COLLOQUIUM MATHEMATICUM

VOL. 145 2016 NO. 1

THE STRUCTURE OF SPLIT REGULAR HOM-POISSON ALGEBRAS

BY

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Abstract. We introduce the class of split regular Hom-Poisson algebras formed by those Hom-Poisson algebras whose underlying Hom-Lie algebras are split and regular. This class is the natural extension of the ones of split Hom-Lie algebras and of split Poisson algebras. We show that the structure theorems for split Poisson algebras can be extended to the more general setting of split regular Hom-Poisson algebras. That is, we prove that an arbitrary split regular Hom-Poisson algebra $\mathfrak{P}$ is of the form $\mathfrak{P} = U + \sum_j I_j$ with U a linear subspace of a maximal abelian subalgebra H and any I_j a well described (split) ideal of $\mathfrak{P}$, satisfying $\{I_j, I_k\} + I_j I_k = 0$ if $j \neq k$. Under certain conditions, the simplicity of $\mathfrak{P}$ is characterized, and it is shown that $\mathfrak{P}$ is the direct sum of the family of its simple ideals.

1. Introduction and first definitions. We recall that a (not necessarily commutative) *Poisson algebra* is a Lie algebra $(\mathcal{P}, \{\cdot,\cdot\})$ over a base field $\mathbb{K}$, endowed with an associative product, denoted by juxtaposition, such that the Leibniz identity

$$\{xy, z\} = \{x, z\}y + x\{y, z\}$$

holds for any $x, y, z \in \mathcal{P}$.

The interest in Poisson algebras has grown in the last years, motivated especially by their applications in geometry and mathematical physics. For instance, we can find them in gauge theories, especially in the study of path integrals in quantum field theory. They can also be seen as a procedure for the quantization of physical systems with symmetries in the Lagrangian formalism (see [5, 14, 17]). As another example, we note that Poisson algebras are the key to recover Hamiltonian mechanics from the coordinate space of the theory [21]. We can list many more applications (see [4, 15, 16, 17, 22]). A *split* Poisson algebra is a Poisson algebra $\mathcal{P}$ whose underlying Lie algebra

2010 *Mathematics Subject Classification*: Primary 17A30, 17B63; Secondary 17A60, 17B22.

Key words and phrases: Hom-algebra, Poisson algebra, root, root space, structure theory.
Received 5 March 2015; revised 9 September 2015.
Published online 10 March 2016.

DOI: 10.4064/cm6568-9-2015

structure is split, that is,

$$\mathcal{P} = H \oplus \bigoplus_{\alpha \in \Lambda} \mathcal{P}_\alpha$$

where H is a maximal abelian subalgebra of the Lie algebra $(\mathcal{P}, \{\cdot, \cdot\})$ and where $\mathcal{P}_\alpha = \{v_\alpha \in \mathcal{P} : \{h, v_\alpha\} = \alpha(h)v_\alpha$ for any $h \in H\}$ for $\alpha \in H^*$ and $\Lambda := \{\alpha \in H^* \backslash \{0\} : \mathcal{P}_\alpha \neq 0\}$. See [6, 24].

On the other hand, a Hom-algebra is an algebra such that a linear homomorphism appears in the identities satisfied by its multiplication. In the case of a Lie algebra, we are dealing with Hom-Lie algebras. This class of algebras appeared in the study of quasi-deformations of Lie algebras of vector fields, in particular quasi-deformations of Witt and Virasoro algebras, [13]. Many authors have been interested in the study of Hom-Lie algebras, motivated in part by their applications in physics; see for instance [2, 3, 10, 11, 12, 18, 19, 20, 23, 25, 26, 28]. A Hom-Poisson algebra is defined as a Hom-Lie algebra $(\mathfrak{P}, \{\cdot, \cdot\})$ over an arbitrary base field $\mathbb{K}$ endowed with a Hom-associative product and with both products compatible via a Hom-Leibniz identity.

Definition 1.1. A *Hom-Lie algebra* $\mathfrak{P}$ is a vector space over a base field $\mathbb{K}$ endowed with a bilinear product

$$\{\cdot, \cdot\} : \mathfrak{P} \times \mathfrak{P} \to \mathfrak{P}$$

and with a linear map $\phi : \mathfrak{P} \to \mathfrak{P}$ such that

(1) $\{x, y\} = -\{y, x\}$,
(2) $\{\{x, y\}, \phi(z)\} + \{\{y, z\}, \phi(x)\} + \{\{z, x\}, \phi(y)\} = 0$ (*Hom-Jacobi identity*),

for any $x, y, z \in \mathfrak{P}$.

Definition 1.2. A *Hom-Poisson algebra* is a Hom-Lie algebra $(\mathfrak{P}, \{\cdot, \cdot\}, \phi)$ endowed with a Hom-associative product, that is, a bilinear product denoted by juxtaposition such that

$$(xy)\phi(z) = \phi(x)(yz)$$

for any $x, y, z \in \mathfrak{P}$, and such that the *Hom-Leibniz identity*

$$\{xy, \phi(z)\} = \{x, z\}\phi(y) + \phi(x)\{y, z\}$$

holds for any $x, y, z \in \mathfrak{P}$.

If ϕ is furthermore a *Poisson automorphism*, that is, a linear bijection such that $\phi(\{x, y\}) = \{\phi(x), \phi(y)\}$ and $\phi(xy) = \phi(x)\phi(y)$ for any $x, y \in \mathfrak{P}$, then $\mathfrak{P}$ is called a *regular* Hom-Poisson algebra.

EXAMPLE 1.3. Consider a 3-dimensional linear space $\mathfrak{P}$ over a base field $\mathbb{K}$ with basis $\{e_1, e_2, e_3\}$. Define the products

$$e_1e_1 = e_1, \quad e_1e_2 = e_2e_1 = e_3,$$
$$\{e_1, e_1\} = ae_2 + be_3, \quad \{e_1, e_3\} = ce_2 + de_3,$$

set the remaining products equal to zero, and define the linear map ϕ as

$$\phi(e_1) = \lambda_1e_2 + \lambda_2e_3, \quad \phi(e_2) = \lambda_3e_2 + \lambda_4e_3, \quad \phi(e_3) = \lambda_5e_2 + \lambda_6e_3,$$

where $a, b, c, d, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in \mathbb{K}$. Then $\mathfrak{P}$ becomes a Hom-Poisson algebra.

