

## Operator positivity and analytic models of commuting tuples of operators

by

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**Abstract.** We study analytic models of operators of class  $C_{.0}$  with natural positivity assumptions. In particular, we prove that for an  $m$ -hypercontraction  $T \in C_{.0}$  on a Hilbert space  $\mathcal{H}$ , there exist Hilbert spaces  $\mathcal{E}$  and  $\mathcal{E}_*$  and a partially isometric multiplier  $\theta \in \mathcal{M}(H^2(\mathcal{E}), A_m^2(\mathcal{E}_*))$  such that

$$\mathcal{H} \cong \mathcal{Q}_\theta = A_m^2(\mathcal{E}_*) \ominus \theta H^2(\mathcal{E}) \quad \text{and} \quad T \cong P_{\mathcal{Q}_\theta} M_z|_{\mathcal{Q}_\theta},$$

where  $A_m^2(\mathcal{E}_*)$  is the  $\mathcal{E}_*$ -valued weighted Bergman space and  $H^2(\mathcal{E})$  is the  $\mathcal{E}$ -valued Hardy space over the unit disc  $\mathbb{D}$ . We then proceed to study analytic models for doubly commuting  $n$ -tuples of operators and investigate their applications to joint shift co-invariant subspaces of reproducing kernel Hilbert spaces over the polydisc. In particular, we completely analyze doubly commuting quotient modules of a large class of reproducing kernel Hilbert modules, in the sense of Arazy and Engliš, over the unit polydisc  $\mathbb{D}^n$ .

### Notation.

$\mathbb{N}$	Set of all natural numbers including 0.
$n$	Natural number $n \geq 2$ .
$\mathbb{N}^n$	$\{\mathbf{k} = (k_1, \dots, k_n) : k_i \in \mathbb{N}, i = 1, \dots, n\}$ .
$\mathbf{z}$	$(z_1, \dots, z_n) \in \mathbb{C}^n$ .
$\mathbf{z}^{\mathbf{k}}$	$z_1^{k_1} \dots z_n^{k_n}$ .
$\mathbf{T}$	$n$ -tuple of commuting operators $(T_1, \dots, T_n)$ .
$\mathbf{T}^{\mathbf{k}}$	$T_1^{k_1} \dots T_n^{k_n}$ .
$\mathbb{D}^n$	Open unit polydisc $\{\mathbf{z} :  z_i  < 1\}$ .

For a closed subspace  $\mathcal{S}$  of a Hilbert space  $\mathcal{H}$ , we denote by  $P_{\mathcal{S}}$  the orthogonal projection of  $\mathcal{H}$  onto  $\mathcal{S}$ . We shall denote the space of all bounded linear operators on  $\mathcal{H}$  by  $\mathcal{B}(\mathcal{H})$ .

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**1. Introduction.** The Sz.-Nagy and Foiaş analytic model theory for contractions on Hilbert spaces is a powerful tool for studying operators on Hilbert spaces and holomorphic function spaces on the open unit disc  $\mathbb{D}$  in  $\mathbb{C}$ . It says that if  $T$  is a contraction (that is,  $I - TT^* \geq 0$ ) on a Hilbert space and in  $C_0$  class (that is,  $T^{*l} \rightarrow 0$  as  $l \rightarrow \infty$  in the strong operator topology) then  $T^*$  is unitarily equivalent to the restriction of the backward shift  $M_z^*$  on a vector-valued Hardy space to an  $M_z^*$ -invariant subspace. More precisely, there exists a coefficient Hilbert space  $\mathcal{E}_*$  and an  $M_z^*$ -invariant closed subspace  $\mathcal{Q}$  of the  $\mathcal{E}_*$ -valued Hardy space  $H^2(\mathcal{E}_*)$  such that

$$T \cong P_{\mathcal{Q}}M_z|_{\mathcal{Q}}.$$

Moreover, there exists a Hilbert space  $\mathcal{E}$  and a  $\mathcal{B}(\mathcal{E}, \mathcal{E}_*)$ -valued inner multiplier  $\theta_T \in H_{\mathcal{B}(\mathcal{E}, \mathcal{E}_*)}^{\infty}(\mathbb{D})$ , also known as the *characteristic function* of  $T$  (see [NF]), such that

$$\mathcal{Q} = H^2(\mathcal{E}_*)/\theta_T H^2(\mathcal{E}).$$

On the other hand, in [Ag] J. Agler introduced and studied hypercontraction operators from the operator positivity point of view. He showed that the vector-valued Hardy space in the dilation space of a contraction can be replaced by a vector-valued weighted Bergman space if the contractivity assumption on the operator is replaced by weighted Bergman-type positivity. Later, Müller and Vasilescu [MV], Curto and Vasilescu [CV], Ambrozie and Timotin [AT02, AT03], Arazy, Engliš and Müller [AEM] and Arazy and Engliš [AE] extended these ideas to a more general class of operators. This viewpoint has proved to be extremely fruitful in studying commuting tuples of operators.

The purpose of this paper is to explore how one might do analytic model theory for a general class of operators and commuting tuples of operators. In particular, we associate a partially isometric multiplier with every operator satisfying a weighted Bergman-type positivity condition (see Theorem 2.4). Another basic result in this direction is the following analytic model: Let  $\mathbf{T} = (T_1, \dots, T_n)$  be a doubly commuting tuple of pure operators on a Hilbert space  $\mathcal{H}$  (that is,  $T_i \in C_0$ ,  $T_i T_j = T_j T_i$  and  $T_p T_q^* = T_q^* T_p$  for all  $i, j = 1, \dots, n$ , and  $1 \leq p < q \leq n$ ). Then  $(T_1^*, \dots, T_n^*)$  is joint unitarily equivalent to the restriction of  $(M_{z_1}^*, \dots, M_{z_n}^*)$  to a joint invariant subspace of a vector-valued weighted Bergman space over  $\mathbb{D}^n$  if and only if  $\mathbf{T}$  satisfies (joint) weighted Bergman-type positivity. Moreover, in this case, the orthocomplement of the co-invariant subspace of the weighted Bergman space is of “Beurling–Lax–Halmos” type (see Theorem 4.5).

Although our method works for more general cases (see Section 6), for simplicity we restrict our discussion to hypercontractions (see Section 2).

Here is a brief description of the paper. In Section 2 we set up notation, recall some basic notions from the theory of hypercontractions and

construct an analytic structure on the model space. Our main tool here is the Agler dilation theorem for hypercontractions [Ag] combined with a Beurling–Lax–Halmos type representation of shift invariant subspaces of analytic reproducing kernel Hilbert spaces ([BB], [S15]). In Section 3 we discuss a dilation theory for a class of doubly commuting operator tuples satisfying a weighted Bergman-type positivity condition. In Section 4 we formulate a version of Sz.-Nagy and Foiaş analytic model for doubly commuting tuple of hypercontractions. In Section 5, we analyze doubly commuting quotient modules of scalar valued weighted Bergman spaces. Finally, in Section 6 we study  $K$ -contractive tuples of operators in the spirit of Arazy and Enĭliĥ [AE].

**2. Functional models for hypercontractions.** The main purpose of this section is to develop an analytic functional model for hypercontractions on Hilbert spaces.

We first recall the definition of weighted Bergman spaces and review the construction of dilation maps for hypercontractions. We refer the reader to Agler’s paper [Ag] for more details.

The *weighted Bergman kernel* on the open unit disc  $\mathbb{D}$  with weight  $\alpha > 0$  is, by definition, the kernel function

$$B_\alpha(z, w) = (1 - z\bar{w})^{-\alpha} \quad (z, w \in \mathbb{D}).$$

For each  $\alpha > 0$ , we let  $A_\alpha^2$  denote the weighted Bergman space corresponding to the kernel  $B_\alpha$ . For any Hilbert space  $\mathcal{E}$ , the  $\mathcal{E}$ -valued weighted Bergman space  $A_\alpha^2(\mathcal{E})$  with reproducing kernel  $(z, w) \in \mathbb{D} \times \mathbb{D} \mapsto B_\alpha(z, w)I_\mathcal{E}$  can be canonically identified with the Hilbert space tensor product  $A_\alpha^2 \otimes \mathcal{E}$ . In order to simplify notation, we often identify  $A_\alpha^2 \otimes \mathcal{E}$  with  $A_\alpha^2(\mathcal{E})$ . It also follows that  $\{B_\alpha(\cdot, w)\eta : w \in \mathbb{D}, \eta \in \mathcal{E}\}$  is a total set in  $A_\alpha^2(\mathcal{E})$  and  $\langle f, B_\alpha(\cdot, w)\eta \rangle_{A_\alpha^2(\mathcal{E})} = \langle f(w), \eta \rangle_\mathcal{E}$  where  $f \in A_\alpha^2(\mathcal{E})$ ,  $w \in \mathbb{D}$  and  $\eta \in \mathcal{E}$ . Moreover, it is easy to see that the shift operator  $M_z$  on  $A_\alpha^2(\mathcal{E})$ ,  $\alpha \geq 1$ , is a  $C_0$ -contraction, where

$$(M_z f)(w) = wf(w) \quad (f \in A_\alpha^2(\mathcal{E}), w \in \mathbb{D}).$$

In the following discussion, we shall mostly use weighted Bergman spaces with integer weights. Let us point out an important special case:  $A_1^2 = H^2$ , the Hardy space over  $\mathbb{D}$ .

