ANNALES
POLONICI MATHEMATICI
116.3 (2016)

# Existence of two positive solutions for a class of semilinear elliptic equations with singularity and critical exponent

Jia-Feng Liao (Chongqing and Zunyi), Jiu Liu (Chongqing),
Peng Zhang (Zunyi) and Chun-Lei Tang (Chongqing)

**Abstract.** We study the following singular elliptic equation with critical exponent

$$\begin{cases} -\Delta u = Q(x)u^{2^*-1} + \lambda u^{-\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a smooth bounded domain, and $\lambda > 0$, $\gamma \in (0,1)$ are real parameters. Under appropriate assumptions on $Q$, by the constrained minimizer and perturbation methods, we obtain two positive solutions for all $\lambda > 0$ small enough.

**1. Introduction and main result.** Consider the following singular elliptic equation with the Dirichlet boundary value conditions:

$$\begin{cases} -\Delta u = Q(x)u^{2^*-1} + \lambda u^{-\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, $\lambda > 0$ is a real parameter, and $\gamma \in (0,1)$ is a constant. The coefficient function $Q \in C(\overline{\Omega})$ is positive. The exponent $2^*-1 = (N+2)/(N-2)$ is the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$ into $L^q(\Omega)$ for every $q \in [1, 2N/(N-2)]$. Here $H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\| = (\int_\Omega (\nabla u, \nabla u)\,dx)^{1/2}$.

2010 *Mathematics Subject Classification*: Primary 35J15; Secondary 35A15, 35B09.
*Key words and phrases*: singular elliptic equation, critical exponent, positive solution, perturbation method.
Received 20 January 2015; revised 27 October 2015.
Published online 16 March 2016.

DOI: 10.4064/ap3606-10-2015  

In many papers the following problem has been studied:

$$
(1.2)\qquad \begin{cases} -\Delta u = \mu u^p + \lambda k(x)u^{-\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}
$$

where $\mu \geq 0$, $1 < p \leq 2^* - 1$ and $k$ is a nonzero nonnegative function. In [7], [8], [9], [12], [13], the authors have studied problem (1.2) with $\mu \equiv 0$, under different assumptions on $k$, and obtained the existence of positive solutions. The case of $\mu > 0$ has also been discussed in [2], [3], [6], [10], [11], [14]–[17], [19]. When the exponent satisfies $1 < p < 2^* - 1$, problem (1.2) has at least two positive solutions under some appropriate conditions via variational methods [2], [3], [6], [11], [14], [16]. When $p$ is the critical exponent, several papers have discussed problem (1.2) (see [10], [11], [15], [17], [19]), also establishing the multiplicity of positive solutions of problem (1.2). In [10], [11] and [19], two positive solutions of problem (1.2) with $\mu = 1$ were obtained by different variational methods for $\lambda > 0$ small enough, while [15] and [17] obtained two positive solutions with $\lambda = 1$ and $\mu > 0$ small enough by the Nehari method. In particular, [11] considered problem (1.2) with $k(x) = \lambda = 1$. Combining the sub-supersolution method and a linking theorem, it was shown that there exists $\Lambda > 0$ such that for every $\mu \in (0, \Lambda)$, problem (1.2) has at least two positive solutions, for $\mu = \Lambda$, it has at least one positive solution, and for $\mu > \Lambda$, it has no positive solution.

When $Q(x) \not\equiv \text{const}$, the analysis of compactness turns out to be more difficult. A natural question is whether problem (1.1) has two positive solutions for $\lambda > 0$ small enough. In the present note, we give a positive answer. First, we obtain a local minimizer solution; next, we study a sequence of mountain-pass solutions of a perturbation problem, and prove that its limit is a positive solution of problem (1.1). Moreover, we distinguish the two solutions by their different values of the corresponding variational functional. Here, we would like to point out some difficulties we will encounter. The first one is the lack of compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, which we overcome by using the Brézis–Lieb Lemma. The second problem is the existence of the second positive solution, where we get rid of the singularity by a perturbation method.

In this paper, we assume that $Q$ satisfies the following condition:

($Q_0$) There exists $x_0 \in \Omega$ such that $Q(x_0) = Q_M = \max_{x\in\overline{\Omega}} Q(x)$ and

$$
Q(x) - Q(x_0) = o(|x - x_0|^{N-2}) \quad \text{as } x \to x_0.
$$

Let $S$ be the best Sobolev constant, namely

$$
(1.3)\qquad S := \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx}{(\int_\Omega |u|^{2^*}\,dx)^{2/2^*}}.
$$

For the convenience of the reader, we recall the Brézis–Lieb Lemma (see [4, Theorem 2] or [18, Lemma 1.32]), which plays an important role in proving Theorem 1.1 below.

LEMMA A. *Let $\Omega \subset \mathbb{R}^N$ be an open set and $\{u_n\}$ be a bounded sequence in $L^p(\Omega)$ $(1 \le p < \infty)$ which converges to $u$ almost everywhere in $\Omega$. Then*

$$\lim_{n\to\infty}\Big(\int_\Omega |u_n|^p\,dx - \int_\Omega |u_n - u|^p\,dx\Big) = \int_\Omega |u|^p\,dx.$$

We define

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2^*}\int_\Omega Q(x)(u^+)^{2^*}\,dx - \frac{\lambda}{1-\gamma}\int_\Omega (u^+)^{1-\gamma}\,dx, \qquad \forall u \in H_0^1(\Omega).$$

In general, a function $u$ is called a *weak solution* of problem (1.1) if $u \in H_0^1(\Omega)$ with $(u^+)^{-\gamma}\varphi \in L^1(\Omega)$ for every $\varphi \in H_0^1(\Omega)$ and satisfies

$$\int_\Omega (\nabla u, \nabla\varphi)\,dx - \int_\Omega Q(x)(u^+)^{2^*-1}\varphi\,dx - \lambda\int_\Omega (u^+)^{-\gamma}\varphi\,dx = 0 \tag{1.4}$$

for all $\varphi \in H_0^1(\Omega)$.

Our main result can be stated as follows:

THEOREM 1.1. *Assume that* ($\mathrm{Q}_0$) *holds and* $\gamma \in (0,1)$. *Then there exists* $\tilde{\Lambda} > 0$ *such that problem* (1.1) *has at least two positive solutions for any* $\lambda \in (0, \tilde{\Lambda})$.

REMARK 1.1. To our best knowledge, problem (1.1) with $Q(x) \not\equiv \text{const}$ has not been considered yet. We generalize the corresponding results of [10], [11], [15], [17] and [19] to problem (1.1) with $Q(x) \not\equiv \text{const}$.

REMARK 1.2. Motivated by [16] and [19], we find the first solution $u_*$ as a local minimizer of $I$ on $H_0^1(\Omega)$ with $I(u_*) < 0$ by a minimax method. Here we encounter two difficulties. One is the lack of compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$. We overcome this difficulty by using the Brézis–Lieb Lemma. The other is how to prove that the local minimizer of $I$ on $H_0^1(\Omega)$ is a solution of problem (1.1).

REMARK 1.3. To find the second solution, by a perturbation method we get rid of the singularity. This method is different from the methods of [10], [11], [15], [17] and [19]. First, we obtain a sequence of positive solutions of the approximating problem by the Mountain Pass Lemma; then we prove that the limit of this sequence is a solution of problem (1.1). We distinguish this solution $u_{**}$ from $u_*$ by $I(u_{**}) > 0$.

This paper is organized as follows. In Section 2, we prove the existence of the first solution of problem (1.1). We study the mountain-pass solutions of the approximating problem in Section 3. In Section 4, we give the proof of Theorem 1.1.

Throughout this paper, we make use of the following notation: the norm in $H_0^1(\Omega)$ is denoted by

$$\|u\| = \Big(\int_\Omega (\nabla u, \nabla u)\, dx\Big)^{1/2} = \Big(\int_\Omega |\nabla u|^2\, dx\Big)^{1/2};$$

the norm in $L^p(\Omega)$ is denoted by $\|u\|_p = (\int_\Omega |u|^p\, dx)^{1/p}$; $C, C_0, C_1, C_2, \ldots$ denote positive constants; and $u^+ = \max\{u, 0\}, u^- = \max\{0, -u\}$.

