(1_A) = \int_A h_\alpha\,d\mu \to \varphi_h(1_A) = \int_A h\,d\mu \quad \text{for all } A \in \Sigma. \tag{2.1}$$

On the other hand, (h_α) is bounded in $L^1(\mu)$, because for every α we have

$$\int_\Omega |h_\alpha|\,d\mu = \int_\Omega \mathrm{sign}(h_\alpha)h_\alpha\,d\mu = \varphi_{h_\alpha}(\mathrm{sign}(h_\alpha)) \le \|\mathrm{sign}(h_\alpha)\|_{L^1(\nu)} \le \|\nu\|(\Omega).$$

The boundedness of (h_α) and (2.1) imply that $h_\alpha \to h$ weakly in $L^1(\mu)$, as required. ∎

COROLLARY 2.3. *Let $\nu \in \mathrm{ca}(\Sigma, X)$. Then $\overline{\mathrm{aco}}(\nu(\Sigma))$ is UEC and $\overline{\mathrm{span}}(\nu(\Sigma))$ is isomorphic to a subspace of a Hilbert generated space.*

Proof. The set $K := \{f \in L^1(\nu) : |f| \le 1\ \|\nu\|\text{-a.e.}\}$ is weakly compact in $L^1(\nu)$ (see e.g. [25, Proposition 2.39]), hence it is UEC (by Theorem 2.2(i)). Since the class of UEC spaces is closed under continuous images (see [3], cf. [18, Corollary 6.34]) and the integration operator $I_\nu : L^1(\nu) \to X$ is w-w-continuous, $I_\nu(K)$ is UEC. Notice that $K \supseteq \{1_A : A \in \Sigma\}$ and so $I_\nu(K) \supseteq \nu(\Sigma)$. Since $I_\nu(K)$ is absolutely convex and closed, we get $I_\nu(K) \supseteq \overline{\mathrm{aco}}(\nu(\Sigma))$. It follows that $\overline{\mathrm{aco}}(\nu(\Sigma))$ is UEC as well.

Finally, note that $\overline{I_\nu(L^1(\nu))} = \overline{\mathrm{span}}(\nu(\Sigma)) =: X_0$. Hence $I_\nu^* : X_0^* \to L^1(\nu)^*$ is injective, and so its restriction to $B_{X_0^*}$ is a w^*-w^*-homeomorphism onto its image, which is UEC by Theorem 2.2(ii). It follows that $(B_{X_0^*}, w^*)$ is UEC. ∎

REMARK 2.4. Let $\nu \in \mathrm{ca}(\Sigma, X)$ and Y be the Banach space obtained in Theorem 2.1. The fact that $\overline{\mathrm{aco}}(\nu(\Sigma))$ is UEC ensures that B_Y is UEC (see [2, Lemma 3.5]).

There are order continuous Banach lattices which are WCG, embed isomorphically into a Hilbert generated space, but fail to be isomorphic to any L^1 space of a vector measure. An example of such an space is $\ell^p(\Gamma)$ where Γ is an uncountable set and $1 < p < \infty$, $p \neq 2$ (see [14] and Theorem 2.6 below). Note that, for Γ uncountable, the space $\ell^p(\Gamma)$ is Hilbert generated if and only if $2 \le p < \infty$ (see [14]).

LEMMA 2.5. *Let $\nu \in \mathrm{ca}(\Sigma, X)$, Y be a Banach space and $S : L^1(\nu) \to Y$ an operator. Define $\nu_S : \Sigma \to Y$ by $\nu_S(A) := S(1_A)$ for all $A \in \Sigma$. Then $\nu_S \in \mathrm{ca}(\Sigma, Y)$, $\mathcal{N}(\nu) \subseteq \mathcal{N}(\nu_S)$ and $I_{\nu_S}(f) = S(f)$ for every simple function f.*

Proof. Straightforward. ∎

Note that if Γ is an uncountable set and $1 \le p < \infty$, then $\ell^p(\Gamma)$ fails to have a weak unit, and so it cannot be Banach lattice isomorphic to the L^1 space of a vector measure. The following result improves this assertion.

THEOREM 2.6. *Let Γ be a non-empty set and $1 \le p < \infty$ with $p \neq 2$. If $\ell^p(\Gamma)$ is isomorphic to $L^1(\nu)$ for some $\nu \in \mathrm{ca}(\Sigma, X)$, then Γ is countable.*

Proof. The case $p = 1$ is clear since $\ell^1(\Gamma)$ is not WCG whenever Γ is uncountable. Assume that $1 < p < \infty$. Let $S : L^1(\nu) \to \ell^p(\Gamma)$ be an isomorphism. We shall check that $L^1(\nu)$ is separable. We divide the proof into two cases.