EXAMPLE 1.4. Consider a Poisson algebra $(\mathfrak{P}, \cdot, [\cdot, \cdot])$ and a linear bijection $\phi : \mathfrak{P} \to \mathfrak{P}$ multiplicative for $\cdot$ and $[\cdot, \cdot]$. Then $\mathfrak{P}$ with the Hom-associative product (with respect to ϕ) defined by the composition $\phi \circ \cdot$ and the Hom-Lie product (with respect to ϕ) given by $\phi \circ [\cdot, \cdot]$ becomes a regular Hom-Poisson algebra (with respect to ϕ).

A *subalgebra* A of $\mathfrak{P}$ is a linear subspace such that $\{A, A\} + AA \subset A$ and $\phi(A) = A$. A linear subspace I of $\mathfrak{P}$ is called an *ideal* if $\{I, \mathfrak{P}\} + I\mathfrak{P} + \mathfrak{P}I \subset I$ and $\phi(I) = I$. A Hom-Poisson algebra $\mathfrak{P}$ will be called *simple* if $\{\mathfrak{P}, \mathfrak{P}\} + \mathfrak{P}\mathfrak{P} \neq 0$ and its only ideals are $\{0\}$ and $\mathfrak{P}$. We refer to [20, 27] for a first approach to Hom-Poisson algebras.

In the present paper we introduce the class of split Hom-Poisson algebras $\mathfrak{P}$ formed by those Hom-Poisson algebras whose underlying Hom-Lie algebras are split. We recall that given a Hom-Lie algebra $(\mathfrak{P}, \{\cdot, \cdot\}, \phi)$ and a maximal abelian subalgebra H of $\mathfrak{P}$, for a linear functional

$$\alpha : H \to \mathbb{K},$$

we define the *root space* of $\mathfrak{P}$ (with respect to H) associated to α to be the subspace

$$\mathfrak{P}_\alpha = \{v_\alpha \in \mathfrak{P} : \{h, v_\alpha\} = \alpha(h)\phi(v_\alpha) \text{ for any } h \in H\}.$$

The functionals $\alpha : H \to \mathbb{K}$ satisfying $\mathfrak{P}_\alpha \neq 0$ are called the *roots* of $\mathfrak{P}$ with respect to H and we denote $\Lambda := \{\alpha \in H^* \setminus \{0\} : \mathfrak{P}_\alpha \neq 0\}$. We say that $\mathfrak{P}$ is a *split Hom-Lie algebra* with respect to H if

$$\mathfrak{P} = H \oplus \bigoplus_{\alpha \in \Lambda} \mathfrak{P}_\alpha.$$

We also say that Λ is the *root system* of $\mathfrak{P}$.

To ease notation, the mappings $\phi|_H, \phi|_H^{-1} : H \to H$ will be denoted by ϕ and ϕ^{-1} respectively.

We recall some properties of split regular Hom-Lie algebras that can be found in [1, Lemmas 1.3 and 1.4].

LEMMA 1.5. *Let $(\mathfrak{P}, \{\cdot,\cdot\}, \phi)$ be a split regular Hom-Lie algebra. Then for any $\alpha, \beta \in \Lambda \cup \{0\}$:*

(1) $\phi(\mathfrak{P}_\alpha) \subset \mathfrak{P}_{\alpha\phi^{-1}}$ *and* $\phi^{-1}(\mathfrak{P}_\alpha) \subset \mathfrak{P}_{\alpha\phi}$.
(2) $\{\mathfrak{P}_\alpha, \mathfrak{P}_\beta\} \subset \mathfrak{P}_{\alpha\phi^{-1}+\beta\phi^{-1}}$.
(3) *If* $\alpha \in \Lambda$ *then* $\alpha\phi^{-z} \in \Lambda$ *for any* $z \in \mathbb{Z}$.
(4) $\mathfrak{P}_0 = H$.

DEFINITION 1.6. A *split Hom-Poisson algebra* is a Hom-Poisson algebra in which the Hom-Lie algebra $(\mathcal{P}, \{\cdot,\cdot\})$ is split with respect to a maximal abelian subalgebra H of $(\mathcal{P}, \{\cdot,\cdot\})$.

Note that by taking $\phi = \mathrm{Id}$, split Poisson algebras become examples of split regular Hom-Poisson algebras. Maybe the main topic in the theory of Hom-algebras consists in studying whether a known result for a class of nondeformed algebras still holds true for the corresponding class of Hom-algebras. Following this line, the present paper shows to what extent the structure theorems obtained in [6] for split Poisson algebras also hold for the class of split regular Hom-Poisson algebras. All of the constructions carried out along this paper strongly involve the structure map ϕ which makes the proofs different from the nondeformed Poisson case.

LEMMA 1.7. *Let $\mathfrak{P}$ be a split regular Hom-Poisson algebra. Then for any $\alpha, \beta \in \Lambda \cup \{0\}$ we have $\mathfrak{P}_\alpha\mathfrak{P}_\beta \subset \mathfrak{P}_{\alpha\phi^{-1}+\beta\phi^{-1}}$.*

Proof. Let $h \in H$, $v_\alpha \in \mathfrak{P}_\alpha$ and $v_\beta \in \mathfrak{P}_\beta$, and denote $h' = \phi(h)$. By applying the Hom-Leibniz identity we get

$$\begin{aligned}\{h', v_\alpha v_\beta\} &= \{\phi(h), v_\alpha v_\beta\} \\ &= -\{v_\alpha h\}\phi(v_\beta) - \phi(v_\alpha)\{v_\beta, h\} = \alpha(h)\phi(v_\alpha)\phi(v_\beta) + \beta(h)\phi(v_\alpha)\phi(v_\beta) \\ &= (\alpha+\beta)(h)\phi(v_\alpha)\phi(v_\beta) = (\alpha+\beta)\phi^{-1}(h')\phi(v_\alpha v_\beta).\end{aligned}$$

That is, $v_\alpha v_\beta \in \mathfrak{P}_{\alpha\phi^{-1}+\beta\phi^{-1}}$. ∎

The paper is organized as follows. In §2 we develop connections of roots techniques in the framework of split regular Hom-Poisson algebras $\mathfrak{P}$, and show that any of these algebras is of the form $\mathfrak{P} = U + \sum_j I_j$ with U a linear subspace of H and any I_j a well defined ideal of $\mathfrak{P}$, satisfying $\{I_j, I_k\} + I_jI_k = 0$ if $j \neq k$. Finally, in §3, and under mild conditions, the simplicity of $\mathfrak{P}$ is characterized, and it is shown that $\mathfrak{P}$ is the direct sum of the family of its simple ideals.