**2. Existence of the first solution of problem (1.1).** In this part, our main work is to prove that problem (1.1) has a local minimum solution in $H_0^1(\Omega)$. To obtain the first solution of problem (1.1), we prove the following important lemma.

LEMMA 2.1. *There exists a constant $\lambda^* > 0$ such that the functional $I$ attains a negative minimum in $H_0^1(\Omega)$ for all $0 < \lambda < \lambda^*$, that is, there exists $u_* \in \overline{B_R}$ such that $I(u_*) = m < 0$, where $\overline{B_R} = \{u \in H_0^1(\Omega) : \|u\| \le R\}$ is a closed ball. Moreover, $u_* \in B_R$.*

*Proof.* By the Hölder and Sobolev inequalities, there exists a constant $C_0 > 0$ such that

$$\int_\Omega (u^+)^{1-\gamma}\, dx \le \int_\Omega |u|^{1-\gamma}\, dx \le \|u\|_{2^*}^{1-\gamma} |\Omega|^{(2^*+\gamma-1)/2^*} \le C_0 \|u\|^{1-\gamma}, \tag{2.1}$$

and by the definition of $S$ we have

$$\int_\Omega Q(x)(u^+)^{2^*}\, dx \le Q_M \int_\Omega |u|^{2^*}\, dx \le Q_M S^{-2^*/2} \|u\|^{2^*}. \tag{2.2}$$

From (2.1) and (2.2), one gets

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2^*}\int_\Omega Q(x)(u^+)^{2^*}\, dx - \frac{\lambda}{1-\gamma}\int_\Omega (u^+)^{1-\gamma}\, dx \\ &\ge \frac{1}{2}\|u\|^2 - \frac{Q_M}{2^* S^{2^*/2}}\|u\|^{2^*} - \frac{\lambda C_0}{1-\gamma}\|u\|^{1-\gamma}, \end{aligned} \tag{2.3}$$

which implies that there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, there are $R > 0$ and $\rho > 0$ such that

$$\begin{cases} \frac{1}{2}\|u\|^2 - \frac{1}{2^*}\int_\Omega Q(x)(u^+)^{2^*}\, dx \ge 2\rho, & \forall u \in \partial B_R, \\ \frac{1}{2}\|u\|^2 - \frac{1}{2^*}\int_\Omega Q(x)(u^+)^{2^*}\, dx \ge 0, & \forall u \in \overline{B_R}, \\ I(u) \ge \rho, & \forall u \in \partial B_R, \end{cases} \tag{2.4}$$

where $\partial B_R = \{u \in H_0^1(\Omega) : \|u\| = R\}$. Since $Q(x) \ge 0$ in $\Omega$ and $\lambda > 0$, we have $I(u) \le \frac{1}{2}\|u\|^2$. Combining this with (2.3) shows that $I$ is bounded on $\overline{B_R}$. Fix $\lambda \in (0, \lambda^*)$; then $m = \inf_{u \in \overline{B_R}} I(u)$ is well defined. Since $0 < 1 - \gamma < 1$, for $R$ small enough we have

$$m = \inf_{u \in \overline{B_R}} I(u) < 0.$$

Next, we prove that $I$ attains this minimum $m$ in $\overline{B_R}$. Obviously, there exists a minimizing sequence $\{u_n\}$ in $\overline{B_R}$ such that $\lim_{n\to\infty} I(u_n) = m < 0$. Since $\{u_n\}$ is bounded and $\overline{B_R}$ is a closed convex set, there exists $u_* \in \overline{B_R} \subset H_0^1(\Omega)$ and a subsequence, still denoted by $\{u_n\}$, such that

$$\begin{cases} u_n \rightharpoonup u_* & \text{weakly in } H_0^1(\Omega), \\ u_n \to u_* & \text{strongly in } L^s(\Omega),\ 1 \le s < 2^* - 1, \\ u_n(x) \to u_*(x) & \text{a.e. in } \Omega, \end{cases} \tag{2.5}$$

as $n \to \infty$. We let $w_n = u_n - u_*$.

Since $0 < \gamma < 1$, one has the following standard inequality:

$$|x^{1-\gamma} - y^{1-\gamma}| \le |x - y|^{1-\gamma}, \quad \forall x, y \ge 0.$$

Combining it with the Hölder inequality, we obtain

$$\Big|\int_\Omega (u_n^+)^{1-\gamma}\,dx - \int_\Omega (u_*^+)^{1-\gamma}\,dx\Big| \le \int_\Omega |u_n - u_*|^{1-\gamma}\,dx \le \|w_n\|_2^{1-\gamma}|\Omega|^{(1+\gamma)/2}.$$

Combining the above with (2.5) yields

$$\int_\Omega u_n^{1-\gamma}\,dx = \int_\Omega u_*^{1-\gamma}\,dx + o(1), \tag{2.6}$$

where $o(1)$ is an infinitesimal as $n \to \infty$. Moreover, according to Lemma A and the weak lower semicontinuity of norm, one gets

$$\int_\Omega Q(x)(u_n^+)^{2^*}\,dx = \int_\Omega Q(x)(w_n^+)^{2^*}\,dx + \int_\Omega Q(x)(u_*^+)^{2^*}\,dx + o(1), \tag{2.7}$$

$$\int_\Omega |\nabla u_n|^2\,dx = \int_\Omega |\nabla w_n|^2\,dx + \int_\Omega |\nabla u_*|^2\,dx + o(1). \tag{2.8}$$

Since $\inf_{u\in\partial B_R} I(u) \ge \rho$ and $m < 0$, using (2.3) we obtain $\|u_n\| \le R - \varepsilon_0$ for some $\varepsilon_0 > 0$ independent of $n$, in particular $u_* \in B_R$. Then, from (2.8),

$$w_n = u_n - u_* \in \overline{B_R}$$

for $n$ sufficiently large. According to (2.4) we can deduce that

$$\frac{1}{2}\|w_n\|^2 - \frac{1}{2^*}\int_\Omega Q(x)(w_n^+)^{2^*}\,dx \ge 0. \tag{2.9}$$

From (2.6)–(2.9) and the definition of $m$, one obtains

$$\begin{aligned}
m &= \lim_{n\to\infty}\Big\{\frac{1}{2}\int_\Omega |\nabla u_n|^2\,dx - \frac{1}{2^*}\int_\Omega Q(x)(u_n^+)^{2^*}\,dx - \frac{\lambda}{1-\gamma}\int_\Omega (u_n^+)^{1-\gamma}\,dx\Big\}\\
&= \lim_{n\to\infty}\Big(\frac{1}{2}\|w_n\|^2 - \frac{1}{2^*}\int_\Omega Q(x)(w_n^+)^{2^*}\,dx\Big)\\
&\quad + \frac{1}{2}\|u_*\|^2 - \frac{1}{2^*}\int_\Omega Q(x)(u_*^+)^{2^*}\,dx - \frac{\lambda}{1-\gamma}\int_\Omega (u_*^+)^{1-\gamma}\,dx\\
&\ge I(u_*) + o(1) \ge m + o(1),
\end{aligned}$$

so $m = I(u_*) < 0$. This completes the proof of Lemma 2.1. ∎

Theorem 2.1. *Assume that* $\gamma \in (0,1)$. *Then problem* (1.1) *has at least one positive solution for any* $\lambda \in (0,\lambda^*)$, *where* $\lambda^*$ *comes from Lemma* 2.1.

*Proof.* For all $0 < \lambda < \lambda^*$, we take $u_*$ as provided by Lemma 2.1.

We claim that $u_* \ge 0$ in $\Omega$. Indeed, since $\|u_*\|^2 = \|u_*^+\|^2 + \|u_*^-\|^2$, we have $I(u_*) = I(u_*^+) + \frac{1}{2}\|u_*^-\|^2$. Since $u_* \in \overline{B_R}$, one has $u_*^+ \in \overline{B_R}$. Suppose $u_*^- \ne 0$; then $I(u_*^+) < I(u_*) = m$, which contradicts the definition of $m$. Thus $u_*^- \equiv 0$ and our claim is true.