CASE $1 < p < 2$. Since $\nu_S \in \mathrm{ca}(\Sigma, \ell^p(\Gamma))$ (Lemma 2.5), the set $\nu_S(\Sigma)$ is relatively norm compact in $\ell^p(\Gamma)$ (this follows from [26, p. 211, Remark 2], see e.g. the proof of [25, Lemma 3.53(v)]). In particular, $\nu_S(\Sigma) = \{S(1_A) : A \in \Sigma\}$ is separable, and so $\{1_A : A \in \Sigma\}$ is a separable subset of $L^1(\nu)$. Since simple functions are dense in $L^1(\nu)$, this space is separable.

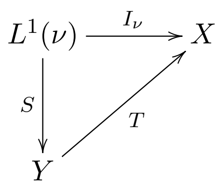
CASE $2 < p < \infty$. Let μ be a Rybakov control measure of ν. Consider the identity operator $i : L^1(\nu) \to L^1(\mu)$ and the composition $T := i \circ S^{-1} : \ell^p(\Gamma) \to L^1(\mu)$. Let $1 < q < 2$ be such that $1/p + 1/q = 1$. The adjoint operator $T^* : L^\infty(\mu) \to \ell^q(\Gamma)$ is compact (see [26, p. 211, Remark 2]) and, by Schauder's theorem (see e.g. [1, Theorem 5.2]), T is compact as well. Therefore, T has separable range and the same holds for $i = T \circ S$. Since $\overline{i(L^1(\nu))} = L^1(\mu)$, it follows that $L^1(\mu)$ is separable, which is equivalent to saying that $L^1(\nu)$ is separable. ∎

QUESTION 2.7. *Is $L^1(\nu)$ Hilbert generated for any $\nu \in \mathrm{ca}(\Sigma, X)$? What about $L^2(\nu)$ when B_X is UEC?*

REMARK 2.8. If $\nu \in \mathrm{ca}(\Sigma, X)$ has finite variation, then $L^1(\nu)$ is Hilbert generated. Indeed, the identity map $L^1(|\nu|) \to L^1(\nu)$ is a well-defined operator with dense range (see e.g. [25, Lemma 3.14]) and $L^1(|\nu|)$ is Hilbert generated.

3. Factorization of integration operators. The following lemma can be found in [21, Lemma 2.2]. We provide another proof which does not rely on [20] and can be more accessible to the reader.

LEMMA 3.1. *Let $\nu \in \mathrm{ca}(\Sigma, X)$. Suppose I_ν factors as*



where Y is a Banach space, S and T are operators. Let $\nu_S \in \mathrm{ca}(\Sigma, Y)$ be as in Lemma 2.5. Then:

(i) *$\nu = T \circ \nu_S$ and $\mathcal{N}(\nu) = \mathcal{N}(\nu_S)$.*
(ii) *Every ν_S-integrable function is ν-integrable.*
(iii) *The identity map $j : L^1(\nu_S) \to L^1(\nu)$ is an operator and $T \circ I_{\nu_S} = I_\nu \circ j$.*

If, in addition, T is injective and $Y \not\supseteq \ell^\infty$, then:

(iv) *Every ν-integrable function is ν_S-integrable.*
(v) *$L^1(\nu_S) = L^1(\nu)$ with equivalent norms.*
(vi) *$S = I_{\nu_S}$.*

Proof. The equality $\nu = T \circ \nu_S$ follows from the very definitions and implies that $\mathcal{N}(\nu) \supseteq \mathcal{N}(\nu_S)$. From Lemma 2.5 we obtain $\mathcal{N}(\nu) = \mathcal{N}(\nu_S)$. Statements (ii) and (iii) also follow from the equality $\nu = T \circ \nu_S$ (see e.g. [25, Lemma 3.27]).

Assume now that T is injective and that $Y \not\supseteq \ell^\infty$. In order to prove (iv), let $f : \Omega \to \mathbb{R}$ be a ν-integrable function. Then we can write $f = g + h$ for some Σ-measurable functions $g, h : \Omega \to \mathbb{R}$ satisfying:

- $|g| \leq 1$ $\|\nu\|$-a.e., so g is both ν-integrable and ν_S-integrable;
- $h = \sum_n \alpha_n 1_{A_n}$, where (α_n) is a sequence of real numbers and (A_n) is a sequence of pairwise disjoint elements of Σ; note that $h = f - g$ is ν-integrable.