Throughout this paper we will denote by $\mathbb{N}$ the set of all nonnegative integers and by $\mathbb{Z}$ the set of all integers. Finally, note that our split regular Hom-Poisson algebras are of arbitrary dimension and over an arbitrary base field $\mathbb{K}$.

2. Decomposition as direct sum of ideals. In the following, $\mathfrak{P}$ denotes a split regular Hom-Poisson algebra and

$$\mathfrak{P} = H \oplus \bigoplus_{\alpha \in \Lambda} \mathfrak{P}_\alpha$$

the corresponding root space decomposition. Given a linear functional $\alpha : H \to \mathbb{K}$, we denote by $-\alpha : H \to \mathbb{K}$ the element in H^* defined by $(-\alpha)(h) := -\alpha(h)$ for all $h \in H$. We also write

$$-\Lambda = \{-\alpha : \alpha \in \Lambda\}.$$

Definition 2.1. Let $\alpha, \beta \in \Lambda$. We will say that α is *connected to* β if either

$$\beta = \epsilon\alpha\phi^z \quad \text{for some } z \in \mathbb{Z} \text{ and } \epsilon \in \{\pm 1\},$$

or there exists $\{\alpha_1, \dots, \alpha_k\} \subset \pm\Lambda$, $k \geq 2$, such that:

(1) $\alpha_1 \in \{\alpha\phi^{-n} : n \in \mathbb{N}\}$.
(2) $\alpha_1\phi^{-1} + \alpha_2\phi^{-1} \in \pm\Lambda$,
$\alpha_1\phi^{-2} + \alpha_2\phi^{-2} + \alpha_3\phi^{-1} \in \pm\Lambda$,
$\alpha_1\phi^{-3} + \alpha_2\phi^{-3} + \alpha_3\phi^{-2} + \alpha_4\phi^{-1} \in \pm\Lambda$,
...
$\alpha_1\phi^{-i} + \alpha_2\phi^{-i} + \alpha_3\phi^{-i+1} + \cdots + \alpha_{i+1}\phi^{-1} \in \pm\Lambda$,
...
$\alpha_1\phi^{-k+2} + \alpha_2\phi^{-k+2} + \alpha_3\phi^{-k+3} + \cdots + \alpha_i\phi^{-k+i} + \cdots + \alpha_{k-1}\phi^{-1} \in \pm\Lambda$.
(3) $\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+1} + \alpha_3\phi^{-k+2} + \cdots + \alpha_i\phi^{-k+i-1} + \cdots + \alpha_k\phi^{-1} \in \{\pm\beta\phi^{-m} : m \in \mathbb{N}\}$.

In this case, we will also say that $\{\alpha_1, \dots, \alpha_k\}$ is a *connection* from α to β.

The proof of the next result is analogous to the one of [1, Proposition 2.4]. For the sake of completeness we give a sketch of the proof.

Proposition 2.2. *The relation $\sim$ in Λ, defined by $\alpha \sim \beta$ if and only if α is connected to β, is an equivalence relation.*

Proof. Clearly $\alpha \sim \alpha$. If $\alpha \sim \beta$, then either $\beta = \epsilon\alpha\phi^z$ for some $z \in \mathbb{Z}$ and $\epsilon \in \{\pm 1\}$, and so β is connected to α; or there exists a connection $\{\alpha_1, \dots, \alpha_k\} \subset \pm\Lambda$, $k \geq 2$, from α to β with

$$\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+1} + \alpha_3\phi^{-k+2} + \cdots + \alpha_k\phi^{-1} = \epsilon\beta\phi^{-m}$$

for some $\epsilon \in \{\pm 1\}$ and some $m \in \mathbb{N}$. Then we can verify that

$$\{\beta\phi^{-m}, -\epsilon\alpha_k\phi^{-1}, -\epsilon\alpha_{k-1}\phi^{-3}, -\epsilon\alpha_{k-2}\phi^{-5}, \dots, -\epsilon\alpha_2\phi^{-2k+3}\}$$

is a connection from β to α and so the relation $\sim$ is symmetric.

Finally, suppose $\alpha \sim \beta$ and $\beta \sim \gamma$. In case $\beta \in \epsilon\alpha\phi^z$ or $\gamma \in \epsilon\beta\phi^z$ for some $z \in \mathbb{Z}$ and $\epsilon \in \{\pm 1\}$, we easily find that α is connected to γ. Hence, suppose $\{\alpha_1,\dots,\alpha_k\}$, $k\geq 2$, is a connection from α to β which satisfies

$$\alpha_1\phi^{-k+1}+\alpha_2\phi^{-k+1}+\alpha_3\phi^{-k+2}+\cdots+\alpha_k\phi^{-1}=\epsilon\beta\phi^{-m}$$

for some $m\in\mathbb{N}$, $\epsilon\in\{\pm1\}$; and $\{h_1,\dots,h_p\}$ is a connection from β to γ. Then $\{\alpha_1,\dots,\alpha_k,\epsilon h_2,\dots,\epsilon h_p\}$ is a connection from α to γ, so the connection relation is also transitive. ∎

From Proposition 2.2 we can consider the quotient set

$$\Lambda/\!\sim\,=\{[\alpha]:\alpha\in\Lambda\},$$

with $[\alpha]$ being the set of nonzero roots connected to α.

