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THE R_2 MEASURE FOR TOTALLY POSITIVE ALGEBRAIC INTEGERS

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Abstract. Let α be a totally positive algebraic integer of degree d, i.e., all of its conjugates $\alpha_1 = \alpha, \dots, \alpha_d$ are positive real numbers. We study the set $\mathcal{R}_2$ of the quantities $(\prod_{i=1}^d (1+\alpha_i^2)^{1/2})^{1/d}$. We first show that $\sqrt{2}$ is the smallest point of $\mathcal{R}_2$. Then, we prove that there exists a number l such that $\mathcal{R}_2$ is dense in (l, ∞). Finally, using the method of auxiliary functions, we find the six smallest points of $\mathcal{R}_2$ in $(\sqrt{2}, l)$. The polynomials involved in the auxiliary function are found by a recursive algorithm.

1. Introduction.

Let $P(x) = a_0x^d + \dots + a_d = a_0(x-\alpha_1)\cdots(x-\alpha_d)$, $a_0 \neq 0$, $P \neq x$, be a polynomial with complex coefficients. M. Langevin [La] defined three families of measures of polynomials, for $p > 0$:

$$\mathrm{M}_p(P) = \Big(\int_0^1 |P(e^{2i\pi t})|^p\, dt\Big)^{1/p},$$

$$\mathrm{L}_p(P) = \Big(\sum_{i=1}^d |a_i|^p\Big)^{1/p},$$

$$\mathrm{R}_p(P) = |a_0| \prod_{i=1}^d (1+|\alpha_i|^p)^{1/p}.$$

Note that $\lim_{p\to 0} \mathrm{M}(P) = \exp(\int_0^1 \log |P(e^{2i\pi t})|\, dt)$ is the well known *Mahler measure* of P and $\mathrm{L}_1(P)$ is the well known *length* of P.

In this paper, we are interested in the R_2 *measure of* P, which is $\mathrm{R}_2(P) = |a_0| \prod_{i=1}^d (1+|\alpha_i|^2)^{1/2}$. If α is an algebraic integer, the R_2 *measure of* α is the R_2 measure of its minimal polynomial. The *absolute* R_2 *measure of* α is the quantity $\mathrm{r}_2(\alpha) = \mathrm{R}_2(\alpha)^{1/\deg(\alpha)}$.

From a well known theorem of Kronecker [Kr], it is easy to prove that if α is an algebraic integer, then $\mathrm{r}_2(\alpha) = \sqrt{2}$ if and only if α is a root of unity.

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Now, we suppose that α is a totally positive algebraic integer (all of its conjugates are positive real numbers). We have

THEOREM 1. *If α is a nonzero totally positive algebraic integer then* $\mathrm{r}_2(\alpha) \geq \sqrt{2}$. *Equality holds if and only if $\alpha = 1$.*

This follows immediately from an inequality due to K. Mahler,

$$\Big(\prod_{i=1}^{d}(u_i+v_i)\Big)^{1/d} \geq \Big(\prod_{i=1}^{d} u_i\Big)^{1/d} + \Big(\prod_{i=1}^{d} v_i\Big)^{1/d} \quad \text{for } u_i, v_i > 0.$$

In order to study the structure of the set $\mathcal{R}_2$ of the quantities $\mathrm{r}_2(\alpha)$, we show the following

THEOREM 2. *$\mathcal{R}_2$ is dense in (l, ∞) where $l = \lim_{n\to\infty} \mathrm{r}_2(\beta_n^2)$.*

Here the β_n^2 were defined by C. J. Smyth [Sm1] as follows:

$$\beta_0^2 = 1, \qquad \beta_n^2 = \beta_{n+1}^2 + \beta_{n+1}^{-2} - 2.$$

β_n^2 is a totally positive algebraic integer of degree 2^n.