Now we prove that $u_*$ is a weak solution of problem (1.1). Let $\phi \in H_0^1(\Omega)$ with $\phi \ge 0$. Since $u_* \in B_R$, there exists $\xi > 0$ such that $u_* + t\phi \in B_R$ for all $t$ with $|t| < \xi$. Since $u_*$ is a minimizer in $\overline{B_R}$, we have

$$\begin{aligned}
0 &\le \frac{I(u_*+t\phi) - I(u_*)}{t}\\
&= \int_\Omega (\nabla u_*, \nabla\phi)\,dx + \frac{t}{2}\int_\Omega |\nabla\phi|^2\,dx\\
&\quad - \frac{1}{2^*}\int_\Omega Q(x)\frac{(u_*+t\phi)^{2^*} - u_*^{2^*}}{t}\,dx\\
&\quad - \frac{\lambda}{1-\gamma}\int_\Omega \frac{(u_*+t\phi)^{1-\gamma} - u_*^{1-\gamma}}{t}\,dx.
\end{aligned}\tag{2.10}$$

Using the Lebesgue Dominated Convergence Theorem, one obtains

$$\lim_{t\to 0^+}\frac{1}{2^*}\int_\Omega Q(x)\frac{(u_*+t\phi)^{2^*} - u_*^{2^*}}{t}\,dx = \int_\Omega Q(x)u_*^{2^*-1}\phi\,dx.\tag{2.11}$$

For any $x \in \Omega$, we denote

$$h(t) = \frac{(u_*+t\phi)^{1-\gamma} - u_*^{1-\gamma}}{(1-\gamma)t}.$$

Since $s \mapsto s^{1-\gamma}$ is a concave function, $h$ is nonincreasing for any $x \in \Omega$. It is clear that $h$ converges pointwise to $u_*^{-\gamma}(x)\phi(x)$ as $t \to 0^+$. Then we can use the Monotone Convergence Theorem (Beppo Levi) and obtain

$$\lim_{t\to 0^+}\frac{1}{1-\gamma}\int_\Omega Q(x)\frac{(u_*+t\phi)^{1-\gamma}-u_*^{1-\gamma}}{t}\,dx=\int_\Omega u_*^{-\gamma}\phi\,dx. \tag{2.12}$$

Therefore, combining (2.10) and (2.11) with (2.12) shows that $u_*^{-\gamma}\phi\in L^1(\Omega)$ and

$$\int_\Omega[(\nabla u_*,\nabla\phi)-Q(x)u_*^{2^*-1}\phi-\lambda u_*^{-\gamma}\phi]\,dx\ge 0 \tag{2.13}$$

for all $\phi\in H_0^1(\Omega)$ with $\phi\ge 0$. In particular, for $u_*$ there exists $\xi'\in(0,1)$ such that $u_*+tu_*\in\overline{B_R}$ for all $|t|\le\xi'$. Let

$$h(t)=I((1+t)u_*);$$

then $h$ attains its minimum at $t=0$. Thus

$$h'(0)=\|u_*\|^2-\int_\Omega Q(x)u_*^{2^*}\,dx-\lambda\int_\Omega u_*^{1-\gamma}\,dx=0. \tag{2.14}$$

Now we claim that (2.13) is true for all $\phi\in H_0^1(\Omega)$. In fact, suppose $\phi\in H_0^1(\Omega)$ and $t>0$, and define $\Psi\in H_0^1(\Omega)$ by

$$\Psi\equiv(u_*+t\phi)^+,$$

where $(u_*+t\phi)^+=\max\{u_*+t\phi,0\}$. Replacing $\phi$ with $\Psi$ in (2.13), coupled with (2.14), one gets

$$\begin{aligned}
0&\le\int_\Omega\big[(\nabla u_*,\nabla\Psi)-Q(x)u_*^{2^*-1}\Psi-\lambda u_*^{-\gamma}\Psi\big]\,dx\\
&=\int_{\{x\,:\,u_*+t\phi\ge 0\}}[(\nabla u_*,\nabla(u_*+t\phi))-Q(x)u_*^{2^*-1}(u_*+t\phi)]\,dx\\
&\quad-\lambda\int_{\{x\,:\,u_*+t\phi\ge 0\}}u_*^{-\gamma}(u_*+t\phi)\,dx\\
&=\Big[\|u_*\|^2-\int_\Omega Q(x)u_*^{2^*}\,dx-\lambda\int_\Omega u_*^{1-\gamma}\,dx\Big]\\
&\quad+t\int_\Omega\big[(\nabla u_*,\nabla\phi)-Q(x)u_*^{2^*-1}\phi-\lambda u_*^{-\gamma}\phi\big]\,dx\\
&\quad-\int_{\{x\,:\,u_*+t\phi<0\}}\big[(\nabla u_*,\nabla(u_*+t\phi))-Q(x)(u_*)^{2^*-1}(u_*+t\phi)\big]\,dx\\
&\quad+\lambda\int_{\{x\,:\,u_*+t\phi<0\}}u_*^{-\gamma}(u_*+t\phi)\,dx\\
&\le t\int_\Omega\big[(\nabla u_*,\nabla\phi)-Q(x)u_*^{2^*-1}\phi-\lambda u_*^{-\gamma}\phi\big]\,dx-t\int_{\{x\,:\,u_*+t\phi<0\}}(\nabla u_*,\nabla\phi)\,dx.
\end{aligned}$$

Since the measure of the domain of integration $\{x:u_*+t\phi<0\}$ tends to zero as $t\to 0^+$, it follows that $\int_{\{x\,:\,u_*+t\phi<0\}}(\nabla u_*,\nabla\phi)\,dx\to 0$ as $t\to 0^+$.

Dividing by $t$ and letting $t \to 0^+$, one infers that

$$\int_\Omega \big[(\nabla u_*, \nabla\phi) - Q(x)u_*^{2^*-1}\phi - \lambda u_*^{-\gamma}\phi\big]\,dx \ge 0.$$

Noticing that $\phi \in H_0^1(\Omega)$ is arbitrary, one has

$$\int_\Omega \big[(\nabla u_*, \nabla\phi) - Q(x)u_*^{2^*-1}\phi - \lambda u_*^{-\gamma}\phi\big]\,dx = 0, \quad \forall\phi \in H_0^1(\Omega), \tag{2.15}$$

which implies that $u_*$ is a weak solution of problem (1.1).

Now, we prove that $u_*$ is a positive solution of problem (1.1). Since $I(u_*) = m < 0$, we have $u_* \not\equiv 0$. Then $u_* \ge 0$ and $u_* \not\equiv 0$. By the strong maximum principle,

$$u_*(x) > 0 \quad \text{a.e. } x \in \Omega.$$

Thus the proof of Theorem 2.1 is finished. ∎

**3. The mountain-pass solution of the perturbation problem.** Seeking the second solution of problem (1.1), we will study the corresponding approximation problem, truncating the singular term so that it becomes a problem with no singularity at the origin; that is, we consider the following perturbation problem:

$$\begin{cases} -\Delta u = Q(x)(u^+)^{2^*-1} + \lambda(u^+ + \alpha)^{-\gamma} & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\alpha > 0$ is small. The solutions of problem (3.1) correspond to the critical points of the $C^1$-functional on $H_0^1(\Omega)$ given by

$$I_\alpha(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2^*}\int_\Omega Q(x)(u^+)^{2^*}\,dx - \frac{\lambda}{1-\gamma}\int_\Omega [(u^+ + \alpha)^{1-\gamma} - \alpha^{1-\gamma}]\,dx.$$

We observe that $I_0 = I$, and

$$I(u) \le I_\alpha(u), \quad \forall u \in H_0^1(\Omega). \tag{3.2}$$

By a *weak solution of problem* (3.1) we mean a function $u \in H_0^1(\Omega)$ such that

$$\int_\Omega (\nabla u, \nabla\phi)\,dx - \int_\Omega Q(x)(u^+)^{2^*-1}\phi\,dx - \lambda\int_\Omega (u^+ + \alpha)^{-\gamma}\phi\,dx = 0 \tag{3.3}$$

for all $\phi \in H_0^1(\Omega)$. A sequence $\{v_n\}$ in $H_0^1(\Omega)$ is called a $(\mathrm{PS})_c$ *sequence* if

$$I_\alpha(v_n) \to c \quad \text{and} \quad I'_\alpha(v_n) \to 0 \quad \text{as } n \to \infty. \tag{3.4}$$

We say $I_\alpha$ satisfies condition $(\mathrm{PS})_c$ if any $(\mathrm{PS})_c$ sequence $\{v_n\} \subset H_0^1(\Omega)$ has a convergent subsequence.