For a multi-index  $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n$  we denote the corresponding weighted Bergman space on  $\mathbb{D}^n$  by  $A_{\mathbf{m}}^2$ . The *weighted Bergman kernel on  $\mathbb{D}^n$*  with weight  $\mathbf{m}$  is, by definition, the reproducing kernel function

$$B_{\mathbf{m}}(\mathbf{z}, \mathbf{w}) = \prod_{i=1}^n B_{m_i}(z_i, w_i) = \prod_{i=1}^n (1 - z_i \bar{w}_i)^{-m_i} \quad (\mathbf{z}, \mathbf{w} \in \mathbb{D}^n).$$

For each  $\mathbf{w} \in \mathbb{D}^n$ , we denote by  $B_{\mathbf{m}}(\cdot, \mathbf{w})$  the kernel function at  $\mathbf{w}$ , where

$$(B_{\mathbf{m}}(\cdot, \mathbf{w}))(\mathbf{z}) = B_{\mathbf{m}}(\mathbf{z}, \mathbf{w}) \quad (\mathbf{z} \in \mathbb{D}^n).$$

*Convention:* Let  $p(\mathbf{z}, \mathbf{w}) = \sum_{\mathbf{p}, \mathbf{q} \in \mathbb{N}^n} a_{\mathbf{p}\mathbf{q}} \mathbf{z}^{\mathbf{p}} \bar{\mathbf{w}}^{\mathbf{q}}$  be a polynomial in  $\{z_1, \dots, z_n\}$  and  $\{\bar{w}_1, \dots, \bar{w}_n\}$ . For a commuting tuple of bounded linear operators  $\mathbf{T} = (T_1, \dots, T_n)$  on a Hilbert space  $\mathcal{H}$  (that is,  $T_i T_j = T_j T_i$  for all  $i, j = 1, \dots, n$ ) we denote by  $p(\mathbf{z}, \mathbf{w})(\mathbf{T}, \mathbf{T}^*)$  the corresponding hereditary functional calculus in the sense of Agler [Ag]:

$$(2.1) \quad p(z, w)(\mathbf{T}, \mathbf{T}^*) = \sum_{\mathbf{p}, \mathbf{q} \in \mathbb{N}^n} a_{\mathbf{p}\mathbf{q}} \mathbf{T}^{\mathbf{p}} \mathbf{T}^{*\mathbf{q}},$$

where  $\mathbf{T}^{\mathbf{k}} = T_1^{k_1} \dots T_n^{k_n}$  and  $\mathbf{T}^{*\mathbf{k}} = T_1^{*k_1} \dots T_n^{*k_n}$  for all  $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$ .

DEFINITION 2.1. A bounded linear operator  $T$  on  $\mathcal{H}$  is said to be  $B_m$ -contractive (or a  $B_m$ -contraction) if  $T$  is of  $C_0$  class and

$$\begin{aligned} B_m^{-1}(z, w)(T, T^*) &= \left( \sum_{k=0}^m (-1)^k \binom{m}{k} z^k \bar{w}^k \right) (T, T^*) \\ &= \sum_{k=0}^m (-1)^k \binom{m}{k} T^k T^{*k} \geq 0. \end{aligned}$$

We also recall that  $T \in \mathcal{B}(\mathcal{H})$  is a hypercontraction of order  $m$  [Ag] if

$$B_p^{-1}(z, w)(T, T^*) \geq 0 \quad \text{for all } 1 \leq p \leq m.$$

Now let  $T$  be a  $B_m$ -contraction on  $\mathcal{H}$ . Since  $T \in C_0$ , it follows from [Ag, Lemma 2.11] that

$$B_p^{-1}(z, w)(T, T^*) \geq 0 \quad (1 \leq p \leq m),$$

that is,  $T$  is a hypercontraction of order  $m$ . In other words, these two notions coincide for  $C_0$  class operators, and hence we will restrict our considerations to  $B_m$ -contractions.

The defect operator and the defect space of a  $B_m$ -contraction  $T \in \mathcal{B}(\mathcal{H})$  are defined by

$$(2.2) \quad D_{m,T} = (B_m^{-1}(z, w)(T, T^*))^{1/2}, \quad \mathcal{D}_{m,T} = \overline{\text{ran}} D_{m,T},$$

respectively. Set

$$(2.3) \quad B_m(z, T) = (I_{\mathcal{H}} - zT^*)^{-m} \quad (z \in \mathbb{D})$$

and

$$(\mathbf{v}_{m,T} f)(z) = D_{m,T} B_m(z, T) f = D_{m,T} (I_{\mathcal{H}} - zT^*)^{-m} f \quad (f \in \mathcal{H}, z \in \mathbb{D}).$$

Then  $\mathbf{v}_{m,T} : \mathcal{H} \rightarrow A_m^2(\mathcal{D}_{m,T})$  is a bounded linear operator with

$$\mathbf{v}_{m,T} T^* = M_z^* \mathbf{v}_{m,T}$$

and

$$\mathbf{v}_{m,T}^*(B_m(\cdot, w)\eta) = B_m(w, T)^* D_{m,T} \eta = (I_{\mathcal{H}} - \bar{w}T)^{-m} D_{m,T} \eta$$

for all  $w \in \mathbb{D}$  and  $\eta \in \mathcal{D}_{m,T}$ . This and the definition of  $\mathbf{v}_{m,T}$  imply

(2.4)

$$(\mathbf{v}_{m,T} \mathbf{v}_{m,T}^* (B_m(\cdot, w) \eta))(z) = D_{m,T} B_m(z, T) B_m(w, T)^* D_{m,T} \eta \quad (z \in \mathbb{D})$$

for all  $w \in \mathbb{D}$  and  $\eta \in \mathcal{D}_{m,T}$ . Furthermore, since  $T$  is a  $B_m$ -contraction,  $\mathbf{v}_{m,T}$  is an isometry and hence a dilation of  $T$  (see Agler [Ag]).

**THEOREM 2.2 (Agler).** *Let  $T \in \mathcal{B}(\mathcal{H})$  be a  $B_m$ -contraction. Then  $T \cong P_{\mathcal{Q}} M_z|_{\mathcal{Q}}$  for some  $M_z^*$ -invariant closed subspace  $\mathcal{Q}$  of  $A_m^2(\mathcal{D}_{m,T})$ .*

We shall now introduce the notion of multipliers on weighted Bergman spaces. Let  $m_1, m_2$  be natural numbers and  $\mathcal{E}_1, \mathcal{E}_2$  be Hilbert spaces. An operator-valued holomorphic map  $\theta : \mathbb{D} \rightarrow \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$  is said to be a *multiplier* from  $A_{m_1}^2(\mathcal{E}_1)$  to  $A_{m_2}^2(\mathcal{E}_2)$  if  $\theta f \in A_{m_2}^2(\mathcal{E}_2)$  for all  $f \in A_{m_1}^2(\mathcal{E}_1)$ . We denote by  $\mathcal{M}(A_{m_1}^2(\mathcal{E}_1), A_{m_2}^2(\mathcal{E}_2))$  the set of all multipliers from  $A_{m_1}^2(\mathcal{E}_1)$  to  $A_{m_2}^2(\mathcal{E}_2)$ . We also use  $M_\theta$ , for  $\theta \in \mathcal{M}(A_{m_1}^2(\mathcal{E}_1), A_{m_2}^2(\mathcal{E}_2))$ , to denote the multiplication operator

$$M_\theta f = \theta f \quad (f \in A_{m_1}^2(\mathcal{E}_1)).$$

A multiplier  $\theta \in \mathcal{M}(A_{m_1}^2(\mathcal{E}_1), A_{m_2}^2(\mathcal{E}_2))$  is said to be *partially isometric* if  $M_\theta$  is a partially isometric operator from  $A_{m_1}^2(\mathcal{E}_1)$  to  $A_{m_2}^2(\mathcal{E}_2)$ .