Our next goal is to associate an ideal $I_{[\alpha]}$ to any $[\alpha]$. Fix $[\alpha]\in\Lambda/\!\sim$. We start by defining

$$I_{H,[\alpha]}:=\operatorname{span}_{\mathbb{K}}\{\{\mathfrak{P}_\beta,\mathfrak{P}_{-\beta}\}+\mathfrak{P}_\beta\mathfrak{P}_{-\beta}:\beta\in[\alpha]\}\subset H.$$

Next, we define

$$V_{[\alpha]}:=\bigoplus_{\beta\in[\alpha]}\mathfrak{P}_\beta.$$

Finally, we denote by $I_{[\alpha]}$ the direct sum of the two subspaces above:

$$I_{[\alpha]}:=I_{H,[\alpha]}\oplus V_{[\alpha]}.$$

Proposition 2.3. *Let $[\alpha]\in\Lambda/\!\sim$. Then:*

(1) $\{I_{[\alpha]},I_{[\alpha]}\}+I_{[\alpha]}I_{[\alpha]}\subset I_{[\alpha]}$.
(2) $\phi(I_{[\alpha]})=I_{[\alpha]}$.
(3) *For any $[\beta]\neq[\alpha]$ we have $\{I_{[\alpha]},I_{[\beta]}\}+I_{[\alpha]}I_{[\beta]}=0$.*

Proof. (1) Let us begin by showing that $\{I_{[\alpha]},I_{[\alpha]}\}\subset I_{[\alpha]}$. We can write

$$\begin{aligned}\{I_{[\alpha]},I_{[\alpha]}\}&=\{I_{H,[\alpha]}+V_{[\alpha]},I_{H,[\alpha]}+V_{[\alpha]}\}\\&\subset\{I_{H,[\alpha]},V_{[\alpha]}\}+\{V_{[\alpha]},V_{[\alpha]}\}.\end{aligned}\tag{2.1}$$

Since $I_{H,[\alpha]}\subset H$, we have $\{I_{H,[\alpha]},V_{[\alpha]}\}\subset V_{[\alpha]}$, and so consider the product $\{V_{[\alpha]},V_{[\alpha]}\}$ in (2.1). If we take $\beta,\gamma\in[\alpha]$ such that $\{\mathfrak{P}_\beta,\mathfrak{P}_\gamma\}\neq0$, in case $\gamma=-\beta$ clearly $\{\mathfrak{P}_\beta,\mathfrak{P}_\gamma\}\subset I_{H,[\alpha]}$. So suppose $\gamma\neq-\beta$. By Lemma 1.5, $\{\mathfrak{P}_\beta,\mathfrak{P}_\gamma\}\subset\mathfrak{P}_{\beta\phi^{-1}+\gamma\phi^{-1}}$, and since $\{\beta,\gamma\}$ is a connection from β to $\beta\phi^{-1}+\gamma\phi^{-1}$, we get $\{\mathfrak{P}_\beta,\mathfrak{P}_\gamma\}\subset V_{[\alpha]}$. Consequently, $\{I_{[\alpha]},I_{[\alpha]}\}\subset I_{[\alpha]}$.

Second, let us verify that $I_{[\alpha]}I_{[\alpha]}\subset I_{[\alpha]}$. We have

$$\begin{aligned}I_{[\alpha]}I_{[\alpha]}&=(I_{H,[\alpha]}+V_{[\alpha]})(I_{H,[\alpha]}+V_{[\alpha]})\\&\subset I_{H,[\alpha]}I_{H,[\alpha]}+I_{H,[\alpha]}V_{[\alpha]}+V_{[\alpha]}I_{H,[\alpha]}+V_{[\alpha]}V_{[\alpha]}.\end{aligned}\tag{2.2}$$

By arguing as above, but now taking into account Lemma 1.7, we have

$$I_{H,[\alpha]}V_{[\alpha]}+V_{[\alpha]}I_{H,[\alpha]}+V_{[\alpha]}V_{[\alpha]}\subset I_{H,[\alpha]}.$$

Hence it just remains to study the product $I_{H,[\alpha]}I_{H,[\alpha]}$ in (2.2). To do so, observe that

$$I_{H,[\alpha]}I_{H,[\alpha]} \subset \Big(\sum_{\beta\in[\alpha]} (\{\mathfrak{P}_\beta, \mathfrak{P}_{-\beta}\} + \mathfrak{P}_\beta\mathfrak{P}_{-\beta})\Big)H \subset \Big(\sum_{\beta\in[\alpha]} \{\mathfrak{P}_\beta, \mathfrak{P}_{-\beta}\}\Big)H + \Big(\sum_{\beta\in[\alpha]} \mathfrak{P}_\beta\mathfrak{P}_{-\beta}\Big)H. \tag{2.3}$$

Consider the first summand on the right hand side of (2.3). By the Hom-Leibniz identity we have

$$\{\mathfrak{P}_\beta, \mathfrak{P}_{-\beta}\}\phi\phi^{-1}(H) \subset \{\mathfrak{P}_\beta\phi^{-1}(H), \phi(\mathfrak{P}_{-\beta})\} + \phi(\mathfrak{P}_\beta)\{\phi^{-1}(H), \mathfrak{P}_{-\beta}\}$$
$$\subset \{\mathfrak{P}_{\beta\phi^{-1}}, \mathfrak{P}_{-\beta\phi^{-1}}\} + \mathfrak{P}_{\beta\phi^{-1}}\mathfrak{P}_{-\beta\phi^{-1}} \subset I_{H,[\alpha]}.$$

Finally, consider the last summand in (2.3). By Hom-associativity,

$$(\mathfrak{P}_\beta\mathfrak{P}_{-\beta})\phi(\phi^{-1}(H)) = \phi(\mathfrak{P}_\beta)(\mathfrak{P}_{-\beta}\phi^{-1}(H)) \subset \mathfrak{P}_{\beta\phi^{-1}}\mathfrak{P}_{-\beta\phi^{-1}} \subset I_{H,[\alpha]}.$$

(2) is a consequence of Lemma 1.5(1).

(3) We have to study the expression $\{I_{[\alpha]}, I_{[\beta]}\} + I_{[\alpha]}I_{[\beta]}$. Observe that

$$\{I_{[\alpha]}, I_{[\beta]}\} = \{I_{H,[\alpha]} \oplus V_{[\alpha]}, I_{H,[\beta]} \oplus V_{[\beta]}\} \subset \{I_{H,[\alpha]}, V_{[\beta]}\} + \{I_{H,[\beta]}, V_{[\alpha]}\} + \{V_{[\alpha]}, V_{[\beta]}\} \tag{2.4}$$

and

$$I_{[\alpha]}I_{[\beta]} = (I_{H,[\alpha]} \oplus V_{[\alpha]})(I_{H,[\beta]} \oplus V_{[\beta])}) \subset I_{H,[\alpha]}I_{H,[\beta]} + I_{H,[\alpha]}V_{[\beta]} + V_{[\alpha]}I_{H,[\beta]} + V_{[\alpha]}V_{[\beta]}. \tag{2.5}$$

We begin by showing that

$$\{V_{[\alpha]}, V_{[\beta]}\} + V_{[\alpha]}V_{[\beta]} = 0. \tag{2.6}$$

Indeed, if there exist $\gamma \in [\alpha]$ and $\rho \in [\beta]$ such that $0 \neq \{\mathfrak{P}_\gamma, \mathfrak{P}_\rho\} + \mathfrak{P}_\gamma\mathfrak{P}_\rho \subset \mathfrak{P}_{\gamma\phi^{-1}+\rho\phi^{-1}}$ then $\{\gamma, \rho, -\gamma\phi^{-1}\}$ would be a connection from γ to ρ, a contradiction.