LEMMA 3.1. *There exists $B > 0$ (depending on $\gamma$, $N$ and $|\Omega|$) such that if $\{v_n\}$ is a $(\mathrm{PS})_c$ sequence of $I_\alpha$ with*

$$c < \frac{S^{N/2}}{NQ_M^{(N-2)/2}} - B\lambda^{2/(1+\gamma)}, \tag{3.5}$$

*then $\{v_n\}$ has a convergent subsequence.*

*Proof.* We claim that $\{v_n\}$ is bounded in $H_0^1(\Omega)$. For contradiction, assume that $\{v_n\}$ is not bounded in $H_0^1(\Omega)$. Then, up to a subsequence, $\|v_n\| \to \infty$, and we have

$$\begin{aligned} c &= I_\alpha(v_n) - \frac{1}{2^*}\langle I'_\alpha(v_n), v_n\rangle + o(1) \\ &\geq \frac{1}{N}\|v_n\|^2 - \frac{\lambda}{1-\gamma}\int_\Omega [(v_n^+ + \alpha)^{1-\gamma} - \alpha^{1-\gamma}]\,dx + \frac{\lambda}{2^*}\int_\Omega (v_n^+ + \alpha)^{-\gamma} v_n\,dx + o(1) \\ &\geq \frac{1}{N}\|v_n\|^2 - C_1\frac{\lambda}{1-\gamma}\|v_n\|^{1-\gamma} + o(1). \end{aligned}$$

Since $0 < 1-\gamma < 1$, the last inequality is absurd. Therefore, $\{v_n\}$ is bounded in $H_0^1(\Omega)$.

Hence there exists a subsequence, still denoted $\{v_n\}$, such that there exists $v_* \in H_0^1(\Omega)$ with

$$\begin{cases} v_n \rightharpoonup v_* & \text{weakly in } H_0^1(\Omega), \\ v_n \to v_* & \text{strongly in } L^p(\Omega),\ 1 < p < 2^*, \\ v_n(x) \to v_*(x) & \text{a.e. in } \Omega, \end{cases}$$

as $n \to \infty$. Now, we only need to prove that $v_n \to v_*$ strongly in $H_0^1(\Omega)$. Let $w_n = v_n - v_*$. Since $I'_\alpha(v_n) \to 0$ in $(H_0^1(\Omega))^*$, we have

$$\|v_n\|^2 - \int_\Omega Q(x)(v_n^+)^{2^*}\,dx - \lambda\int_\Omega (v_n^+ + \alpha)^{-\gamma} v_n^+\,dx = o(1).$$

Consequently, by Lemma A and the Dominated Convergence Theorem,

$$\begin{aligned} \|w_n\|^2 + \|v_*\|^2 - \int_\Omega Q(x)(w_n^+)^{2^*}\,dx - \int_\Omega Q(x)(v_*^+)^{2^*}\,dx \\ - \lambda\int_\Omega (v_*^+ + \alpha)^{-\gamma} v_*^+\,dx = o(1) \end{aligned} \tag{3.6}$$

and

$$\lim_{n\to\infty}\langle I'_\alpha(v_n), v_*\rangle = \|v_*\|^2 - \int_\Omega Q(x)(v_*^+)^{2^*}\,dx - \lambda\int_\Omega (v_*^+ + \alpha)^{-\gamma} v_*^+\,dx = 0. \tag{3.7}$$

From (3.6) and (3.7),

$$\lim_{n\to\infty}\|w_n\|^2 = \lim_{n\to\infty}\int_\Omega Q(x)(w_n^+)^{2^*}\,dx = l.$$

Since

$$\int_{\Omega} |w_n|^{2^*}\,dx \ge \int_{\Omega} \frac{Q(x)}{Q_M}|w_n|^{2^*}\,dx \ge \int_{\Omega} \frac{Q(x)}{Q_M}(w_n^+)^{2^*}\,dx,$$

we have $\|w_n\|_{2^*}^{2^*} \ge l/Q_M$ as $n \to \infty$. Applying the Sobolev inequality, from (1.3) one obtains

$$\|w_n\|^2 \ge S\|w_n\|_{2^*}^2.$$

Then $l \ge S(l/Q_M)^{2/2^*}$, which implies that either $l=0$ or $l \ge S^{N/2}/Q_M^{(N-2)/2}$.

We claim that $l = 0$. For contradiction suppose

$$l \ge \frac{S^{N/2}}{Q_M^{(N-2)/2}}. \tag{3.8}$$

Combining (3.8) with the elementary inequality

$$(a+b)^{1-\gamma} \le a^{1-\gamma} + b^{1-\gamma} \qquad \forall a,b>0,\ 0<\gamma<1,$$

and using (3.6)–(3.8), we get

$$\begin{aligned}
c &= I_\alpha(v_n) - \frac{1}{2^*}\langle I'_\alpha(v_n), v_n\rangle + o(1)\\
&\ge \frac{1}{N}\|v_n\|^2 - \frac{\lambda}{1-\gamma}\int_{\Omega}[(v_n^+ + \alpha)^{1-\gamma} - \alpha^{1-\gamma}]\,dx\\
&\quad + \frac{\lambda}{2^*}\int_{\Omega}(v_n^+ + \alpha)^{-\gamma} v_n\,dx + o(1)\\
&\ge \frac{S^{N/2}}{NQ_M^{(N-2)/2}} + \frac{1}{N}\|v_*\|^2 - \frac{\lambda}{1-\gamma}\int_{\Omega}(v_n^+)^{1-\gamma}\,dx\\
&\ge \frac{S^{N/2}}{NQ_M^{(N-2)/2}} + \frac{1}{N}\|v_*\|^2 - \frac{C_2\lambda}{1-\gamma}\|v_*\|^{1-\gamma} + o(1)\\
&\ge \frac{S^{N/2}}{NQ_M^{(N-2)/2}} - B\lambda^{2/(1+\gamma)},
\end{aligned}$$

where, in the last inequality, $B$ can be chosen using the Young inequality. This contradicts (3.5), so our claim is true.

Thus, $v_n \to v_*$ strongly in $H_0^1(\Omega)$, and the proof of Lemma 3.1 is complete. ■

From now on, $B$ will be as in Lemma 3.1. According to [5], when $\Omega = \mathbb{R}^N$, the infimum in (1.3) is achieved by the function

$$u_\varepsilon(x) = \frac{C_N\varepsilon^{(N-2)/2}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}}, \qquad \forall x \in \mathbb{R}^N,$$

where $C_N = [N(N-2)]^{(N-2)/4}$. Take a cut-off function $\eta \in C_0^\infty(\Omega)$ with $0 \le \eta \le 1$ and

$$\eta(x) = \begin{cases} 1, & |x - x_0| \le \tilde{\delta}/2, \\ 0, & |x - x_0| \ge \tilde{\delta}. \end{cases}$$

Set $v_\varepsilon(x) = \eta(x) u_\varepsilon(x - x_0)$. Then

$$\|v_\varepsilon\|^2 = K_1 + O(\varepsilon^{N-2}), \tag{3.9}$$

$$\|v_\varepsilon\|_{2^*}^2 = K_2 + O(\varepsilon^N), \tag{3.10}$$

where $K_1$ and $K_2$ are positive constants which only depend on $N$ and such that $K_1/K_2 = S$, $O(\varepsilon^{N-2})$ and $O(\varepsilon^N)$ denote quantities such that there exist constants $L_1, L_2 > 0$ such that $|O(\varepsilon^{N-2})/\varepsilon^{N-2}| \le L_1$ and $|O(\varepsilon^N)/\varepsilon^N| \le L_2$ for $\varepsilon$ small enough.