Before proceeding, let us for completeness recall a Beurling–Lax–Halmos type theorem for weighted Bergman shifts (see [BB] and [S15, Theorem 2.3]) upon which much of our discussion in this paper will rest.

**THEOREM 2.3.** *Let  $\mathcal{S}$  be a closed subspace of  $A_m^2(\mathcal{E}_*)$ . Then  $\mathcal{S}$  is  $M_z$ -invariant if and only if there exists a Hilbert space  $\mathcal{E}$  and a partially isometric multiplier  $\theta \in \mathcal{M}(A_1^2(\mathcal{E}), A_m^2(\mathcal{E}_*))$  such that  $\mathcal{S} = \theta A_1^2(\mathcal{E})$ .*

We are now ready to present a functional model for  $B_m$ -contractions.

**THEOREM 2.4.** *Let  $T \in \mathcal{B}(\mathcal{H})$  be a  $B_m$ -contraction. Then there exists a Hilbert space  $\mathcal{E}$  and a partially isometric multiplier  $\theta \in \mathcal{M}(A_1^2(\mathcal{E}), A_m^2(\mathcal{D}_{m,T}))$  such that*

$$T \cong P_{\mathcal{Q}_\theta} M_z|_{\mathcal{Q}_\theta},$$

where  $\mathcal{Q}_\theta = A_m^2(\mathcal{D}_{m,T}) \ominus \theta A_1^2(\mathcal{E})$ .

*Proof.* First, by Theorem 2.2, we realize  $T$  as  $T \cong P_{\mathcal{Q}} M_z|_{\mathcal{Q}}$ . Therefore, it remains to prove the existence of a partially isometric multiplier  $\theta$  such that  $\mathcal{Q} = A_m^2(\mathcal{D}_{m,T}) \ominus \theta A_1^2(\mathcal{E})$ .

Since  $\mathcal{Q} = \text{ran } \mathbf{v}_{m,T}$  is  $M_z^*$ -invariant,  $(\text{ran } \mathbf{v}_{m,T})^\perp$  is an  $M_z$ -invariant closed subspace of  $A_m^2(\mathcal{D}_{m,T})$ . Thus, applying Theorem 2.3 to  $(\text{ran } \mathbf{v}_{m,T})^\perp$ , we obtain a coefficient Hilbert space  $\mathcal{E}$  and a partially isometric multiplier  $\theta \in \mathcal{M}(A_1^2(\mathcal{E}), A_m^2(\mathcal{D}_{m,T}))$  such that

$$(\text{ran } \mathbf{v}_{m,T})^\perp = \theta A_1^2(\mathcal{E}),$$

that is,  $\mathcal{Q} = A_m^2(\mathcal{D}_{m,T}) \ominus \theta A_1^2(\mathcal{E})$ . ■

The following observation was pointed out to us by R. G. Douglas: Let  $T \in \mathcal{B}(\mathcal{H})$  be a  $B_m$ -contraction. Then by Theorem 2.4, we have

$$H^2(\mathcal{E}) \xrightarrow{M_\theta} A_m^2(\mathcal{D}_{m,T}) \xrightarrow{\pi} \mathcal{H} \rightarrow 0,$$

where  $\pi = \mathbf{v}_{m,T}^*$  (recall that  $A_1^2(\mathcal{E}) = H^2(\mathcal{E})$ ). Note that since  $M_\theta M_z = M_z M_\theta$ ,  $\mathcal{S} := \ker M_\theta$  is an  $M_z$ -invariant subspace of  $H^2(\mathcal{E})$ . Then by the Beurling–Lax–Halmos theorem there exists a Hilbert space  $\mathcal{E}_*$  and an inner (or isometric) multiplier  $\psi \in H_{\mathcal{B}(\mathcal{E}_*, \mathcal{E})}^\infty(\mathbb{D})$  such that  $\mathcal{S} = \psi H^2(\mathcal{E}_*)$ . Consequently, we have a natural chain complex of Hilbert spaces

$$0 \rightarrow H^2(\mathcal{E}_*) \xrightarrow{M_\psi} H^2(\mathcal{E}) \xrightarrow{M_\theta} A_m^2(\mathcal{D}_{m,T}) \xrightarrow{\pi} \mathcal{H} \rightarrow 0.$$

**3. Dilations of commuting hypercontractions.** In this section, we prove that a doubly commuting tuple of hypercontractions can be dilated to the tuple of shift operators on a suitable weighted Bergman space over  $\mathbb{D}^n$  (see [AEM, AT02, AT03, AE]). We begin with a definition.

DEFINITION 3.1. A commuting tuple of operators  $\mathbf{T} = (T_1, \dots, T_n)$  on  $\mathcal{H}$  is said to be  $B_m$ -contractive if  $T_i$  is a  $B_{m_i}$ -contraction,  $i = 1, \dots, n$ , and

$$B_{\mathbf{m}}^{-1}(\mathbf{z}, \mathbf{w})(\mathbf{T}, \mathbf{T}^*) = \left( \prod_{i=1}^n B_{m_i}^{-1}(z_i, w_i) \right) (\mathbf{T}, \mathbf{T}^*) \geq 0.$$

We denote the (joint-)defect operator and defect space of a  $B_m$ -contraction  $\mathbf{T}$  as

$$D_{\mathbf{m}, \mathbf{T}} := (B_{\mathbf{m}}^{-1}(\mathbf{z}, \mathbf{w})(\mathbf{T}, \mathbf{T}^*))^{1/2} \quad \text{and} \quad \mathcal{D}_{\mathbf{m}, \mathbf{T}} = \overline{\text{ran}} D_{\mathbf{m}, \mathbf{T}},$$

respectively.

For the rest of the paper we shall be dealing with a fixed natural number  $n \geq 2$ , and a multi-index  $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n$ ,  $m_j \geq 1$ ,  $j = 1, \dots, n$ .

Let  $\mathbf{T}$  be a doubly commuting  $B_m$ -contractive tuple on  $\mathcal{H}$ . Then

$$T_i(B_{m_j}^{-1}(z, w)(T_j, T_j^*)) = (B_{m_j}^{-1}(z, w)(T_j, T_j^*))T_i$$

for all  $i \neq j$ . This also implies that

$$\begin{aligned} (B_{m_i}^{-1}(z_i, w_i)(T_i, T_i^*)) (B_{m_j}^{-1}(z_j, w_j)(T_j, T_j^*)) \\ = (B_{m_j}^{-1}(z_j, w_j)(T_j, T_j^*)) (B_{m_i}^{-1}(z_i, w_i)(T_i, T_i^*)). \end{aligned}$$

The above observations yield the following:

LEMMA 3.2. Let  $\mathbf{T}$  be an  $n$ -tuple of doubly commuting operators on  $\mathcal{H}$  and suppose each  $T_i$  is a  $B_{m_i}$ -contraction,  $i = 1, \dots, n$ . Then  $\mathcal{D}_{m_j, T_j}$  is a  $T_i$ -reducing subspace of  $\mathcal{H}$  and

$$T_i D_{m_j, T_j} = D_{m_j, T_j} T_i, \quad D_{m_j, T_j} D_{m_i, T_i} = D_{m_i, T_i} D_{m_j, T_j},$$

for all  $i \neq j$ . Moreover,  $\mathbf{T}$  is a  $B_{\mathbf{m}}$ -contraction and

$$D_{\mathbf{m},\mathbf{T}} = \prod_{i=1}^n D_{m_i, T_i}.$$

Now, we shall construct, by induction, a dilation map for a doubly commuting  $B_{\mathbf{m}}$ -contractive tuple  $\mathbf{T}$  on  $\mathcal{H}$ .