Towards determining the structure of $\mathcal{R}_2$ in the gap $(\sqrt{2}, l)$, we prove the following

THEOREM 3. *If α is a totally positive algebraic integer whose minimal polynomial is different from $x-1$, x^2-3x+1, $x^4-7x^3+13x^2-7x+1$, $x^8-15x^7+83x^6-220x^5+303x^4-220x^3+83x^2-15x+1$, $x^6-11x^5+41x^4-63x^3+41x^2-11x+1$ and $x^8-15x^7+84x^6-225x^5+311x^4-225x^3+84x^2-15x+1$, then*

$$\mathrm{r}_2(\alpha) \geq 1.866755.$$

COROLLARY 4. *The six smallest points of $\mathcal{R}_2$ in $(\sqrt{2}, l)$ are:*

$1.4142136\ldots = \mathrm{r}_2(x-1) = \mathrm{r}_2(\beta_0^2)$,

$1.7320508\ldots = \mathrm{r}_2(x^2-3x+1) = \mathrm{r}_2(\beta_1^2)$,

$1.8211603\ldots = \mathrm{r}_2(x^4-7x^3+13x^2-7x+1) = \mathrm{r}_2(\beta_2^2)$,

$1.8530061\ldots = \mathrm{r}_2(x^8-15x^7+83x^6-220x^5+303x^4-220x^3+83x^2-15x+1) = \mathrm{r}_2(\beta_3^2)$,

$1.8569376\ldots = \mathrm{r}_2(x^6-11x^5+41x^4-63x^3+41x^2-11x+1)$,

$1.8628205\ldots = \mathrm{r}_2(x^8-15x^7+84x^6-225x^5+311x^4-225x^3+84x^2-15x+1)$.

We conjecture that the next point has minimal polynomial $x^{14}-27x^{13}+308x^{12}-1963x^{11}+7790x^{10}-20307x^9+35763x^8-43131x^7+35763x^6-20307x^5+7790x^4-1963x^3+308x^2-27x+1$ and R_2 measure 1.8698925.

Section 2 deals with the denseness of the set $\mathcal{R}_2$. In Section 3, we describe the method of explicit auxiliary functions. We link these functions with the integer transfinite diameter. Then, we give a recursive algorithm which enables us to obtain the constant of Theorem 3. All the computations were done on a MacBookPro with the languages Pascal and Pari.

2. Denseness of the set $\mathcal{R}_2$

2.1. Study of the sequence $(\mathrm{r}_2(\beta_n^2))_{n\geq 0}$**.** We first prove the following

LEMMA 5.

$$r_2(\beta_n^2) = \Big(2\prod_{i=1}^{n-1}(1+\lambda_i)^{1/2^i}\Big)^{1/2}$$

where

$$\lambda_0 = \frac{1}{2} \quad \textit{and} \quad \lambda_{i+1} = \frac{\lambda_i}{(1+\lambda_i)^2} \quad \textit{for } i \geq 0.$$

Proof. For $n \geq 0$, we set $\gamma_n = \beta_n^2$, so $\gamma_n = \gamma_{n+1} + \gamma_{n+1}^{-1} - 2$ and $\gamma_{n+1}^2 + \gamma_{n+1}^{-2} = \gamma_n^2 + 4\gamma_n + 2$. Therefore, we can write

$$\mathrm{R}_2(\beta_n^2) = \mathrm{R}_2(\gamma_n) = \prod_{i=1}^{2^n}(1+\gamma_{n,i}^2)^{1/2}$$

where, for $1 \leq i \leq 2^n$, $\gamma_{n,i}$ denote the conjugates of γ_n. Then we have

$$\begin{aligned}\mathrm{R}_2(\beta_n^2) &= \prod_{i=1}^{2^{n-1}}\big((1+\gamma_{n,i}^2)(1+\gamma_{n,i}^{-2})\big)^{1/2} = \prod_{i=1}^{2^{n-1}}\big(2+\gamma_{n,i}^2+\gamma_{n,i}^{-2}\big)^{1/2}\\ &= \prod_{i=1}^{2^{n-1}}(2+\gamma_{n-1,i}^2+4\gamma_{n-1,i}+2)^{1/2} = \prod_{i=1}^{2^{n-1}}(\gamma_{n-1,i}+2)\\ &= 2^{2^{n-1}}\prod_{i=1}^{2^{n-1}}\Big(1+\frac{1}{2}\gamma_{n-1,i}\Big).\end{aligned}$$