LEMMA 3.2. *Suppose* ($Q_0$) *holds. Then for every* $\gamma \in (0,1)$ *there exists* $u_0 \in H_0^1(\Omega)$ *such that*

$$\sup_{t \ge 0} I_\alpha(tu_0) < \frac{S^{N/2}}{N Q_M^{(N-2)/2}} - B\lambda^{2/(1+\gamma)} \tag{3.11}$$

*for all* $\lambda > 0$ *small enough.*

*Proof.* For $t \ge 0$, let

$$I_\alpha(tv_\varepsilon) = \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{1}{2^*} t^{2^*} \int_\Omega Q(x) v_\varepsilon^{2^*}\,dx - \frac{\lambda}{1-\gamma} \int_\Omega [(tv_\varepsilon + \alpha)^{1-\gamma} - \alpha^{1-\gamma}]\,dx.$$

Then

$$\lim_{t \to +0} I_\alpha(tv_\varepsilon) = 0 \qquad \text{uniformly for all } 0 < \varepsilon < \varepsilon_0,$$
$$\lim_{t \to \infty} I_\alpha(tv_\varepsilon) = -\infty \qquad \text{uniformly for all } 0 < \varepsilon < \varepsilon_0,$$

where $\varepsilon_0 > 0$ is a small constant. Let

$$I_{\varepsilon,1}(t) = \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2^*}}{2^*} \int_\Omega Q(x) v_\varepsilon^{2^*}\,dx,$$

$$I_{\varepsilon,2}(t) = \frac{1}{1-\gamma} \int_\Omega [\alpha^{1-\gamma} - (t_\varepsilon v_\varepsilon + \alpha)^{1-\gamma}]\,dx.$$

Then $I'_{\varepsilon,1}(t) = t(\|v_\varepsilon\|^2 - t^{2^*-2}\|v_\varepsilon\|_{2^*}^{2^*})$. Considering $I'_{\varepsilon,1}(t) = 0$, one sees that

$$T_\varepsilon = \left( \frac{\|v_\varepsilon\|^2}{\int_\Omega Q(x) v_\varepsilon^{2^*}\,dx} \right)^{1/(2^*-2)}$$

is such that $I'_{\varepsilon,1}(t) > 0$ for all $0 < t < T_\varepsilon$, and $I'_{\varepsilon,1}(t) < 0$ for all $t > T_\varepsilon$, so $I_{\varepsilon,1}(t)$ attains its maximum at $T_\varepsilon$. So for $\lambda > 0$ small, $I_\alpha(T_\varepsilon v_\varepsilon) > 0$. Therefore, using the behavior of $I_\alpha(tv_\varepsilon)$ at $t = 0$ and $t = \infty$, one finds that $\sup_{t \ge 0} I_\alpha(tv_\varepsilon)$ is attained for some $t_\varepsilon > 0$.

Moreover, we claim that there exist constants $t_0, T_0 > 0$, independent of $\varepsilon$, such that $t_0 < t_\varepsilon < T_0$. In fact, from $\lim_{t\to+0} I_\alpha(tv_\varepsilon) = 0$ uniformly for all $\varepsilon$, we choose $\epsilon = I_\alpha(t_\varepsilon v_\varepsilon)/4 > 0$; then there exists $t_0 > 0$ such that $|I_\alpha(t_0 v_\varepsilon)| = |I_\alpha(t_0 v_\varepsilon) - I_\alpha(0)| < \epsilon$. By the monotonicity of $I_\alpha(tv_\varepsilon)$ near $t = 0$, we have $t_\varepsilon > t_0$. Similarly, we can show that $t_\varepsilon < T_0$.

We claim that from $(\mathrm{Q}_0)$ we have, as $\varepsilon \to 0^+$,

$$\Big(\int_\Omega Q(x)v_\varepsilon^{2^*}\,dx\Big)^{2/2^*} = Q_M^{2/2^*}\|v_\varepsilon\|_{2^*}^2 + o(\varepsilon^{N-2}). \tag{3.12}$$

In fact, for all $\varepsilon > 0$,

$$\begin{aligned}\Big|\int_\Omega Q(x)v_\varepsilon^{2^*}\,dx - \int_\Omega Q_M {v_\varepsilon}^{2^*}\,dx\Big| &\le \int_\Omega |Q(x)-Q(x_0)|v_\varepsilon^{2^*}\,dx \\ &\le \int_{\Omega'} |Q(x)-Q(x_0)|v_\varepsilon^{2^*}\,dx,\end{aligned} \tag{3.13}$$

where $\Omega' = \{x \in \Omega : |x-x_0| \le \tilde\delta\}$. From $(\mathrm{Q}_0)$, for all $\zeta > 0$, there exists $\delta > 0$ such that

$$|Q(x) - Q(x_0)| < \zeta|x-x_0|^{N-2} \quad \text{for all } 0 < |x-x_0| < \delta.$$

When $\varepsilon > 0$ is small enough, for $\delta > \varepsilon^{1/2}$ it follows from (3.13) and $(\mathrm{Q}_0)$ that

$$\begin{aligned}&\Big|\int_\Omega Q(x)v_\varepsilon^{2^*}\,dx - \int_\Omega Q_M v_\varepsilon^{2^*}\,dx\Big| \\ &\quad\le \int_{\{x\in\Omega\,:\,|x-x_0|\le\tilde\delta\}} |Q(x)-Q(x_0)|v_\varepsilon^{2^*}\,dx \\ &\quad< \int_{\{x\in\Omega\,:\,|x-x_0|\le\delta\}} \zeta|x-x_0|^{N-2}\frac{[N(N-2)\varepsilon^2]^{N/2}}{[\varepsilon^2+|x-x_0|^2]^N}\,dx \\ &\qquad+ \int_{\{x\in\Omega\,:\,\delta<|x-x_0|\le\tilde\delta\}} \frac{[N(N-2)\varepsilon^2]^{N/2}}{[\varepsilon^2+|x-x_0|^2]^N}\,dx \\ &\quad= c_N\zeta\int_0^\delta r\frac{\varepsilon^N}{(\varepsilon^2+r^2)^N}\,dr + c_N\int_\delta^{\tilde\delta}\frac{\varepsilon^N r^{N-1}}{(\varepsilon^2+r^2)^N}\,dr \\ &\quad= c_N\zeta\varepsilon^{N-2}\int_0^{\delta/\varepsilon}\frac{r}{(1+r^2)^N}\,dr + c_N\int_{\delta/\varepsilon}^{\tilde\delta/\varepsilon}\frac{r^{N-1}}{(1+r^2)^N}\,dr \\ &\quad\le C_2\zeta\varepsilon^{N-2} + C_3\varepsilon^N,\end{aligned}$$

where $c_N = [N(N-2)]^{N/2}$. Consequently,

$$\frac{|\int_\Omega Q(x)v_\varepsilon^{2^*}\,dx - \int_\Omega Q_M v_\varepsilon^{2^*}\,dx|}{\varepsilon^{N-2}} \le C_2\zeta + C_3\varepsilon^2,$$

which implies that

$$\limsup_{\varepsilon\to 0^+} \frac{|\int_\Omega Q(x)v_\varepsilon^{2^*}\,dx - \int_\Omega Q_M v_\varepsilon^{2^*}\,dx|}{\varepsilon^{N-2}} \le C_2\zeta.$$

Then from the arbitrariness of $\zeta$, we obtain (3.12).

Combining (3.10) and (3.12) gives

$$\Big(\int_\Omega Q(x)v_\varepsilon^{2^*}\,dx\Big)^{2/2^*} = Q_M^{2/2^*}K_2 + o(\varepsilon^{N-2}). \tag{3.14}$$

From (3.14) and (3.10), one has

$$\begin{aligned} I_{\varepsilon,1}(T_\varepsilon) &= \Big(\frac{\|v_\varepsilon\|^2}{\int_\Omega Q(x)v_\varepsilon^{2^*}\,dx}\Big)^{2/(2^*-2)}\Big(\frac{\|v_\varepsilon\|^2}{2} - \frac{\|v_\varepsilon\|^2}{2^*}\Big) \\ &= \frac{1}{N}\Big(\frac{\|v_\varepsilon\|^2}{\int_\Omega Q(x)v_\varepsilon^{2^*}\,dx}\Big)^{N/2} \le \frac{S^{N/2}}{NQ_M^{(N-2)/2}} + C_4\varepsilon^{N-2}, \end{aligned} \tag{3.15}$$

where $C_4 > 0$ is a constant.