In what follows, for  $j \in \{2, \dots, n\}$ ,  $\mathbf{m}_j$  denotes the  $j$ -tuple  $(m_1, \dots, m_j)$  in  $\mathbb{N}^j$  and  $\mathbf{T}_j$  the doubly commuting  $B_{\mathbf{m}_j}$ -contraction  $(T_1, \dots, T_j)$  on  $\mathcal{H}$ . For each  $j \in \{2, \dots, n\}$ , one checks easily that  $\mathcal{D}_{\mathbf{m}_{j-1}, \mathbf{T}_{j-1}}$  is a  $T_j$ -reducing closed subspace of  $\mathcal{H}$  and

$$D_{T_j|_{\mathcal{D}_{\mathbf{m}_{j-1}, \mathbf{T}_{j-1}}}} = D_{T_j}|_{\mathcal{D}_{\mathbf{m}_{j-1}, \mathbf{T}_{j-1}}}, \quad \mathcal{D}_{T_j|_{\mathcal{D}_{\mathbf{m}_{j-1}, \mathbf{T}_{j-1}}}} = \mathcal{D}_{\mathbf{m}_j, \mathbf{T}_j},$$

where  $\mathcal{D}_{\mathbf{m}_1, \mathbf{T}_1} = \mathcal{D}_{m_1, T_1}$ . Now we set  $V_1 := \mathbf{v}_{m_1, T_1} : \mathcal{H} \rightarrow A_{m_1}^2(\mathcal{D}_{m_1, T_1})$  and define

$$V_2 : A_{m_1}^2 \otimes \mathcal{D}_{m_1, T_1} \cong A_{m_1}^2(\mathcal{D}_{m_1, T_1}) \rightarrow A_{m_1}^2 \otimes A_{m_2}^2 \otimes \mathcal{D}_{\mathbf{m}_2, \mathbf{T}_2} \cong A_{\mathbf{m}_2}^2(\mathcal{D}_{\mathbf{m}_2, \mathbf{T}_2})$$

by

$$V_2 := I_{A_{m_1}^2} \otimes \mathbf{v}_{m_2, T_2|_{\mathcal{D}_{m_1, T_1}}},$$

where  $\mathbf{v}_{m_2, T_2|_{\mathcal{D}_{m_1, T_1}}} : \mathcal{D}_{m_1, T_1} \rightarrow A_{m_2}^2 \otimes \mathcal{D}_{\mathbf{m}_2, \mathbf{T}_2}$  is the dilation of  $T_2|_{\mathcal{D}_{m_1, T_1}} \in \mathcal{B}(\mathcal{D}_{m_1, T_1})$ . It follows that, for all  $l \in \mathbb{N}$ ,  $h \in \mathcal{D}_{m_1, T_1}$  and  $z_1, z_2 \in \mathbb{D}$ ,

$$(V_2(z^l h))(z_1, z_2) = z_1^l (\mathbf{v}_{m_2, T_2} h)(z_2).$$

Continuing, one can define bounded linear operators  $\{V_j\}_{j=2}^n$  by

$$V_j = I_{A_{\mathbf{m}_{j-1}}^2} \otimes \mathbf{v}_{m_j, T_j|_{\mathcal{D}_{\mathbf{m}_{j-1}, \mathbf{T}_{j-1}}}} : A_{\mathbf{m}_{j-1}}^2(\mathcal{D}_{\mathbf{m}_{j-1}, \mathbf{T}_{j-1}}) \rightarrow A_{\mathbf{m}_j}^2(\mathcal{D}_{\mathbf{m}_j, \mathbf{T}_j}),$$

where

$$(3.1) \quad (V_j(z_1^{k_1} \dots z_{j-1}^{k_{j-1}} h))(z_1, \dots, z_j) = z_1^{k_1} \dots z_{j-1}^{k_{j-1}} (\mathbf{v}_{m_j, T_j|_{\mathcal{D}_{\mathbf{m}_{j-1}, \mathbf{T}_{j-1}}}} h)(z_j)$$

for all  $h \in \mathcal{D}_{\mathbf{m}_{j-1}, \mathbf{T}_{j-1}}$ ,  $\{z_1, \dots, z_j\} \subseteq \mathbb{D}$  and  $j = 2, \dots, n$ . Consequently, we have the following sequence of maps:

$$0 \rightarrow \mathcal{H} \xrightarrow{V_1} A_{m_1}^2(\mathcal{D}_{m_1, T_1}) \xrightarrow{V_2} A_{\mathbf{m}_2}^2(\mathcal{D}_{\mathbf{m}_2, \mathbf{T}_2}) \xrightarrow{V_3} \dots \xrightarrow{V_n} A_{\mathbf{m}}^2(\mathcal{D}_{\mathbf{m}, \mathbf{T}}).$$

Let us denote by  $V_{\mathbf{T}}$  the composition of  $\{V_j\}_{j=1}^n$ :

$$(3.2) \quad V_{\mathbf{T}} := V_n \circ \dots \circ V_2 \circ V_1 : \mathcal{H} \rightarrow A_{\mathbf{m}}^2(\mathcal{D}_{\mathbf{m}, \mathbf{T}}).$$

Then  $V_{\mathbf{T}} \in \mathcal{B}(\mathcal{H}, A_{\mathbf{m}}^2(\mathcal{D}_{\mathbf{m}, \mathbf{T}}))$  is an isometric dilation of  $\mathbf{T}$ :

**THEOREM 3.3.** *Let  $\mathbf{T}$  be a doubly commuting  $B_{\mathbf{m}}$ -contractive tuple on  $\mathcal{H}$ . Then  $V_{\mathbf{T}}$  is an isometry and*

$$(V_{\mathbf{T}} h)(z) = \left( \prod_{i=1}^n D_{m_i, T_i} B_{m_i}(z_i, T_i) \right) h \quad (h \in \mathcal{H}, z \in \mathbb{D}^n).$$

Moreover,  $V_{\mathbf{T}}T_i^* = M_{z_i}^*V_{\mathbf{T}}$ ,  $i = 1, \dots, n$ , and for  $\mathbf{z}, \mathbf{w} \in \mathbb{D}^n$  and  $\eta \in \mathcal{D}_{\mathbf{m}, \mathbf{T}}$ ,

$$((V_{\mathbf{T}}V_{\mathbf{T}}^*)(B_{\mathbf{m}}(\cdot, \mathbf{w})\eta))(\mathbf{z}) = \prod_{i=1}^n D_{m_i, T_i} B_{m_i}(z_i, T_i) B_{m_i}(w_i, T_i)^* D_{m_i, T_i} \eta.$$

*Proof.* Clearly  $V_1^*V_1 = I_{\mathcal{H}}$  and for each  $j = 2, \dots, n$ , we have

$$(\mathbf{v}_{m_j, T_j | \mathcal{D}_{m_{j-1}, T_{j-1}}})^* \mathbf{v}_{m_j, T_j | \mathcal{D}_{m_{j-1}, T_{j-1}}} = I_{A_{m_j}^2(\mathcal{D}_{m_j, T_j})},$$

from which we immediately deduce  $V_j^*V_j = I_{A_{m_j}^2(\mathcal{D}_{m_{j-1}, T_{j-1}})}$ , and finally  $V_{\mathbf{T}}^*V_{\mathbf{T}} = I_{\mathcal{H}}$ . Now by (3.1), we have

$$\begin{aligned} V_{\mathbf{T}}h &= V_n \cdots V_2(V_1h) = V_n \cdots V_3(V_2D_{m_1, T_1}B_{m_1}(z_1, T_1)h) \\ &= V_n \cdots V_3(D_{m_2, T_2}B_{m_2}(z_2, T_2)D_{m_1, T_1}B_{m_1}(z_1, T_1)h) \\ &= V_n \cdots V_3(D_{m_1, T_1}D_{m_2, T_2}B_{m_1}(z_1, T_1)B_{m_2}(z_2, T_2)h) \end{aligned}$$

for all  $h \in \mathcal{H}$ . Continuing, we have

$$(V_{\mathbf{T}}h)(\mathbf{z}) = \prod_{i=1}^n D_{m_i, T_i} B_{m_i}(z_i, T_i) h \quad (h \in \mathcal{H}, \mathbf{z} \in \mathbb{D}^n).$$

A direct computation (or see [AEM] or [CV]) now readily implies the intertwining property of  $V_{\mathbf{T}}$  and

$$V_{\mathbf{T}}^*(B_{\mathbf{m}}(\cdot, \mathbf{w})\eta) = \prod_{i=1}^n B_{m_i}(w_i, T_i)^* D_{m_i, T_i} \eta \quad (\mathbf{w} \in \mathbb{D}^n, \eta \in \mathcal{D}_{\mathbf{m}, \mathbf{T}}),$$

which in turn yields the last assertion of the theorem. ■

The above theorem is a doubly commuting version and a particular case of [CV, Theorem 3.16] by Curto and Vasilescu and [AEM, Corollary 16] by Ambrozie, Engliš and Müller (see also [BNS]). However, the present approach is based on the idea of “simple tensor products of one-variable dilation maps”. Moreover, our construction of an explicit dilation map is especially useful in analytic model theory (see Section 4).