Consider now the first summand $\{I_{H,[\alpha]}, V_{[\beta]}\}$ on the right hand side of (2.4) and the second one, $I_{H,[\alpha]}V_{[\beta]}$, of (2.5); and suppose there exist $\gamma \in [\alpha]$ and $\rho \in [\delta]$ such that

$$\{\{\mathfrak{P}_\gamma, \mathfrak{P}_{-\gamma}\}, \mathfrak{P}_\rho\} + \{\mathfrak{P}_\gamma\mathfrak{P}_{-\gamma}, \mathfrak{P}_\rho\} + \{\mathfrak{P}_\gamma, \mathfrak{P}_{-\gamma}\}\mathfrak{P}_\rho + (\mathfrak{P}_\gamma\mathfrak{P}_{-\gamma})\mathfrak{P}_\rho \neq 0.$$

Then some of the four summands are different from zero.

If

$$\{\{\mathfrak{P}_\gamma, \mathfrak{P}_{-\gamma}\}, \mathfrak{P}_\rho\} \neq 0,$$

then the Hom-Jacobi identity gives

$$0 \neq \{\{\mathfrak{P}_\gamma, \mathfrak{P}_{-\gamma}\}, \phi(\phi^{-1}(\mathfrak{P}_\rho)\}$$
$$\subset \{\{\mathfrak{P}_{-\gamma}, \phi^{-1}(\mathfrak{P}_\rho)\}, \phi(\mathfrak{P}_\gamma)\} + \{\{\phi^{-1}(\mathfrak{P}_\rho), \mathfrak{P}_\gamma\}, \phi(\mathfrak{P}_{-\gamma})\}.$$

Hence

$$\{\mathfrak{P}_{-\gamma}, \phi^{-1}(\mathfrak{P}_\rho)\} + \{\mathfrak{P}_\gamma, \phi^{-1}(\mathfrak{P}_\rho)\} \neq 0,$$

which contradicts (2.6). Hence, $\{\{\mathfrak{P}_\gamma, \mathfrak{P}_{-\gamma}\}, \mathfrak{P}_\rho\} = 0$.

If the second, third or fourth summand were nonzero, we can argue as above but using the Hom-Leibniz or Hom-associativity identities to show that these products are zero. Consequently,

$$\{I_{H,[\alpha]}, V_{[\beta]}\} + I_{H,[\alpha]}V_{[\beta]} = 0.$$

In a similar way we prove that the remaining summands in (2.4) and (2.5) are zero, and the proof is complete. ■

Lemma 2.4. *For any* $[\alpha] \in \Lambda/\!\sim$ *we have* $I_{H,[\alpha]}H + HI_{H,[\alpha]} \subset I_{H,[\alpha]}$.

Proof. Fix any $\beta \in [\alpha]$. On the one hand, by the Hom-Leibniz identity we get

$$\begin{aligned}
&\{\mathfrak{P}_\beta, \mathfrak{P}_{-\beta}\}H + H\{\mathfrak{P}_\beta, \mathfrak{P}_{-\beta}\} = \{\mathfrak{P}_\beta, \mathfrak{P}_{-\beta}\}\phi(H) + \phi(H)\{\mathfrak{P}_\beta, \mathfrak{P}_{-\beta}\}\\
&\subset \{\mathfrak{P}_\beta H, \phi(\mathfrak{P}_{-\beta})\} + \phi(\mathfrak{P}_\beta)\{H, \mathfrak{P}_{-\beta}\} + \{H\mathfrak{P}_\beta, \phi(\mathfrak{P}_{-\beta})\} + \{H, \mathfrak{P}_{-\beta}\}\phi(\mathfrak{P}_\beta)\\
&\subset \{\mathfrak{P}_{\beta\phi^{-1}}, \mathfrak{P}_{-\beta\phi^{-1}}\} + \mathfrak{P}_{\beta\phi^{-1}}\mathfrak{P}_{-\beta\phi^{-1}} + \mathfrak{P}_{-\beta\phi^{-1}}\mathfrak{P}_{\beta\phi^{-1}} \subset I_{H,[\alpha]}.
\end{aligned}$$

On the other hand, by Hom-associativity,

$$\begin{aligned}
(\mathfrak{P}_\beta\mathfrak{P}_{-\beta})H + H(\mathfrak{P}_\beta\mathfrak{P}_{-\beta}) &= (\mathfrak{P}_\beta\mathfrak{P}_{-\beta})\phi(H) + \phi(H)(\mathfrak{P}_\beta\mathfrak{P}_{-\beta})\\
&\subset \phi(\mathfrak{P}_\beta)(\mathfrak{P}_{-\beta}H) + (H\mathfrak{P}_\beta)\phi(\mathfrak{P}_{-\beta}) \subset \mathfrak{P}_{\beta\phi^{-1}}\mathfrak{P}_{-\beta\phi^{-1}} \subset I_{H,[\alpha]}.
\end{aligned}$$

Since $I_{H,[\alpha]} = \sum_{\beta\in[\alpha]}(\{\mathfrak{P}_\beta, \mathfrak{P}_{-\beta}\} + \mathfrak{P}_\beta\mathfrak{P}_{-\beta})$ the proof is complete. ■

Theorem 2.5.

(1) *For any* $[\alpha] \in \Lambda/\!\sim$, *the linear subspace* $I_{[\alpha]} = I_{H,[\alpha]} \oplus V_{[\alpha]}$ *of* $\mathfrak{P}$ *associated to* $[\alpha]$ *is an ideal of* $\mathfrak{P}$.
(2) *If* $\mathfrak{P}$ *is simple, then there exists a connection from* α *to* β *for any* $\alpha, \beta \in \Lambda$ *and* $H = \sum_{\alpha\in\Lambda}(\{\mathfrak{P}_\alpha, \mathfrak{P}_{-\alpha}\} + \mathfrak{P}_\alpha\mathfrak{P}_{-\alpha})$.