Next, we concentrate on the estimate of $I_{\varepsilon,2}$. We claim that

$$\alpha^{1-\gamma} - (\alpha+\beta)^{1-\gamma} \le -(1-\gamma)\beta^{(1-\gamma)/4}\alpha^{3(1-\gamma)/4}, \qquad \beta > \alpha > 0. \tag{3.16}$$

In fact, dividing (3.16) by $\beta^{1-\gamma}$ and setting $t = \alpha/\beta$, for $t > 0$ small enough, one has

$$t^{1-\gamma} - (1+t)^{1-\gamma} \le -(1-\gamma)t^{3(1-\gamma)/4}. \tag{3.17}$$

Let

$$f(t) = t^{1-\gamma} - (1+t)^{1-\gamma} + (1-\gamma)t^{3(1-\gamma)/4}.$$

We only need to prove that $f(t) \le 0$ for $t > 0$ small. Indeed, since $f(0) = -1$, we obtain $f(t) \le 0$ for $t > 0$ small by continuity. So (3.17) is true, thus (3.16) holds.

According to (3.16),

$$\begin{aligned} I_{\varepsilon,2}(t) &\le \frac{1}{1-\gamma}\int_{\{x\,:\,|x-x_0|\le\varepsilon^{(1-\gamma)/8}\}} [\alpha^{1-\gamma} - (t_\varepsilon v_\varepsilon + \alpha)^{1-\gamma}]\,dx \\ &\le -C_5\int_{\{x\,:\,|x-x_0|\le\varepsilon^{(1-\gamma)/8}\}} (t_\varepsilon v_\varepsilon)^{(1-\gamma)/4}\,dx \\ &= -C_5\int_{\{x\,:\,|x-x_0|\le\varepsilon^{(1-\gamma)/8}\}} \Big[\frac{t_\varepsilon\varepsilon^{(N-2)/2}\eta(x)}{(\varepsilon^2+|x-x_0|^2)^{(N-2)/2}}\Big]^{(1-\gamma)/4}\,dx \\ &\le -C_6\varepsilon^{(1-\gamma)(7N+\gamma N-6-2\gamma)/32}. \end{aligned}$$

By the above inequality and (3.15), there exists $\lambda^{**} > 0$ small enough such that

$$\begin{aligned} I_\alpha(t_\varepsilon v_\varepsilon) &= I_{\varepsilon,1}(t_\varepsilon) + \lambda I_{\varepsilon,2}(t_\varepsilon) \\ &\le \frac{S^{N/2}}{NQ_M^{(N-2)/2}} + C_4\varepsilon^{N-2} - C_6\lambda\varepsilon^{(1-\gamma)(7N+\gamma N-6-2\gamma)/32} \\ &\le \frac{S^{N/2}}{NQ_M^{(N-2)/2}} + C_4\lambda^{2/(1+\gamma)} - C_6\lambda^{\frac{(1-\gamma)(7N+\gamma N-6-2\gamma)+16(N-2)(1+\gamma)}{16(N-2)(1+\gamma)}} \\ &\le \frac{S^{N/2}}{NQ_M^{(N-2)/2}} + C_4\lambda^{2/(1+\gamma)}\big(1 - C_7\lambda^{\frac{(1-\gamma)[(N-2)\gamma-9N+26]}{16(N-2)(1+\gamma)}}\big) \\ &< \frac{S^{N/2}}{NQ_M^{(N-2)/2}} - B\lambda^{2/(1+\gamma)}, \end{aligned}$$

where we choose $\varepsilon = \lambda^{2/((1+\gamma)(N-2))}$ for $0 < \lambda < \lambda^{**}$ and $(N-2)\gamma - 9N + 26 < 0$ for all $0 < \gamma < 1$ and every $N \ge 3$. This implies that (3.11) holds for all $0 < \lambda < \lambda^{**}$. Thus Lemma 3.2 holds. ∎

PROPOSITION 3.1. *For every $\alpha > 0$, problem* (3.1) *has a positive mountain-pass solution $v_\alpha$ for all $0 < \lambda < \min\{\lambda^*, \lambda^{**}\}$.*

*Proof.* Let $\omega \in H_0^1(\Omega)$, $\omega \not\equiv 0$. Then for all $t > 0$,

$$\begin{aligned} I_\alpha(t\omega) &= \frac{t^2}{2}\|\omega\|^2 - \frac{t^{2^*}}{2^*}\int_\Omega Q(x)(\omega^+)^{2^*}\,dx \\ &\quad - \frac{\lambda}{1-\gamma}\int_\Omega \big[(t\omega^+ + \alpha)^{1-\gamma} - \alpha^{1-\gamma}\big]\,dx \\ &\le \frac{t^2}{2}\|\omega\|^2 - \frac{t^{2^*}}{2^*}\int_\Omega Q(x)(\omega^+)^{2^*}\,dx, \end{aligned}$$

which implies that there exists $t_0 > 0$ such that $I_\alpha(t_0\omega) < 0$ and $\|t_0\omega\| > R$, where the choice of $t_0\omega$ is independent of $\lambda$ and $\varepsilon$. Setting $\omega_0 = t_0\omega$ and

$$\Gamma = \{h \in C([0,1], H_0^1(\Omega)) : h(0) = 0,\ h(1) = \omega_0\},$$

we can define the mountain-pass level for $I_\alpha$,

$$c = \inf_{h\in\Gamma}\max_{t\in(0,1)} I_\alpha(h(t)).$$

Consequently, from (2.4) and (3.2),

$$0 < \rho < c \le \max_{t\in[0,1]} I_\alpha(t\omega) \le \sup_{t\ge0} I_\alpha(t\omega) < \frac{S^{N/2}}{NQ_M^{(N-2)/2}} - B\lambda^{2/(1+\gamma)} \tag{3.18}$$

for all $0 < \lambda < \min\{\lambda^*, \lambda^{**}\}$ and $\alpha > 0$. By Lemmas 3.1 and 3.2, $I_\alpha$ satisfies the geometry of the Mountain-Pass Theorem [1]. According to Lemma 3.1, $\{v_n\} \subset H_0^1(\Omega)$ has a convergent subsequence, still denoted by $\{v_n\}$, such that $v_n \to v_\alpha$ in $H_0^1(\Omega)$ as $n \to \infty$. Hence, it follows from (3.18) that

$$I_\alpha(v_\alpha) = \lim_{n\to\infty} I_\alpha(v_n) = c > \rho > 0. \tag{3.19}$$

So from (3.19), $v_\alpha \not\equiv 0$. Furthermore, from the continuity of $I'_\alpha$, we find that $v_\alpha$ is a solution of problem (3.1), namely

$$\int_\Omega (\nabla v_\alpha, \nabla\varphi)\,dx - \int_\Omega Q(x)(v_\alpha^+)^{2^*-1}\varphi\,dx - \lambda\int_\Omega \frac{\varphi}{(v_\alpha^+ + \alpha)^\gamma}\,dx = 0 \tag{3.20}$$

for all $\varphi \in H_0^1(\Omega)$. Taking the test function $\varphi = v_\alpha^-$ in (3.20), one has

$$-\|v_\alpha^-\|^2 = \lambda\int_\Omega \frac{v_\alpha^-}{(v_\alpha^+ + \alpha)^\gamma}\,dx \ge 0;$$

this implies that $v_\alpha^- = 0$, therefore $v_\alpha \ge 0$ and $v_\alpha \not\equiv 0$. Hence, by the strong maximum principle, $v_\alpha$ is a positive solution of problem (3.1). This completes the proof of Proposition 3.3. ∎

**4. Proof of Theorem 1.1.** Let $\gamma \in (0,1)$ and $\tilde{\Lambda} = \min\{\lambda^*, \lambda^{**}\}$; then our lemmas and proposition all hold for all $0 < \lambda < \tilde{\Lambda}$. Hence Theorem 2.1 also holds. Thus problem (1.1) has a solution $u_*$, which is a local minimum for the corresponding functional $I$.