Recall that a pair of commuting tuples  $\mathbf{T} = (T_1, \dots, T_n)$  on  $\mathcal{H}$  and  $\mathbf{S} = (S_1, \dots, S_n)$  on  $\mathcal{K}$  are said to be *jointly unitarily equivalent*, denoted by  $\mathbf{T} \cong \mathbf{S}$ , if there exists a unitary map  $U : \mathcal{H} \rightarrow \mathcal{K}$  such that  $UT_i = S_iU$ ,  $i = 1, \dots, n$ . The following dilation result is an easy consequence of Theorem 3.3.

**THEOREM 3.4.** *Let  $\mathbf{T}$  be a doubly commuting  $B_{\mathbf{m}}$ -contractive tuple on  $\mathcal{H}$ . Then there exists a joint  $(M_{z_1}^*, \dots, M_{z_n}^*)$ -invariant closed subspace  $\mathcal{Q} \subseteq A_{\mathbf{m}}^2(\mathcal{D}_{\mathbf{m}, \mathbf{T}})$  such that*

$$(T_1, \dots, T_n) \cong (P_{\mathcal{Q}}M_{z_1}|_{\mathcal{Q}}, \dots, P_{\mathcal{Q}}M_{z_n}|_{\mathcal{Q}}).$$

*Proof.* Let  $\mathcal{Q} = \text{ran } V_{\mathbf{T}}$ , where  $V_{\mathbf{T}}$  is the dilation map of  $\mathbf{T}$  as in Theorem 3.3. Then  $\mathcal{Q}$  is a joint  $(M_{z_1}^*, \dots, M_{z_n}^*)$ -invariant subspace of  $A_{\mathbf{m}}^2(\mathcal{H})$  and



$U_{\mathbf{T}} := V_{\mathbf{T}} : \mathcal{H} \rightarrow \mathcal{Q}$  is a unitary map. Moreover,

$$U_{\mathbf{T}} T_j^* = V_{\mathbf{T}} T_j^* = M_{z_j}^* V_{\mathbf{T}} = M_{z_j}^* V_{\mathbf{T}} V_{\mathbf{T}}^* V_{\mathbf{T}} = (M_{z_j}^*|_{\mathcal{Q}}) V_{\mathbf{T}} = (M_{z_j}^*|_{\mathcal{Q}}) U_{\mathbf{T}}$$

for all  $j = 1, \dots, n$ . Hence

$$U_{\mathbf{T}} T_j = P_{\mathcal{Q}} M_{z_j}|_{\mathcal{Q}} U_{\mathbf{T}} \quad (j = 1, \dots, n). \quad \blacksquare$$

**4. Analytic model.** We begin with the following lemma, the relevance of which to our purpose will become apparent in connection with the analytic model of doubly commuting tuples of operators.

LEMMA 4.1. *Let  $\mathbf{T}$  be a doubly commuting  $B_{\mathbf{m}}$ -contractive tuple on  $\mathcal{H}$  and  $1 \leq j \leq n$ . Then  $A_{\mathbf{m}}^2(\mathcal{D}_{\mathbf{m}, \mathbf{T}}) \subseteq A_{\mathbf{m}}^2(\mathcal{D}_{m_j, T_j})$  and  $A_{\mathbf{m}}^2(\mathcal{D}_{\mathbf{m}, \mathbf{T}})$  is a reducing subspace for*

$$\left( \bigotimes_{\substack{i=1 \\ i \neq j}}^n I_{A_{m_i}^2} \right) \otimes \mathbf{v}_{m_j, T_j} \mathbf{v}_{m_j, T_j}^* \in \mathcal{B}(A_{\mathbf{m}}^2(\mathcal{D}_{m_j, T_j})).$$

*Proof.* The first statement follows from the inclusion  $\mathcal{D}_{\mathbf{m}, \mathbf{T}} \subseteq \mathcal{D}_{m_j, T_j}$ . For the second, it is enough to prove that  $X_j(B_{\mathbf{m}}(\cdot, \mathbf{w})\eta) \in A_{\mathbf{m}}^2(\mathcal{D}_{\mathbf{m}, \mathbf{T}})$  where  $\mathbf{w} \in \mathbb{D}^n$ ,  $\eta \in \mathcal{D}_{\mathbf{m}, \mathbf{T}}$  and

$$X_j := \left( \bigotimes_{\substack{i=1 \\ i \neq j}}^n I_{A_{m_i}^2} \right) \otimes \mathbf{v}_{m_j, T_j} \mathbf{v}_{m_j, T_j}^*.$$

To this end, for each  $z_j, w_j \in \mathbb{D}$ , we compute

$$\begin{aligned} & D_{m_j, T_j} B_{m_j}(z_j, T_j) B_{m_j}(w_j, T_j)^* D_{m_j, T_j} \left( \prod_{i=1}^n D_{m_i, T_i} \right) \\ &= D_{m_j, T_j} B_{m_j}(z_j, T_j) B_{m_j}(w_j, T_j)^* \left( \prod_{\substack{i=1 \\ i \neq j}}^n D_{m_i, T_i} \right) D_{m_j, T_j}^2 \\ &= D_{m_j, T_j} \left( \prod_{\substack{i=1 \\ i \neq j}}^n D_{m_i, T_i} \right) B_{m_j}(z_j, T_j) B_{m_j}(w_j, T_j)^* D_{m_j, T_j}^2 \\ &= \left( \prod_{i=1}^n D_{m_i, T_i} \right) B_{m_j}(z_j, T_j) B_{m_j}(w_j, T_j)^* D_{m_j, T_j}^2. \end{aligned}$$

In particular,

$$(D_{m_j, T_j} B_{m_j}(z_j, T_j) B_{m_j}(w_j, T_j)^* D_{m_j, T_j}) \mathcal{D}_{\mathbf{m}, \mathbf{T}} \subseteq \mathcal{D}_{\mathbf{m}, \mathbf{T}} \quad (z_j, w_j \in \mathbb{D}).$$

If  $\mathbf{w} \in \mathbb{D}^n$ ,  $\eta = \prod_{i=1}^n D_{m_i, T_i} h \in \mathcal{D}_{\mathbf{m}, \mathbf{T}}$  and  $h \in \mathcal{H}$ , then (2.4) gives

$$X_j(B_{\mathbf{m}}(\cdot, \mathbf{w})\eta) = \left( \prod_{\substack{k=1 \\ k \neq j}}^n B_{m_k}(z_k, w_k) \right) (D_{m_j, T_j} B_{m_j}(z_j, T_j) B_{m_j}(w_j, T_j)^* D_{m_j, T_j} \eta) \in A_{\mathbf{m}}^2(\mathcal{D}_{\mathbf{m}, \mathbf{T}}). \blacksquare$$

Therefore,  $R_j \in \mathcal{B}(A_{\mathbf{m}}^2(\mathcal{D}_{\mathbf{m}, \mathbf{T}}))$ ,  $j = 1, \dots, n$ , where

$$R_j = \left( \bigotimes_{\substack{i=1 \\ i \neq j}}^n I_{A_{m_i}^2} \right) \otimes (\mathbf{v}_{m_j, T_j} \mathbf{v}_{m_j, T_j}^*)|_{A_{m_j}^2(\mathcal{D}_{\mathbf{m}, \mathbf{T}})}.$$

By (2.4) we have in particular, for  $\mathbf{w} \in \mathbb{D}^n$ ,  $\eta \in \mathcal{D}_{\mathbf{m}, \mathbf{T}}$ , and  $j = 1, \dots, n$ ,

$$(4.1) \quad R_j(B_{\mathbf{m}}(\cdot, \mathbf{w})\eta) = \left( \prod_{\substack{i=1 \\ i \neq j}}^n B_{m_i}(z_i, w_i) \right) (D_{m_j, T_j} B_{m_j}(z_j, T_j) B_{m_j}(w_j, T_j)^* D_{m_j, T_j} \eta).$$

CLAIM.  $\{R_1, \dots, R_n\}$  is a family of commuting orthogonal projections.