It only remains to show that h is ν_S-integrable. To this end, it suffices to check that for every $B_n \subseteq A_n$, $B_n \in \Sigma$, the series $\sum_n \alpha_n \nu_S(B_n)$ is unconditionally convergent in Y (see e.g. [25, Theorem 3.5]). Since $\{x^* \circ T : x^* \in X^*\} \subseteq Y^*$ separates the points of Y (because T is injective) and $Y \not\supseteq \ell^\infty$, in order to prove that $\sum_n \alpha_n \nu_S(B_n)$ is unconditionally convergent it is enough to check (see [10], cf. [12, p. 23, Corollary 7]) that for every $P \subseteq \mathbb{N}$ there is $y_P \in Y$ such that

$$(x^* \circ T)(y_P) = \sum_{n \in P} (x^* \circ T)(\alpha_n \nu_S(B_n)) = \sum_{n \in P} \alpha_n x^*(\nu(B_n)) \tag{3.1}$$

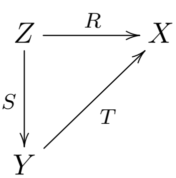
for all $x^* \in X^*$ (the series being absolutely convergent). Equality (3.1) holds by taking $B := \bigcup_{n \in P} B_n \in \Sigma$ and $y_P := S(h 1_B)$, because h is ν-integrable and so

$$T(y_P) = I_\nu(h 1_B) = \sum_{n \in P} \alpha_n \nu(B_n),$$

the series being unconditionally convergent in X. This shows that h is ν_S-integrable and the proof of (iv) is complete.

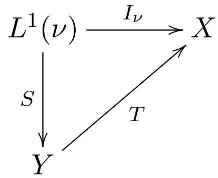
The equality $L^1(\nu_S) = L^1(\nu)$ is now clear. The equivalence of the norms $\|\cdot\|_{L^1(\nu_S)}$ and $\|\cdot\|_{L^1(\nu)}$ follows from the Open Mapping Theorem and the fact that the identity $j : L^1(\nu_S) \to L^1(\nu)$ is a bijective operator. Finally, (vi) is a consequence of the density of simple functions in $L^1(\nu) = L^1(\nu_S)$ and Lemma 2.5. ∎

Let $C \subseteq X$ be an absolutely convex bounded set. The DFJP method applied to C (see e.g. [1, Theorem 5.37]) generates a Banach space Y and an injective operator $T : Y \to X$ with $T(B_Y) \supseteq C$ satisfying some relevant properties, e.g. Y is reflexive if (and only if) C is relatively weakly compact. When the DFJP method is applied to a set of the form $C = R(B_Z)$, where Z is a Banach space and $R : Z \to X$ is an operator, we get the *DFJP factorization* of R as



where $S : Z \to Y$ is an operator. Recall that an operator between Banach spaces is called *completely continuous* (or *Dunford–Pettis*) if it maps weakly convergent sequences to norm convergent ones.

LEMMA 3.2. *Let $\nu \in \mathrm{ca}(\Sigma, X)$ and let*



be the DFJP factorization of I_ν. Let $\nu_S \in \mathrm{ca}(\Sigma, Y)$ be as in Lemma 2.5. Then:

(i) *$\nu_S(\Sigma)$ is relatively norm compact if and only if $\nu(\Sigma)$ is relatively norm compact.*
(ii) *If, in addition, $Y \not\supseteq \ell^\infty$, then I_{ν_S} is completely continuous if and only if I_ν is completely continuous.*

Proof. (i) If $\nu_S(\Sigma)$ is relatively norm compact, then so is $\nu(\Sigma) = T(\nu_S(\Sigma))$. On the other hand, note that $\nu(\Sigma)$ is contained in a multiple of the set $I_\nu(B_{L^1(\nu)})$ inducing the DFJP factorization of I_ν. Hence if $\nu(\Sigma)$ is relatively norm compact in X, then $T^{-1}(\nu(\Sigma)) = \nu_S(\Sigma)$ is relatively norm compact in Y (see e.g. [1, Theorem 5.40]).