Now, we only need to prove that problem (1.1) has another positive solution. Since $\{v_\alpha\}$ are solutions of problem (3.1), one has

$$\|v_\alpha\|^2 - \int_\Omega Q(x)v_\alpha^{2^*}\,dx - \lambda\int_\Omega (v_\alpha + \alpha)^{-\gamma}v_\alpha\,dx = 0. \tag{4.1}$$

According to Proposition 3.3, by (4.1) we obtain

$$\begin{aligned}
\frac{S^{N/2}}{NQ_M^{(N-2)/2}} - B\lambda^{2/(1+\gamma)} &> I_\alpha(v_\alpha) \\
&= \frac{1}{N}\|v_\alpha\|^2 + \frac{\lambda}{2^*}\int_\Omega (v_\alpha + \alpha)^{-\gamma}v_\alpha\,dx \\
&\quad - \frac{\lambda}{1-\gamma}\int_\Omega [(v_\alpha+\alpha)^{1-\gamma} - \alpha^{1-\gamma}]\,dx \\
&\ge \frac{1}{N}\|v_\alpha\|^2 - \frac{\lambda}{1-\gamma}\int_\Omega [(v_\alpha+\alpha)^{1-\gamma} - \alpha^{1-\gamma}]\,dx \\
&\ge \frac{1}{N}\|v_\alpha\|^2 - C_8\|v_\alpha\|^{1-\gamma}.
\end{aligned}$$

Since $\gamma \in (0,1)$, it follows that $\{v_\alpha\}$ is bounded in $H_0^1(\Omega)$. Going if necessary to a subsequence, also denoted by $\{v_\alpha\}$, there exists $u_{**} \in H_0^1(\Omega)$ such that

$$(4.2)\qquad \begin{cases} v_\alpha \rightharpoonup u_{**} & \text{weakly in } H_0^1(\Omega), \\ v_\alpha \to u_{**} & \text{strongly in } L^p(\Omega),\ 1 < p < 2^*, \\ v_\alpha(x) \to u_{**}(x) & \text{a.e. in } \Omega. \end{cases}$$

Next, we prove that $u_{**}$ is a solution of problem (1.1). According to Proposition 3.1, we have $u_{**}(x) \geq 0$ in $\Omega$.

We claim that $\{v_\alpha\}$ has a uniform lower bound. Indeed, set

$$h(t) = t^{2^*-1} + \frac{\lambda}{(t+1)^\gamma}.$$

Then for $0 < t < 1$,

$$h(t) \geq \frac{\lambda}{(1+1)^\gamma} = \frac{\lambda}{2^\gamma},$$

while for $t \geq 1$ we have $h(t) \geq 1$. Therefore, for any $\alpha \in (0,1)$ and $t \geq 0$,

$$t^{2^*-1} + \frac{\lambda}{(t+\alpha)^\gamma} \geq t^{2^*-1} + \frac{\lambda}{(t+1)^\gamma} \geq \min\{1, \lambda/2^\gamma\}.$$

Noticing that $v_\alpha$ satisfies problem (3.1), we have

$$-\Delta v_\alpha = Q(x)v_\alpha^{2^*-1} + \frac{\lambda}{(v_\alpha+\alpha)^\gamma} \geq \min\{1, Q_{\mathrm{m}}\}\min\{1, \lambda/2^\gamma\},$$

where $Q_{\mathrm{m}} = \min_{x\in\Omega} Q(x) > 0$. Denote by $e$ the positive solution of

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $e(x) > 0$ in $\Omega$. By the maximum principle,

$$(4.3)\qquad v_\alpha \geq \min\{1, Q_{\mathrm{m}}\}\min\{1, \lambda/2^\gamma\}e > 0$$

in $\Omega$. Thus our claim is true.

Notice that $v_\alpha \rightharpoonup u_{**}$ as $\alpha \to 0^+$. Take $v = \phi \in H_0^1(\Omega) \cap C_0(\Omega)$ as a test function in (3.20), where $C_0(\Omega)$ is the subset of $C(\Omega)$ consisting of functions with compact support in $\Omega$. Letting $\alpha \to 0^+$ and applying (4.3), one obtains $u_{**} \geq \min\{1, Q_{\mathrm{m}}\}\min\{1, \lambda/2^\gamma\}e > 0$ and

$$(4.4)\qquad \int_\Omega (\nabla u_{**}, \nabla\phi)\,dx = \int_\Omega Q(x)u_{**}^{2^*-1}\phi\,dx + \lambda\int_\Omega u_{**}^{-\gamma}\phi\,dx.$$

We claim that (4.4) holds for any $\phi \in H_0^1(\Omega)$. Indeed, let $\phi_n \in H_0^1(\Omega) \cap C_0(\Omega)$ satisfy

$$(4.5)\qquad \int_\Omega (\nabla u_{**}, \nabla\phi_n)\,dx = \int_\Omega Q(x)u_{**}^{2^*-1}\phi_n\,dx + \lambda\int_\Omega u_{**}^{-\gamma}\phi_n\,dx.$$

Since $H_0^1(\Omega) \cap C_0(\Omega)$ is dense in $H_0^1(\Omega)$, for any $\phi \in H_0^1(\Omega)$ there exists a sequence $\phi_n \in H_0^1(\Omega) \cap C_0(\Omega)$ such that $\phi_n \to \phi$ as $n \to \infty$. Replacing $\phi_n$

with $\phi_n - \phi_m$ in (4.5), one has

$$(4.6)\quad \int_\Omega (\nabla u_{**}, \nabla |\phi_n - \phi_m|)\, dx = \int_\Omega Q(x) u_{**}^{2^*-1} |\phi_n - \phi_m|\, dx + \lambda \int_\Omega u_{**}^{-\gamma} |\phi_n - \phi_m|\, dx.$$

Since $\phi_n \to \phi$ as $n \to \infty$, from (4.6) we infer that $\{\phi_n/u_{**}^\gamma\}$ is a Cauchy sequence in $L^1(\Omega)$, hence there exists $\nu \in L^1(\Omega)$ such that $\phi_n/u_{**}^\gamma \to \nu$ as $n \to \infty$, which implies that $\phi_n/u_{**}^\gamma \to \nu$ as $n \to \infty$ in measure. By the Riesz Theorem, going if necessary to a subsequence of $\{\phi_n/u_{**}^\gamma\}$, still denoted by $\{\phi_n/u_{**}^\gamma\}$, we get

$$(4.7)\quad \phi_n(x)/u_{**}^\gamma(x) \to \nu(x) \quad \text{a.e. } x \in \Omega,$$

as $n \to \infty$. On the other hand, since $\phi_n(x)/u_{**}^\gamma(x) \to \phi(x)/u_{**}^\gamma(x)$ a.e. in $\Omega$, by (4.7) we obtain $\nu = \phi/u_{**}^\gamma$. Consequently, $\int_\Omega (\phi_n/u_{**}^\gamma)\, dx \to \int_\Omega (\phi/u_{**}^\gamma)\, dx$ as $n \to \infty$. From (4.5) one gets

$$(4.8)\quad \int_\Omega (\nabla u_{**}, \nabla \phi)\, dx = \int_\Omega Q(x) u_{**}^{2^*-1} \phi\, dx + \lambda \int_\Omega u_{**}^{-\gamma} \phi\, dx, \quad \forall \phi \in H_0^1(\Omega).$$

Therefore, our claim is true, and $u_{**}$ is a solution of problem (1.1).

Finally, we prove that $u_{**}$ is different from $u_*$. We claim that $v_\alpha \to u_{**}$ as $\alpha \to 0^+$ in $H_0^1(\Omega)$. Indeed, setting $\omega_\alpha = v_\alpha - u_{**}$, we need to prove $\|\omega_\alpha\| \to 0$ as $\alpha \to 0^+$. Suppose there exists a subsequence, still denoted by $\omega_\alpha$, such that $\lim_{\alpha\to 0} \|\omega_\alpha\|^2 = l > 0$. Since $0 \le v_\alpha/(v_\alpha + \alpha)^\gamma \le v_\alpha^{1-\gamma}$, by the Hölder inequality and subadditivity, from (4.2) one has

$$\begin{aligned}\int_\Omega \frac{v_\alpha}{(v_\alpha + \alpha)^\gamma}\, dx &\le \int_\Omega v_\alpha^{1-\gamma}\, dx \le \int_\Omega |w_\alpha|^{1-\gamma}\, dx + \int_\Omega u_{**}^{1-\gamma}\, dx \\ &\le \|w_\alpha\|_2^{1-\gamma} |\Omega|^{(1+\gamma)/2} + \int_\Omega u_{**}^{1-\gamma}\, dx \\ &\le \int_\Omega u_{**}^{1-\gamma}\, dx + o(1).\end{aligned}$$

Similarly,

$$\int_\Omega u_{**}^{1-\gamma}\, dx \le \int_\Omega \frac{v_\alpha}{(v_\alpha + \alpha)^\gamma}\, dx + o(1).$$

Hence

$$(4.9)\quad \lim_{\alpha \to 0^+} \int_\Omega \frac{v_\alpha}{(v_\alpha + \alpha)^\gamma}\, dx = \int_\Omega u_{**}^{1-\gamma}\, dx.$$