*Proof of the claim.* Since  $\mathbf{v}_{m_j, T_j}$  is an isometry, we deduce from the definition of  $R_j$  that  $R_j = R_j^* = R_j^2$ ,  $1 \leq j \leq n$ , that is,  $\{R_j\}_{j=1}^n$  is a family of orthogonal projections. Now let  $p, q = 1, \dots, n$ ,  $p \neq q$ ,  $\mathbf{w} \in \mathbb{D}^n$  and  $\eta \in \mathcal{D}_{\mathbf{m}, \mathbf{T}}$ . Using (4.1), we obtain

$$\begin{aligned} R_p R_q (B_{\mathbf{m}}(\cdot, \mathbf{w})\eta) &= R_p \left( \prod_{\substack{i=1 \\ i \neq q}}^n B_{m_i}(z_i, w_i) (D_{m_q, T_q} B_{m_q}(z_q, T_q) B_{m_q}(w_q, T_q)^* D_{m_q, T_q} \eta) \right) \\ &= \prod_{\substack{i=1 \\ i \neq p, q}}^n B_{m_i}(z_i, w_i) (D_{m_p, T_p} B_{m_p}(z_p, T_p) B_{m_p}(w_p, T_p)^* D_{m_p, T_p}) \\ &\quad \cdot (D_{m_q, T_q} B_{m_q}(z_q, T_q) B_{m_q}(w_q, T_q)^* D_{m_q, T_q} \eta) \\ &= \prod_{\substack{i=1 \\ i \neq p, q}}^n B_{m_i}(z_i, w_i) (D_{m_p, T_p} D_{m_q, T_q} B_{m_p}(z_p, T_p) B_{m_q}(z_q, T_q) \\ &\quad \cdot B_{m_p}(w_p, T_p)^* B_{m_q}(w_q, T_q)^* D_{m_p, T_p} D_{m_q, T_q}) \eta \\ &= R_q R_p (B_{\mathbf{m}}(\cdot, \mathbf{w})\eta). \end{aligned}$$

Therefore  $R_p R_q = R_q R_p$  for all  $p, q = 1, \dots, n$ , proving the claim.

We now investigate the product  $\prod_{j=1}^n R_j$ . For computational simplicity, assume that, for each  $i = 1, \dots, n$ ,

$$f_i(z, w) := B_{m_i}(z, T_i) B_{m_i}(w, T_i)^* \quad (z, w \in \mathbb{D}).$$

For each  $\mathbf{w} \in \mathbb{D}^n$  and  $\eta \in \mathcal{D}_{\mathbf{m}, \mathbf{T}}$ , we have

$$\begin{aligned} \left(\prod_{j=1}^n R_j\right)(B_{\mathbf{m}}(\cdot, \mathbf{w})\eta) &= \prod_{\substack{j=1 \\ j \neq 1}}^n R_j(R_1 B_{\mathbf{m}}(\cdot, \mathbf{w})\eta) \\ &= \prod_{\substack{j=1 \\ j \neq 1}}^n R_j \left( \prod_{\substack{i=1 \\ i \neq 1}}^n B_{m_i}(\cdot, w_i) D_{m_1, T_1} f_1(\cdot, w_1) D_{m_1, T_1} \eta \right) \\ &= \prod_{\substack{j=1 \\ j \neq 1, 2}}^n R_j \left( \prod_{\substack{i=1 \\ i \neq 1, 2}}^n B_{m_i}(\cdot, w_i) D_{m_1, T_1} D_{m_2, T_2} f_1(\cdot, w_1) f_2(\cdot, w_2) D_{m_1, T_1} D_{m_2, T_2} \eta \right). \end{aligned}$$

Continuing, we have

$$\begin{aligned} \left(\prod_{j=1}^n R_j\right)(B_{\mathbf{m}}(\cdot, \mathbf{w})\eta) &= \prod_{i=1}^n D_{m_i, T_i} f_i(\cdot, w_i) D_{m_i, T_i} \eta \\ &= \prod_{i=1}^n D_{m_i, T_i} B_{m_i}(\cdot, T_i) B_{m_i}(w_i, T_i)^* D_{m_i, T_i} \eta, \end{aligned}$$

and hence Theorem 3.3 yields  $V_{\mathbf{T}} V_{\mathbf{T}}^* = \prod_{i=1}^n R_i$ . Summing up, we obtain:

**THEOREM 4.2.** *Let  $\mathbf{T}$  be a doubly commuting  $B_{\mathbf{m}}$ -contractive tuple on  $\mathcal{H}$ . Then  $\{R_i\}_{i=1}^n$  is a family of commuting orthogonal projections and*

$$V_{\mathbf{T}} V_{\mathbf{T}}^* = \prod_{i=1}^n R_i.$$

We need to introduce one more piece of notation. For  $\mathbf{m} \in \mathbb{N}^n$  and  $j = 1, \dots, n$ , set

$$\hat{\mathbf{m}}_j := (m_1, \dots, m_{j-1}, \underbrace{1}_{j \text{th}}, m_{j+1}, \dots, m_n).$$

In particular,

$$A_{\hat{\mathbf{m}}_j}^2 = A_{m_1}^2 \otimes \dots \otimes A_{m_{j-1}}^2 \otimes H^2 \otimes A_{m_{j+1}}^2 \otimes \dots \otimes A_{m_n}^2.$$

Now let  $\mathbf{T}$  be a doubly commuting  $B_{\mathbf{m}}$ -contractive tuple on  $\mathcal{H}$ , and let  $1 \leq j \leq n$ . Then

$$\begin{aligned} \text{ran } R_j &= \left( \left( \bigotimes_{\substack{i=1 \\ i \neq j}}^n A_{m_i}^2 \right) \otimes \text{ran}(\mathbf{v}_{m_j, T_j} \mathbf{v}_{m_j, T_j}^*) \right) \cap \left( \left( \bigotimes_{\substack{i=1 \\ i \neq j}}^n A_{m_i}^2 \right) \otimes A_{m_j}^2(\mathcal{D}_{\mathbf{m}, \mathbf{T}}) \right) \\ &= \left( \bigotimes_{\substack{i=1 \\ i \neq j}}^n A_{m_i}^2 \right) \otimes \mathcal{Q}_j, \end{aligned}$$

where  $\mathcal{Q}_j = \text{ran}(\mathbf{v}_{m_j, T_j} \mathbf{v}_{m_j, T_j}^*) \cap A_{m_j}^2(\mathcal{D}_{\mathbf{m}, \mathbf{T}})$ . But  $\text{ran}(\mathbf{v}_{m_j, T_j} \mathbf{v}_{m_j, T_j}^*)$  is an  $M_z^*$ -invariant subspace of  $A_{m_j}^2(\mathcal{D}_{m_j, T_j})$ , hence  $\mathcal{Q}_j$  is an  $M_z^*$ -invariant closed subspace of  $A_{m_j}^2(\mathcal{D}_{\mathbf{m}, \mathbf{T}})$ . By Theorem 2.3 there exists an auxiliary Hilbert space  $\mathcal{E}_j$  and a partially isometric multiplier  $\theta_j \in \mathcal{M}(A_1^2(\mathcal{E}_j), A_{m_j}^2(\mathcal{D}_{\mathbf{m}, \mathbf{T}}))$  such that

$$R_j = I_{A_{m_j}^2(\mathcal{D}_{\mathbf{m}, \mathbf{T}})} - M_{\theta_j} M_{\theta_j}^*,$$

where

$$(4.2) \quad M_{\theta_j} = \left( \bigotimes_{\substack{i=1 \\ i \neq j}}^n I_{A_{m_i}^2} \right) \otimes M_{\theta_j}.$$

Notice that  $\theta_j \in \mathcal{M}(A_{m_j}^2(\mathcal{E}_j), A_{m_j}^2(\mathcal{D}_{\mathbf{m}, \mathbf{T}}))$  is a partially isometric multiplier and

$$\Theta_j(\mathbf{z}) = \theta_j(z_j) \quad (\mathbf{z} \in \mathbb{D}^n).$$

Together with Theorem 4.2, this yields the following:

**THEOREM 4.3.** *Let  $\mathbf{T}$  be a doubly commuting  $B_{\mathbf{m}}$ -contractive tuple on  $\mathcal{H}$ . Then there exist Hilbert spaces  $\mathcal{E}_k$  and partially isometric multipliers  $\theta_k \in \mathcal{M}(A_1^2(\mathcal{E}_k), A_{m_k}^2(\mathcal{D}_{\mathbf{m}, \mathbf{T}}))$ ,  $k = 1, \dots, n$ , such that*

$$(M_{\theta_i} M_{\theta_i}^*)(M_{\theta_j} M_{\theta_j}^*) = (M_{\theta_j} M_{\theta_j}^*)(M_{\theta_i} M_{\theta_i}^*)$$

for all  $i, j, k = 1, \dots, n$ . Moreover

$$V_{\mathbf{T}} V_{\mathbf{T}}^* = \prod_{i=1}^n (I_{A_{m_i}^2(\mathcal{D}_{\mathbf{m}, \mathbf{T}})} - M_{\theta_i} M_{\theta_i}^*).$$

In order to formulate our functional model for  $B_{\mathbf{m}}$ -contractive tuples, we need to recall the following result concerning commuting orthogonal projections (cf. [S14, Lemma 1.5]):

**LEMMA 4.4.** *Let  $\{P_i\}_{i=1}^n$  be a collection of commuting orthogonal projections on a Hilbert space  $\mathcal{H}$ . Then  $\mathcal{L} := \sum_{i=1}^n \text{ran } P_i$  is closed and the orthogonal projection of  $\mathcal{H}$  onto  $\mathcal{L}$  is given by  $P_{\mathcal{L}} = I - \prod_{i=1}^n (I - P_i)$ .*

We are now ready to present the main theorem of this section.