(ii) We shall use the following fact (see [5, Theorem 5.8]):

FACT. *Let Z be a Banach space and $\xi \in \mathrm{ca}(\Sigma, Z)$. Then I_ξ is completely continuous if and only if $L^1(\xi)$ has the* positive Schur property (*i.e. weakly null positive sequences in $L^1(\xi)$ are norm null*) *and $\xi(\Sigma)$ is relatively norm compact.*

Suppose now that $Y \not\supseteq \ell^\infty$. By Lemma 3.1, we have $L^1(\nu_S) = L^1(\nu)$ with equivalent norms and $S = I_{\nu_S}$. Hence $I_\nu = T \circ I_{\nu_S}$ is completely continuous whenever I_{ν_S} is. Conversely, assume that I_ν is completely continuous. According to the Fact, this is equivalent to saying that $L^1(\nu)$ has the positive Schur property and $\nu(\Sigma)$ is relatively norm compact. Since $L^1(\nu_S) = L^1(\nu)$ with equivalent norms, $L^1(\nu_S)$ has the positive Schur property as well. Bearing in mind that $\nu_S(\Sigma)$ is also relatively norm compact (by (i)), another appeal to the Fact ensures that I_{ν_S} is completely continuous. ∎

We arrive at the main result of this section. An operator between Banach spaces is said to be an *Asplund operator* if it factors through an Asplund space. This concept has its origin in [27]. Recall that a Banach space Z is called *Asplund* if every separable subspace of Z has separable dual, or equivalently, Z^* has the Radon–Nikodým property [12, p. 198].

THEOREM 3.3. *Let $\nu \in \mathrm{ca}(\Sigma, X)$ be such that I_ν is completely continuous and Asplund. Then ν has finite variation and $L^1(\nu) = L^1(|\nu|)$ with equivalent norms.*

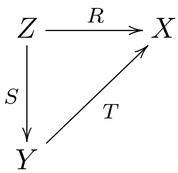
Proof. Let us consider the DFJP factorization of I_ν as in Lemma 3.2. Since I_ν is an Asplund operator, Y is an Asplund space (see e.g. [13, Theorem 1.4.4]). In particular, $Y \not\supseteq \ell^\infty$. By Lemma 3.2, the integration operator I_{ν_S} is completely continuous. An appeal to [7, Theorem 1.3] ensures that ν_S has finite variation, and so does $\nu = T \circ \nu_S$. The last statement follows from [23, Proposition 1.1] applied to the operator ideal of all completely continuous Asplund operators. ∎

All compact operators and all p-summing operators $(1 \le p < \infty)$ are completely continuous and weakly compact (hence Asplund); see e.g. [11, Theorem 2.17]. Thus, Theorem 3.3 gives a unified approach to the following known results.

COROLLARY 3.4 ([22]). *Let $\nu \in \mathrm{ca}(\Sigma, X)$ be such that I_ν is compact. Then ν has finite variation and $L^1(\nu) = L^1(|\nu|)$ with equivalent norms.*

COROLLARY 3.5 ([6, 23]). *Let $\nu \in \mathrm{ca}(\Sigma, X)$ be such that I_ν is p-summing, $1 \le p < \infty$. Then ν has finite variation and $L^1(\nu) = L^1(|\nu|)$ with equivalent norms.*

There remains an open question, raised in [23]: whether $\nu \in \mathrm{ca}(\Sigma, X)$ has finite variation whenever I_ν is completely continuous and $X \not\supseteq \ell^1$. In order to reformulate this question we need some terminology. Let $R : Z \to X$ be an operator, where Z is a Banach space. Recall that R is said to be *weakly precompact* if every sequence in $R(B_Z)$ admits a weakly Cauchy subsequence. By Rosenthal's ℓ^1-theorem (see e.g. [1, Theorem 4.72]), R is weakly precompact whenever $X \not\supseteq \ell^1$. On the other hand, it is known that if R is weakly precompact and



is the DFJP factorization of R, then $Y \not\supseteq \ell^1$ (see e.g. [17, Theorem 5.3.6]). Summing up, it follows that an operator is weakly precompact if and only if it factors through a Banach space not containing subspaces isomorphic to ℓ^1. In particular, Asplund operators are weakly precompact. The proof of Theorem 3.3 can be adapted to show that the aforementioned question in [23] is equivalent to the following:

QUESTION 3.6. *Let Y be a Banach space and $\nu \in \mathrm{ca}(\Sigma, Y)$ be such that I_ν is completely continuous and weakly precompact. Does ν have finite variation?*

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José Rodríguez
Departamento de Matemática Aplicada
Facultad de Informática
Universidad de Murcia
30100 Espinardo (Murcia), Spain
E-mail: joserr@um.es