Then the result follows immediately from the following more general lemma that we proved in [F]:

LEMMA 6. *Under the above notation,*

$$\prod_{i=0}^{2^n}(1+\lambda_0\gamma_{n,i}) = \Big(\prod_{i=0}^{n}(1+\lambda_i)^{1/2^i}\Big)^{2^n}.$$

The lemma shows that the sequence $(\mathrm{r}_2(\beta_n^2))_{n\geq 0}$ is increasing. Furthermore, as $\log(1+x) \leq x$ for all $x \geq 0$, we have $\log \mathrm{r}_2(\beta_n^2) \leq \frac{1}{2} + \sum_{i=0}^{n-1}\frac{\lambda_i}{2^i}$. The series $\sum_{i=0}^{n-1}\frac{\lambda_i}{2^i}$ is convergent because $0 \leq \lambda_i \leq 1$ for $i \geq 0$.

Thus, the sequence $(\mathrm{r}_2(\beta_n^2))_{n\geq 0}$ is also convergent and its limit is $l = 1.874348\ldots$. Note that l gives an upper bound for the first accumulation point of $\mathcal{R}_2$.

2.2. Proof of Theorem 2. The proof and notation follow those of C. J. Smyth [Sm1]. For a given function $g : [0,\infty) \to \mathbb{R}$, let $\mathcal{M}(g)$ be the set of all means

$$\mathrm{M}_g(\alpha) = \frac{1}{d}\sum_{i=1}^{d} g(|\alpha_i|)$$

for α a totally real algebraic integer, i.e., all its conjugates $\alpha_1 = \alpha, \dots, \alpha_d$ are real numbers. When the limits exist, set

$$a(g) = \lim_{n\to\infty} \mathrm{M}_g(\beta_n) \quad \text{and} \quad c(g) = \lim_{n\to\infty} \mathrm{M}_g(2\cos(2\pi/n)).$$

Here a convenient choice for g is $g : x \mapsto \frac{1}{2}\log(1+x^4)$ because then $\mathrm{M}_g(\alpha) = \log \mathrm{r}_2(\alpha^2)$.

The proof consists of two parts.

2.2.1. *First step of the proof.* C. J. Smyth [Sm1] proved the following

THEOREM 7. *Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function, zero on $[0,1]$, such that*

$$\lim_{x\to\infty} g(x+1)/g(x) = 1$$

and the values of $\log_2 g(2k+1) \bmod 1$ $(k = 0, 1, \dots)$ are everywhere dense in $(0,1)$. Then the limit $a(g)$ exists and $\mathcal{M}(g)$ is dense in $(a(g), \infty)$.

We replace the function g by the function g^* which satisfies the hypothesis of Theorem 7:

$$g^*(x) = \begin{cases} g(x) + g(1/x) & \text{if } x > 1, \\ 0 & \text{if } 0 \le x \le 1. \end{cases}$$

As $\beta_{n,i}^{-1}$ or $-\beta_{n,i}^{-1}$ is a conjugate of $\beta_{n,i}$, we have

$$\mathrm{M}_g(\beta_n) = \frac{1}{2^n}\sum_{i=1}^{2^n} g(\beta_{n,i}) = \frac{1}{2^n}\sum_{i=1}^{2^{n-1}} \bigl(g(\beta_{n,i}) + g(\beta_{n,i}^{-1})\bigr) = \mathrm{M}_{g^*}(\beta_n).$$

Thus, the existence of $a(g^*)$ implies that of $a(g)$, and $a(g^*) = a(g)$.