Since $v_\alpha$ is a positive solution of problem (3.1), one has

$$\int_\Omega |\nabla v_\alpha|^2\, dx - \int_\Omega Q(x) v_\alpha^{2^*}\, dx - \lambda \int_\Omega \frac{v_\alpha}{(v_\alpha + \alpha)^\gamma}\, dx = 0.$$

Consequently, according to Lemma A and (4.9),

$$\|\omega_\alpha\|^2+\|u_{**}\|^2-\int_\Omega Q(x)\omega_\alpha^{2^*}\,dx-\int_\Omega Q(x)u_{**}^{2^*}\,dx-\lambda\int_\Omega u_{**}^{1-\gamma}\,dx=o(1). \tag{4.10}$$

Taking $\phi=u_{**}$ in (4.8), one has

$$\|u_{**}\|^2-\int_\Omega Q(x)u_{**}^{2^*}\,dx-\lambda\int_\Omega u_{**}^{1-\gamma}\,dx=0,$$

and (4.10) implies that

$$\|\omega_\alpha\|^2-\int_\Omega Q(x)\omega_\alpha^{2^*}\,dx=o(1). \tag{4.11}$$

Thus

$$\lim_{\alpha\to0^+}\|\omega_\alpha\|^2=\lim_{\alpha\to0^+}\int_\Omega Q(x)\omega_\alpha^{2^*}\,dx=l>0.$$

Since

$$\int_\Omega|\omega_\alpha|^{2^*}\,dx\ge\int_\Omega\frac{Q(x)}{Q_M}|\omega_\alpha|^{2^*}\,dx\ge\int_\Omega\frac{Q(x)}{Q_M}(\omega_\alpha^+)^{2^*}\,dx,$$

we get $\|\omega_\alpha\|_{2^*}^{2^*}\ge l/Q_M$ as $\alpha\to0^+$. Applying the Sobolev inequality, from (1.3) one obtains

$$\|\omega_\alpha\|^2\ge S\|\omega_\alpha\|_{2^*}^2.$$

Then $l\ge S(l/Q_M)^{2/2^*}$, which implies that $l\ge S^{(N)/2}/Q_M^{(N-2)/2}$. On the one hand,

$$\begin{aligned}I(u_{**})&=\frac12\|u_{**}\|^2-\frac1{2^*}\int_\Omega Q(x)u_{**}^{2^*}\,dx-\frac{\lambda}{1-\gamma}\int_\Omega u_{**}^{1-\gamma}\,dx\\&=\frac1N\|u_{**}\|^2-\lambda\left(\frac1{1-\gamma}-\frac1{2^*}\right)\int_\Omega u_{**}^{1-\gamma}\,dx\\&\ge\frac1N\|u_{**}\|^2-C_9\lambda\|u_{**}\|^{1-\gamma}\ge-B\lambda^{2/(1+\gamma)},\end{aligned} \tag{4.12}$$

where the last inequality follows from the Young inequality. On the other hand, from

$$I_\alpha(v_\alpha)<\frac{S^{N/2}}{NQ_M^{(N-2)/2}}-B\lambda^{2/(1+\gamma)}\quad\text{and}\quad l\ge\frac{S^{N/2}}{Q_M^{(N-2)/2}},$$

it follows by (4.10) and (4.11) that

$$\begin{aligned}I(u_{**})&=I_\alpha(v_\alpha)-\frac1N\|\omega_\alpha\|^2+o(1)\\&\le\frac1N\left(\frac{S^{N/2}}{NQ_M^{(N-2)/2}}-l\right)-B\lambda^{2/(1+\gamma)}\le-B\lambda^{2/(1+\gamma)},\end{aligned}$$

which contradicts (4.12). Thus our claim holds.

Consequently, from (3.19), we obtain

$$I(u_{**}) = \lim_{\alpha \to 0} I_\alpha(v_\alpha) > \rho > 0.$$

Therefore, $u_*$ and $u_{**}$ are two different solutions of problem (1.1). Moreover, $u_{**} \not\equiv 0$. Combining this with $u_{**}(x) \geq 0$ and (4.8), by the strong maximum principle one has $u_{**}(x) > 0$ a.e. $x \in \Omega$. Thus $u_{**}$ is a positive solution of problem (1.1). This completes the proof of Theorem 1.1. ∎

**Acknowledgements.** The authors express their gratitude to the reviewers for careful reading and helpful suggestions which led to an improvement of the original manuscript.

This research was partly supported by National Natural Science Foundation of China (No. 11471267) and the Science Foundation of Education of Guizhou Province (No. LKZS[2014]30; No. LH[2015]7049; No. LH[2015] 7001).

## References

[1] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. 14 (1973), 349–381.

[2] D. Arcoya and L. Moreno-Mérida, *Multiplicity of solutions for a Dirichlet problem with a strongly singular nonlinearity*, Nonlinear Anal. 95 (2014), 281–291.

[3] L. Boccardo, *A Dirichlet problem with singular and supercritical nonlinearities*, Nonlinear Anal. 75 (2012), 4436–4440.

[4] H. Brézis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. 88 (1983), 486–490.

[5] H. Brézis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. 36 (1983), 437–477.

[6] M. M. Coclite and G. Palmieri, *On a singular nonlinear Dirichlet problem*, Comm. Partial Differential Equations 14 (1989), 1315–1327.

[7] M. G. Crandall, P. H. Rabinowitz and L. Tartar, *On a Dirichlet problem with a singular nonlinearity*, Comm. Partial Differential Equations 2 (1977), 193–222.

[8] A. Edelson, *Entire solutions of singular elliptic equations*, J. Math. Anal. Appl. 139 (1989), 523–532.

[9] C. F. Gui and F. H. Lin, *Regularity of an elliptic problem with a singular nonlinearity*, Proc. Roy. Soc. Edinburgh Sect. A 123 (1993), 1021–1029.

[10] N. Hirano, C. Saccon and N. Shioji, *Existence of multiple positive solutions for singular elliptic problems with concave and convex nonlinearities*, Adv. Differential Equations 9 (2004), 197–220.

[11] N. Hirano, C. Saccon and N. Shioji, *Brezis–Nirenberg type theorems and multiplicity of positive solutions for a singular elliptic problem*, J. Differential Equations 245 (2008), 1997–2037.

[12] A. V. Lair and A. W. Shaker, *Classical and weak solutions of a singular semilinear elliptic problem*, J. Math. Anal. Appl. 211 (1997), 371–385.

[13] A. C. Lazer and P. J. McKenna, *On a singular nonlinear elliptic boundary value problem*, Proc. Amer. Math. Soc. 111 (1991), 721–730.

[14] Y. J. Sun and S. J. Li, *Structure of ground state solutions of singular semilinear elliptic equations*, Nonlinear Anal. 55 (2003), 399–417.
[15] Y. J. Sun and S. P. Wu, *An exact estimate result for a class of singular equations with critical exponents*, J. Funct. Anal. 260 (2011), 1257–1284.
[16] Y. J. Sun, S. P. Wu and Y. M. Long, *Combined effects of singular and superlinear nonlinearities in some singular boundary value problems*, J. Differential Equations 176 (2001), 511–531.
[17] X. Wang, L. Zhao and P. H. Zhao, *Combined effects of singular and critical nonlinearities in elliptic problems*, Nonlinear Anal. 87 (2013), 1–10.
[18] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.
[19] H. T. Yang, *Multiplicity and asymptotic behavior of positive solutions for a singular semilinear elliptic problem*, J. Differential Equations 189 (2003), 487–512.

Jia-Feng Liao
School of Mathematics and Statistics
Southwest University
400715 Chongqing
People's Republic of China
and
School of Mathematics
and Computational Science
Zunyi Normal College
563002 Zunyi, Guizhou
People's Republic of China
E-mail: liaojiafeng@163.com

Jiu Liu, Chun-Lei Tang (corresponding author)
School of Mathematics and Statistics
Southwest University
400715 Chongqing
People's Republic of China
E-mail: jiuliu2011@163.com
tangcl@swu.edu.cn

Peng Zhang
School of Mathematics
and Computational Science
Zunyi Normal College
563002 Zunyi, Guizhou
People's Republic of China
E-mail: gzzypd@sina.com