Proof. (1) Since $\{I_{[\alpha]}, H\} \subset I_{[\alpha]}$, Proposition 2.3 shows that

$$\{I_{[\alpha]}, \mathfrak{P}\} = \Big\{I_{[\alpha]}, H \oplus \Big(\bigoplus_{\beta\in[\alpha]} \mathfrak{P}_\beta\Big) \oplus \Big(\bigoplus_{\gamma\notin[\alpha]} \mathfrak{P}_\gamma\Big)\Big\} \subset I_{[\alpha]}.$$

By Lemma 2.4 and Proposition 2.3 we also have

$$\begin{aligned}
&I_{[\alpha]}\mathfrak{P} + \mathfrak{P}I_{[\alpha]}\\
&= I_{[\alpha]}\Big(H \oplus \Big(\bigoplus_{\beta\in[\alpha]} \mathfrak{P}_\beta\Big) \oplus \Big(\bigoplus_{\gamma\notin[\alpha]} \mathfrak{P}_\gamma\Big)\Big) + \Big(H \oplus \Big(\bigoplus_{\beta\in[\alpha]} \mathfrak{P}_\beta\Big) \oplus \Big(\bigoplus_{\gamma\notin[\alpha]} \mathfrak{P}_\gamma\Big)\Big)I_{[\alpha]} \subset I_{[\alpha]}.
\end{aligned}$$

As by Proposition 2.3(2) also $\phi(I_{[\alpha]}) = I_{[\alpha]}$, we conclude that $I_{[\alpha]}$ is an ideal of I.

(2) The simplicity of $\mathfrak{P}$ implies $I_{[\alpha]} = \mathfrak{P}$. Hence it is clear that $[\alpha] = \Lambda$ and $H = \sum_{\alpha\in\Lambda}(\{\mathfrak{P}_\alpha, \mathfrak{P}_{-\alpha}\} + \mathfrak{P}_\alpha\mathfrak{P}_{-\alpha})$. ∎

THEOREM 2.6. *We have*

$$\mathfrak{P} = U + \sum_{[\alpha]\in\Lambda/\sim} I_{[\alpha]},$$

where U is a linear complement in H of $\mathrm{span}_{\mathbb{K}}\{\{\mathfrak{P}_\alpha, \mathfrak{P}_{-\alpha}\} + \mathfrak{P}_\alpha\mathfrak{P}_{-\alpha} : \alpha \in \Lambda\}$ *and any $I_{[\alpha]}$ is one of the ideals of $\mathfrak{P}$ described in Theorem* 2.5(1), *satisfying $\{I_{[\alpha]}, I_{[\beta]}\} + I_{[\alpha]}I_{[\beta]} = 0$ if $[\alpha] \neq [\beta]$.*

Proof. $I_{[\alpha]}$ is well defined and, by Theorem 2.5(1), an ideal of $\mathfrak{P}$, since it is clear that

$$\mathfrak{P} = H \oplus \Big(\bigoplus_{\alpha\in\Lambda} \mathfrak{P}_\alpha\Big) = U + \sum_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}.$$

Finally Proposition 2.3(3) gives $\{I_{[\alpha]}, I_{[\beta]}\} + I_{[\alpha]}I_{[\beta]} = 0$ if $[\alpha] \neq [\beta]$. ∎

Denote by $\mathcal{Z}(\mathfrak{P}) = \{v \in \mathfrak{P} : \{v, \mathfrak{P}\} + v\mathfrak{P} + \mathfrak{P}v = 0\}$ the *center* of $\mathfrak{P}$.

COROLLARY 2.7. *If $\mathcal{Z}(\mathfrak{P}) = 0$ and $H = \sum_{\alpha\in\Lambda}(\{\mathfrak{P}_\alpha, \mathfrak{P}_{-\alpha}\} + \mathfrak{P}_\alpha\mathfrak{P}_{-\alpha})$, then $\mathfrak{P}$ is the direct sum of the ideals given in Theorem* 2.5,

$$\mathfrak{P} = \bigoplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]},$$

with $\{I_{[\alpha]}, I_{[\beta]}\} + I_{[\alpha]}I_{[\beta]} = 0$ if $[\alpha] \neq [\beta]$.

Proof. Since $H = \sum_{\alpha\in\Lambda}(\{\mathfrak{P}_\alpha, \mathfrak{P}_{-\alpha}\} + \mathfrak{P}_\alpha\mathfrak{P}_{-\alpha})$ we get $\mathfrak{P} = \sum_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$. To verify that the sum is direct, take some $v \in I_{[\alpha]} \cap \sum_{[\beta]\in\Lambda/\sim,\, [\beta]\neq[\alpha]} I_{[\beta]}$. Since $v \in I_{[\alpha]}$, the fact that $\{I_{[\alpha]}, I_{[\beta]}\} + I_{[\alpha]}I_{[\beta]} = 0$ when $[\alpha] \neq [\beta]$ gives

$$\Big\{v, \sum_{[\beta]\in\Lambda/\sim,\, [\beta]\neq[\alpha]} I_{[\beta]}\Big\} + v\Big(\sum_{[\beta]\in\Lambda/\sim,\, [\beta]\neq[\alpha]} I_{[\beta]}\Big) + \Big(\sum_{[\beta]\in\Lambda/\sim,\, [\beta]\neq[\alpha]} I_{[\beta]}\Big)v = 0.$$

In a similar way, since $v \in \sum_{[\beta]\in\Lambda/\sim,\, [\beta]\neq[\alpha]} I_{[\beta]}$ we get $\{v, I_{[\alpha]}\} + vI_{[\alpha]} + I_{[\alpha]}v = 0$. That is, $v \in \mathcal{Z}(\mathfrak{P})$ and so $v = 0$. ∎

3. The simple components. In this section we are going to present a framework in which the decomposition of $\mathfrak{P}$ given in Corollary 2.7 is actually by means of the family of its minimal (simple) ideals, thus getting a second Wedderburn type theorem for the class of split regular Hom-Poisson algebras. We recall that a root system Λ of a split regular Hom-Poisson algebra $\mathfrak{P}$ is called *symmetric* if $\alpha \in \Lambda$ implies $-\alpha \in \Lambda$. From now on we will suppose Λ is symmetric.