**THEOREM 4.5.** *Let  $\mathbf{T}$  be a doubly commuting  $B_{\mathbf{m}}$ -contractive tuple on  $\mathcal{H}$ . Then there exist Hilbert spaces  $\mathcal{E}_k$  and partial isometric multipliers  $\theta_k \in \mathcal{M}(A_1^2(\mathcal{E}_k), A_{m_k}^2(\mathcal{D}_{\mathbf{m}, \mathbf{T}}))$ ,  $k = 1, \dots, n$ , such that*

$$\mathcal{H} \cong \mathcal{Q}_{\Theta} := A_{\mathbf{m}}^2(\mathcal{D}_{\mathbf{m}, \mathbf{T}}) / \sum_{i=1}^n \Theta_i A_{m_i}^2(\mathcal{E}_i)$$

and

$$(T_1, \dots, T_n) \cong (P_{\mathcal{Q}_{\Theta}} M_{z_1}|_{\mathcal{Q}_{\Theta}}, \dots, P_{\mathcal{Q}_{\Theta}} M_{z_n}|_{\mathcal{Q}_{\Theta}}),$$

where  $\Theta_i$  is the one-variable multiplier corresponding to  $\theta_i$ ,  $i = 1, \dots, n$ , as defined in (4.2).

*Proof.* We keep the notation of Theorem 4.3. Set  $P_i := M_{\Theta_i} M_{\Theta_i}^*$ ,  $i = 1, \dots, n$ . By Theorem 4.3 we have

$$I_{A_m^2(\mathcal{D}_{m,T})} - V_T V_T^* = I_{A_m^2(\mathcal{D}_{m,T})} - \prod_{i=1}^n (I_{A_m^2(\mathcal{D}_{m,T})} - P_i).$$

Now by Lemma 4.4, it follows that

$$(\operatorname{ran} V_T)^\perp = \sum_{i=1}^n \operatorname{ran} M_{\Theta_i} = \sum_{i=1}^n \Theta_i A_{\tilde{m}_i}^2(\mathcal{E}_i).$$

Therefore,

$$\mathcal{Q}_\Theta := \operatorname{ran} V_T = \left( \sum_{i=1}^n \Theta_i A_{\tilde{m}_i}^2(\mathcal{E}_i) \right)^\perp \cong A_m^2(\mathcal{D}_{m,T}) / \sum_{i=1}^n \Theta_i A_{\tilde{m}_i}^2(\mathcal{E}_i).$$

Now using the argument from the proof of Theorem 3.4 one can prove that  $(T_1, \dots, T_n) \cong (P_{\mathcal{Q}_\Theta} M_{z_1}|_{\mathcal{Q}_\Theta}, \dots, P_{\mathcal{Q}_\Theta} M_{z_n}|_{\mathcal{Q}_\Theta})$ . ■

In the special case of  $\mathbf{m} = (1, \dots, 1)$  we recover the functional model for doubly commuting tuples of pure contractions [BNS]. However, the methods used here are different from those in [BNS].

**5. Quotient modules of  $A_m^2$ .** We have a particular interest in tuples of operators  $(M_{z_1}, \dots, M_{z_n})$  compressed to joint  $(M_{z_1}^*, \dots, M_{z_n}^*)$ -invariant subspaces of reproducing kernel Hilbert spaces over  $\mathbb{D}^n$ . Let  $\mathcal{Q}$  be a joint  $(M_{z_1}^*, \dots, M_{z_n}^*)$ -invariant closed subspace of  $A_m^2$  and

$$C_{z_i} = P_{\mathcal{Q}} M_{z_i}|_{\mathcal{Q}}, \quad i = 1, \dots, n.$$

Then  $\mathcal{Q}$  is called a *doubly commuting quotient module* of  $A_m^2$  if

$$C_{z_i}^* C_{z_j} - C_{z_j} C_{z_i}^* = 0 \quad (1 \leq i < j \leq n).$$

First, we compute the defect operator of a given doubly commuting quotient module  $\mathcal{Q}$  of  $A_m^2$ :

$$D_{\mathbf{m}, C_z}^2 = B_m^{-1}(z, \mathbf{w})(C_z, C_z^*) = P_{\mathcal{Q}}(B_m^{-1}(z, \mathbf{w})(M_z, M_z^*))|_{\mathcal{Q}}.$$

On the other hand, it is easy to see that (cf. [CDS, Theorem 3.3])

$$D_{\mathbf{m}, M_z}^2 = B_m^{-1}(z, \mathbf{w})(M_z, M_z^*) = P_{\mathbb{C}},$$

where  $P_{\mathbb{C}}$  is the orthogonal projection of  $A_m^2$  onto the one-dimensional subspace of all constant functions. Consequently,  $D_{\mathbf{m}, C_z}^2 = P_{\mathcal{Q}} P_{\mathbb{C}}|_{\mathcal{Q}}$ , and hence

$$(5.1) \quad \operatorname{rank} D_{\mathbf{m}, C_z} \leq 1.$$

**THEOREM 5.1.** *Let  $\mathcal{Q}$  be a quotient module of  $A_{\mathbf{m}}^2$ . Then the following conditions are equivalent:*

- (i)  $\mathcal{Q}$  is doubly commuting.
- (ii) There exist  $M_z^*$ -invariant closed subspaces  $\mathcal{Q}_i$  of  $A_{m_i}^2$ ,  $i = 1, \dots, n$ , such that

$$\mathcal{Q} \cong \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n.$$

- (iii) There exist Hilbert spaces  $\mathcal{E}_i$  and partially isometric multipliers  $\theta_i \in \mathcal{M}(A_1(\mathcal{E}_i), A_{m_i}^2)$ ,  $i = 1, \dots, n$ , such that

$$\mathcal{Q} \cong \mathcal{Q}_{\theta_1} \otimes \cdots \otimes \mathcal{Q}_{\theta_n},$$

where  $\mathcal{Q}_{\theta_j} = A_{m_j}^2 / \theta_j A_1(\mathcal{E}_j)$ ,  $j = 1, \dots, n$ .

*Proof.* Let us begin by observing that the operator  $C_{z_i}$ ,  $i = 1, \dots, n$ , on  $\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n$  can be represented as

$$\begin{aligned} C_{z_i} &= P_{\mathcal{Q}}(I_{A_{m_1}^2} \otimes \cdots \otimes I_{A_{m_{i-1}}^2} \otimes M_z \otimes I_{A_{m_{i+1}}^2} \otimes \cdots \otimes I_{\mathcal{Q}_n})|_{\mathcal{Q}} \\ &= I_{\mathcal{Q}_1} \otimes \cdots \otimes I_{\mathcal{Q}_{i-1}} \otimes P_{\mathcal{Q}_i} M_z|_{\mathcal{Q}_i} \otimes I_{\mathcal{Q}_{i+1}} \otimes \cdots \otimes I_{\mathcal{Q}_n}. \end{aligned}$$

This yields (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i). The implication (ii) $\Rightarrow$ (iii) follows from Theorem 2.3, and (iii) $\Rightarrow$ (ii) is trivial. Hence it suffices to show (i) $\Rightarrow$ (iii). Assume (i). Then by Theorem 4.5, there exist Hilbert spaces  $\mathcal{E}_i$  and one-variable partially isometric multipliers  $\Theta_i \in \mathcal{M}(A_{m_i}^2(\mathcal{E}_i), A_{\mathbf{m}}^2(\mathcal{D}_{\mathbf{m}, \mathbf{C}_z}))$ ,  $i = 1, \dots, n$ , such that

$$(C_{z_1}, \dots, C_{z_n})|_{\mathcal{Q}} \cong (P_{\mathcal{Q}_{\Theta}} M_{z_1}|_{\mathcal{Q}_{\Theta}}, \dots, P_{\mathcal{Q}_{\Theta}} M_{z_n}|_{\mathcal{Q}_{\Theta}}),$$

and

$$\mathcal{Q}_{\Theta} = A_{\mathbf{m}}^2(\mathcal{D}_{\mathbf{m}, \mathbf{C}_z}) / \sum_{i=1}^n \Theta_i A_{m_i}^2(\mathcal{E}_i).$$