LEMMA 3.1. *Suppose $H = \sum_{\alpha\in\Lambda}(\{\mathfrak{P}_\alpha, \mathfrak{P}_{-\alpha}\} + \mathfrak{P}_\alpha\mathfrak{P}_{-\alpha})$. If I is an ideal of $\mathfrak{P}$ such that $I \subset H$, then $I \subset \mathcal{Z}(\mathfrak{P})$.*

Proof. Observe that $\{I,H\}\subset\{H,H\}=0$ and

$$\Big\{I,\bigoplus_{\alpha\in\Lambda}\mathfrak{P}_\alpha\Big\}+I\Big(\bigoplus_{\alpha\in\Lambda}\mathfrak{P}_\alpha\Big)+\Big(\bigoplus_{\alpha\in\Lambda}\mathfrak{P}_\alpha\Big)I \subset I\cap\Big(\bigoplus_{\alpha\in\Lambda}\mathfrak{P}_\alpha\Big)\subset H\cap\Big(\bigoplus_{\alpha\in\Lambda}\mathfrak{P}_\alpha\Big)=0.$$

Since $H=\sum_{\alpha\in\Lambda}(\{\mathfrak{P}_\alpha,\mathfrak{P}_{-\alpha}\}+\mathfrak{P}_\alpha\mathfrak{P}_{-\alpha})$, we also infer, by Hom-associativity, the Hom-Leibniz identity, and the above observation, that $HI+IH=0$. Consequently, $I\subset\mathcal{Z}(\mathfrak{P})$. ■

Let us introduce the concepts of root-multiplicativity and maximal length in the framework of split Hom-Poisson algebras, in a similar way to the cases of split Lie algebras, split 3-Lie algebras, split Lie superalgebras or split Poisson algebras (see [6, 7, 8, 9] for these notions and examples).

Definition 3.2. A split regular Hom-Poisson algebra $\mathfrak{P}$ is *root-multiplicative* if whenever $\alpha,\beta\in\Lambda$ are such that $\alpha\phi^{-1}+\beta\phi^{-1}\in\Lambda$, then $\{\mathfrak{P}_\alpha,\mathfrak{P}_\beta\}+\mathfrak{P}_\alpha\mathfrak{P}_\beta+\mathfrak{P}_\beta\mathfrak{P}_\alpha\neq 0$.

Definition 3.3. A split regular Hom-Lie algebra $\mathfrak{P}$ is of *maximal length* if $\dim\mathfrak{P}_\alpha=1$ for any $\alpha\in\Lambda$.

Theorem 3.4. *Let $\mathfrak{P}$ be a split regular Hom-Poisson algebra of maximal length and root-multiplicative. Then $\mathfrak{P}$ is simple if and only $\mathcal{Z}(\mathfrak{P})=0$, $H=\sum_{\alpha\in\Lambda}(\{\mathfrak{P}_\alpha,\mathfrak{P}_{-\alpha}\}+\mathfrak{P}_\alpha\mathfrak{P}_{-\alpha})$ and Λ has all of its elements connected.*

Proof. Suppose $\mathfrak{P}$ is simple. Since $\mathcal{Z}(\mathfrak{P})$ is an ideal of $\mathfrak{P}$, we have $\mathcal{Z}(\mathfrak{P})=0$. Now Theorem 2.5(2) completes the proof of the direct implication.

To prove the converse, consider a nonzero ideal I of $\mathfrak{P}$. Since I is also an ideal of the split regular Hom-Lie algebra $(\mathfrak{P},\{\cdot,\cdot\})$, by [1, Lemma 4.3] we can write $I=(I\cap H)\oplus(\bigoplus_{\alpha\in\Lambda}I_\alpha)$, where $I_\alpha=I\cap\mathfrak{P}_\alpha$. By the maximal length of $\mathfrak{P}$, if we denote $\Lambda_I:=\{\alpha\in\Lambda : I_\alpha\neq 0\}$ we can write $I=(I\cap H)\oplus(\bigoplus_{\alpha\in\Lambda_I}\mathfrak{P}_\alpha)$, where $\Lambda_I\neq\emptyset$ as a consequence of Lemma 3.1. Pick $\alpha_0\in\Lambda_I$ with $0\neq\mathfrak{P}_{\alpha_0}\subset I$. Since $\phi(I)=I$, Lemma 1.5(1) allows us to assert that

$$\text{if } \alpha\in\Lambda_I \text{ then } \{\alpha\phi^z : z\in\mathbb{Z}\}\subset\Lambda_I, \tag{3.1}$$

that is,

$$\{\mathfrak{P}_{\alpha_0\phi^z} : z\in\mathbb{Z}\}\subset I. \tag{3.2}$$

Now, take any $\beta\in\Lambda$ satisfying $\beta\notin\{\pm\alpha_0\phi^z : z\in\mathbb{Z}\}$. Since α_0 and β are connected, we have a connection $\{\alpha_1,\dots,\alpha_k\}$, $k\geq 2$, from α_0 to β satisfying:

(i) $\alpha_1 = \alpha_0\phi^{-n}$ for some $n \in \mathbb{N}$.
(ii) $\alpha_1\phi^{-1} + \alpha_2\phi^{-1} \in \Lambda$,
$\alpha_1\phi^{-2} + \alpha_2\phi^{-2} + \alpha_3\phi^{-1} \in \Lambda$,
$\ldots$
$\alpha_1\phi^{-k+2} + \alpha_2\phi^{-k+2} + \alpha_3\phi^{-k+3} + \cdots + \alpha_i\phi^{-k+i} + \cdots + \alpha_{k-1}\phi^{-1} \in \Lambda$.
(iii) $\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+1} + \alpha_3\phi^{-k+2} + \cdots + \alpha_i\phi^{-k+i-1} + \cdots + \alpha_k\phi^{-1} = \epsilon\beta\phi^{-m}$ for some $m \in \mathbb{N}$ and $\epsilon \in \{\pm 1\}$.

Taking into account that $\alpha_1, \alpha_2 \in \Lambda$ and $\alpha_1\phi^{-1} + \alpha_2\phi^{-1} \in \Lambda$, the root-multiplicativity and maximal length of $\mathfrak{P}$ allow us to assert that either $0 \neq \{\mathfrak{P}_{\alpha_1}, \mathfrak{P}_{\alpha_2}\} = \mathfrak{P}_{\alpha_1\phi^{-1}+\alpha_2\phi^{-1}}$ or $0 \neq \mathfrak{P}_{\alpha_1}\mathfrak{P}_{\alpha_2} + \mathfrak{P}_{\alpha_2}\mathfrak{P}_{\alpha_1} = \mathfrak{P}_{\alpha_1\phi^{-1}+\alpha_2\phi^{-1}}$.

Since $0 \neq \mathfrak{P}_{\alpha_1} \subset I$ as a consequence of (3.2), we get

$$0 \neq \mathfrak{P}_{\alpha_1\phi^{-1}+\alpha_2\phi^{-1}} \subset I.$$

A similar argument applied to $\alpha_1\phi^{-1} + \alpha_2\phi^{-1}$, α_3 and

$$(\alpha_1\phi^{-1} + \alpha_2\phi^{-1})\phi^{-1} + \alpha_3\phi^{-1} = \alpha_1\phi^{-2} + \alpha_2\phi^{-2} + \alpha_3\phi^{-1}$$

gives $0 \neq \mathfrak{P}_{\alpha_1\phi^{-2}+\alpha_2\phi^{-2}+\alpha_3\phi^{-1}} \subset I$. Continuing, we get

$$0 \neq \mathfrak{P}_{\alpha_1\phi^{-k+1}+\alpha_2\phi^{-k+1}+\alpha_3\phi^{-k+2}+\cdots+\alpha_k\phi^{-1}} \subset I,$$

and so

$$\text{either} \quad \mathfrak{P}_{\beta\phi^{-m}} \subset I \quad \text{or} \quad \mathfrak{P}_{-\beta\phi^{-m}} \subset I.$$

From (3.1) and (3.2), we now get

$$\text{either } \{\mathfrak{P}_{\alpha\phi^{-z}} : z \in \mathbb{Z}\} \subset I \text{ or } \{\mathfrak{P}_{-\alpha\phi^{-z}} : z \in \mathbb{Z}\} \subset I \text{ for any } \alpha \in \Lambda. \tag{3.3}$$

This can be reformulated by saying that for any $\alpha \in \Lambda$ either $\{\alpha\phi^{-z} : z \in \mathbb{Z}\}$ or $\{-\alpha\phi^{-z} : z \in \mathbb{Z}\}$ is contained in Λ_I. Taking into account $H = \sum_{\alpha\in\Lambda}(\{\mathfrak{P}_\alpha, \mathfrak{P}_{-\alpha}\} + \mathfrak{P}_\alpha\mathfrak{P}_{-\alpha})$ we have

$$H \subset I. \tag{3.4}$$

Now for any $\alpha \in \Lambda$, since $\mathfrak{P}_\alpha = \{H, \mathfrak{P}_\alpha\}$ by the maximal length of $\mathfrak{P}$, (3.4) gives $\mathfrak{P}_\alpha \subset I$, and so $I = \mathfrak{P}$. That is, $\mathfrak{P}$ is simple. ∎

THEOREM 3.5. *Let $\mathfrak{P}$ be a split regular Hom-Poisson algebra of maximal length, root-multiplicative, with $\mathcal{Z}(\mathfrak{P}) = 0$ and satisfying*

$$H = \sum_{\alpha\in\Lambda}(\{\mathfrak{P}_\alpha, \mathfrak{P}_{-\alpha}\} + \{\mathfrak{P}_\alpha, \mathfrak{P}_{-\alpha}\}).$$

Then

$$\mathfrak{P} = \bigoplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]},$$

where any $I_{[\alpha]}$ is a simple (split) ideal whose root system, $\Lambda_{I_{[\alpha]}}$, has all of its elements $\Lambda_{I_{[\alpha]}}$-connected.

Proof. By Corollary 2.7 we can write $\mathfrak{P}$ as the direct sum $\bigoplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$ of the family of ideals

$$I_{[\alpha]} = I_{H,[\alpha]} \oplus V_{[\alpha]} = \mathrm{span}_{\mathbb{K}}\{\{\mathfrak{P}_\beta, \mathfrak{P}_{-\beta}\} + \mathfrak{P}_\beta\mathfrak{P}_{-\beta} : \beta \in [\alpha]\} \oplus \bigoplus_{\beta\in[\alpha]} \mathfrak{P}_\beta,$$

where each $I_{[\alpha]}$ is a split regular Hom-Poisson algebra with root system $\Lambda_{I_{[\alpha]}} = [\alpha]$. To make use of Theorem 3.4 in each $I_{[\alpha]}$, we observe that the root-multiplicativity of $\mathfrak{P}$ and Proposition 2.3(3) show that $\Lambda_{I_{[\alpha]}}$ has all of its elements $\Lambda_{I_{[\alpha]}}$-connected, that is, connected through connections contained in $\Lambda_{I_{[\alpha]}}$. Moreover, each $I_{[\alpha]}$ is root-multiplicative by the root-multiplicativity of $\mathfrak{P}$. Clearly $I_{[\alpha]}$ is of maximal length, and finally $\mathcal{Z}_{I_{[\alpha]}}(I_{[\alpha]}) = 0$ (where $\mathcal{Z}_{I_{[\alpha]}}(I_{[\alpha]})$ denotes the center of $I_{[\alpha]}$ in $I_{[\alpha]}$), because $\{I_{[\alpha]}, I_{\{\beta\}}\} + I_{[\alpha]}I_{\{\beta\}} = 0$ if $[\alpha] \neq [\beta]$ (Theorem 2.6) and $\mathcal{Z}(\mathfrak{P}) = 0$. We can apply Theorem 3.4 to any $I_{[\alpha]}$ to conclude that $I_{[\alpha]}$ is simple. It is clear that the decomposition $\mathfrak{P} = \bigoplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$ satisfies the assertions of the theorem. ∎

Acknowledgements. This research was supported by the PCI of the UCA 'Teoría de Lie y Teoría de Espacios de Banach', by the PAI with project numbers FQM298, FQM7156 and by the Spanish Ministerio de Educación y Ciencia, project MTM2013-41208-P.

We would like to thank the referee for the detailed reading of this work and for the suggestions which have improved the final version.

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