Now by (5.1) we have  $\mathcal{D}_{\mathbf{m}, \mathbf{C}_z} \cong \{0\}$ , or  $\mathbb{C}$ . To avoid trivialities we assume that  $\mathcal{D}_{\mathbf{m}, \mathbf{C}_z} \cong \mathbb{C}$ . Then

$$\mathcal{Q} \cong \mathcal{Q}_{\Theta} = A_{\mathbf{m}}^2 / \sum_{i=1}^n \Theta_i A_{m_i}^2(\mathcal{E}_i).$$

In particular,

$$P_{\mathcal{Q}_{\Theta}} = \prod_{i=1}^n (I_{A_{m_i}^2} - M_{\Theta_i} M_{\Theta_i}^*) = \bigotimes_{i=1}^n (I_{A_{m_i}^2} - M_{\Theta_i} M_{\Theta_i}^*),$$

which implies  $\mathcal{Q} \cong \mathcal{Q}_{\Theta} = \mathcal{Q}_{\theta_1} \otimes \cdots \otimes \mathcal{Q}_{\theta_n}$  and concludes the proof.  $\blacksquare$

The implication (i) $\Rightarrow$ (ii) in the previous theorem was obtained in [CDS]. For the Hardy space case  $H^2(\mathbb{D}^n)$ , that is, for  $\mathbf{m} = (1, \dots, 1)$ , this was observed in [BNS] and [S14]. Moreover, as we shall see in the next section, the same result holds for more general reproducing kernel Hilbert spaces over  $\mathbb{D}^n$ .

**6.  $1/K$ -calculus and  $K$ -contractivity.** The key concept in our approach is the natural connections between (i) operator positivity, implemented by the inverse of a positive definite kernel function on  $\mathbb{D}$ , and a dilation map, again in terms of the kernel function, (ii) tensor product structure of reproducing kernel Hilbert spaces on  $\mathbb{D}^n$ , and (iii) operator positivity, implemented by the product of  $n$  positive definite kernel functions on  $\mathbb{D}$ , of a doubly commuting  $n$ -tuple of operators. Consequently, our considerations can be applied even for a more general framework (in the sense of Arazy and Engliš [AE]).

Let  $k$  be a positive definite kernel function on  $\mathbb{D}$  and suppose  $k(z, w)$  is holomorphic in  $z$  and anti-holomorphic in  $w$ , and  $k(z, w) \neq 0$  for all  $z, w \in \mathbb{D}$ . Let  $\mathcal{R}_k \subseteq \mathcal{O}(\mathbb{D}, \mathbb{C})$  be the corresponding reproducing kernel Hilbert space. Moreover, assume that

- (i)  $\mathbb{C}[z]$  is dense in  $\mathcal{R}_k$ ,
- (ii) the multiplication operator  $M_z$  on  $\mathcal{R}_k$  is a contraction,
- (iii) there exists a sequence  $\{p_k\}_{k=0}^\infty \in \mathbb{C}[z, \bar{w}]$  of polynomials such that

$$p_k(z, \bar{w}) \rightarrow \frac{1}{k(z, w)} \quad (z, w \in \mathbb{D})$$

and

$$\sup_k \|p_k(M_z, M_z^*)\| < \infty.$$

We will call such a reproducing kernel Hilbert space a *standard reproducing kernel Hilbert space*, or just SRKH for short.

Let  $\mathcal{R}_k$  be a SRKH and, by (i) above, let  $\{\psi_k\}_{k=0}^\infty \subseteq \mathbb{C}[z]$  be an orthonormal basis of  $\mathcal{R}_k$ . For any nonnegative operator  $C$  and bounded linear operator  $T$  on a Hilbert space  $\mathcal{H}$ , set

$$f_{k,C}(T) = I_{\mathcal{H}} - \sum_{0 \leq m < k} \psi_m(T) C \psi_m(T)^*.$$

**DEFINITION 6.1.** Let  $\mathcal{R}_k$  be a SRKH and  $T \in \mathcal{B}(\mathcal{H})$ . Then  $T$  is said to be  *$k$ -contractive* if  $\sup_k \|p_k(T, T^*)\| < \infty$  and

$$C := \text{WOT-} \lim_{k \rightarrow \infty} p_k(T, T^*)$$

exists and nonnegative, and

$$\text{SOT-} \lim_{k \rightarrow \infty} f_{k,C}(T) = 0.$$

We are now ready to state the Arazy–Engliš dilation result (see [AE, Corollary 3.2]).

**THEOREM 6.2.** *Let  $\mathcal{R}_k$  be a SRKH and  $T \in \mathcal{B}(\mathcal{H})$  be a  $k$ -contraction, and set  $\mathcal{D} = \overline{\text{ran}} C$ . Then there exists an  $M_z^*$ -invariant closed subspace  $\mathcal{Q}$  of  $\mathcal{R}_k \otimes \mathcal{D}$  such that  $T \cong P_{\mathcal{Q}} M_z|_{\mathcal{Q}}$ .*

In this case, the dilation map  $V_T$  is given by (see [AE, (1.5)])

$$(V_T h)(z) = \sum_k \psi_k(z) \otimes C^{1/2} \psi_k(T)^* h \quad (h \in \mathcal{H}).$$

Finally, note that the statement of Theorem 4.5 can be generalized to this framework as follows (see [S15, Theorem 2.3]): Let  $\mathcal{H}$  be a Hilbert space and  $\mathcal{S}$  be a closed subspace of  $\mathcal{R}_k \otimes \mathcal{H}$ . Then  $\mathcal{S}$  is  $M_z$ -invariant if and only if  $\mathcal{S} = \Theta H^2(\mathcal{E})$  for some Hilbert space  $\mathcal{E}$  and a partially isometric multiplier  $\Theta \in \mathcal{M}(H^2(\mathcal{E}), \mathcal{R}_k \otimes \mathcal{H})$ .

Consequently, Theorem 2.4 holds for the class of  $k$ -contractions.

**THEOREM 6.3.** *Let  $\mathcal{R}_k$  be a SRKH and  $T \in \mathcal{B}(\mathcal{H})$  be a  $k$ -contraction, and set  $\mathcal{D} = \overline{\text{ran}} C$ . Then there exists a Hilbert space  $\mathcal{E}$  and a partially isometric multiplier  $\theta \in \mathcal{M}(H^2(\mathcal{E}), \mathcal{R}_k \otimes \mathcal{D})$  such that  $T \cong P_{\mathcal{Q}_\theta} M_z|_{\mathcal{Q}_\theta}$  where  $\mathcal{Q}_\theta = (\mathcal{R}_k \otimes \mathcal{D}) \ominus \theta H^2(\mathcal{E})$ .*

Now let  $\mathcal{R}_{k_i}, i = 1, \dots, n$ , be  $n$  standard reproducing kernel Hilbert spaces over  $\mathbb{D}$ , and set

$$\mathcal{R}_K := \mathcal{R}_{k_1} \otimes \cdots \otimes \mathcal{R}_{k_n}.$$

Then  $\mathcal{R}_K$  is a reproducing kernel Hilbert space (see Tomerlin [To]) and

$$K(\mathbf{z}, \mathbf{w}) = \prod_{i=1}^n k_i(z_i, w_i) \quad (\mathbf{z}, \mathbf{w} \in \mathbb{D}^n).$$

Let  $\mathbf{T}$  be a doubly commuting tuple of operators on  $\mathcal{H}$  and let  $T_i$  be a  $k_i$ -contraction,  $i = 1, \dots, n$ . Set

$$C_i = \text{WOT-} \lim_{k \rightarrow \infty} p_{i,k}(T_i, T_i^*),$$

where  $p_{i,k}(z, \bar{w}) \rightarrow 1/k_i(z, w), i = 1, \dots, n$ . In a similar way to Lemma 3.1, one can prove that  $C_i C_j = C_j C_i$  for all  $i, j = 1, \dots, n$ , and

$$C_{\mathbf{T}} := \prod_{i=1}^n C_i \geq 0.$$

In view of this observation, a doubly commuting tuple  $\mathbf{T}$  is called  $K$ -contractive if  $T_i$  is  $k_i$ -contractive for all  $i = 1, \dots, n$  (see the remark at the end of Lemma 3.2). Consequently, all the results and proofs in this paper hold verbatim for doubly commuting  $K$ -contractive tuples as well.

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