It is easy to see that g^* satisfies the first hypothesis of Theorem 7. So, it is sufficient to study the denseness of the set $\mathcal{F} = \{\log_2 g(2k+1) \bmod 1 \colon k \in \mathbb{N}\}$.

Let $t \in [0,1]$ and $\epsilon > 0$. Does there exist $f \in \mathcal{F}$ such that $|f - t| < \epsilon$? We search for n and k satisfying

$$|\log_2 g^*(2k+1) - t - n| < \epsilon,$$

i.e.,

$$|\log g^*(2k+1) - t' - n\log 2| < \epsilon'. \tag{2.1}$$

The uniform continuity of log on $[1,\infty)$ gives

$$\forall \epsilon' > 0\ \exists \eta(\epsilon')\ \forall x, y > 0, \quad |x - y| < \eta(\epsilon') \Rightarrow |\log x - \log y| < \epsilon'.$$

We choose n with $2^{-n} < \eta(\epsilon')$ and k such that $|(2k+1)-(g^*)^{-1}(2^n e^{t'})| \le 1$. As $(g^*)'$ is bounded by 1, the mean value theorem for g^* on $(1,\infty)$ gives

$$|g^*(2k+1) - 2^n e^{t'}| \le 1,$$

i.e.,

$$|2^{-n} g^*(2k+1) - e^{t'}| \le 2^{-n} < \eta(\epsilon'),$$

and the inequality (2.1) follows immediately. Thus, we have proved that $\mathcal{M}(g)$ is dense in $(a(g^*),\infty) = (a(g),\infty)$.

2.2.2. *Second step of the proof.* C. J. Smyth [Sm1] established the following

THEOREM 8. *Let $g:\mathbb{R}_+ \to \mathbb{R}_+$ be a function such that $\lim_{x\to\infty} g(x) = \infty$ and which satisfies a Lipschitz condition*

$$|g(x) - g(y)| < B(\lambda)|x-y|$$

for $x,y \in [0,\lambda]$, for each $\lambda > 0$. Then $\mathcal{M}(g)$ is dense in $(c(g),\infty)$, where

$$c(g) = \frac{2}{\pi}\int_0^{\pi/2} g(2\cos\theta)\,d\theta.$$

It is easy to see that, for our function g, the Lipschitz condition is satisfied with $B(\lambda) = 4\lambda^3$.

2.2.3. *Conclusion.* We have shown that $\mathcal{M}(g)$ is dense in the interval $(\min(a(g),c(g)),\infty)$, which means that $\mathcal{R}_2$ is dense in (l,∞), where $l = \lim_{n\to\infty} r_2(\beta_n^2) = 1.874348\ldots$.

3. Proof of Theorem 3

3.1. The explicit auxiliary function. The auxiliary function involved in Theorem 3 is of the following type:

$$f(x) = \frac{1}{2}\log(1+x^2) - c_0\log x - \sum_{1\le j\le J} c_j \log|Q_j(x)| \quad \text{for } x > 0, \tag{3.1}$$

where the c_j are positive real numbers and the Q_j are nonzero polynomials in $\mathbb{Z}[x]$.

Let α be a totally positive algebraic integer with conjugates $\alpha_1 = \alpha,\ldots,\alpha_d$ and minimal polynomial P. Then

$$\sum_{i=1}^{d} f(\alpha_i) \ge md,$$

where m denotes the minimum of the function f, i.e.,

$$\log \mathrm{R}_2(\alpha) \ge md + \sum_{1\le j\le J} c_j \log\Big|\prod_{i=1}^{d} Q_j(\alpha_i)\Big|.$$

We assume that P does not divide any Q_j. Then $\prod_{i=1}^{d} Q_j(\alpha_i)$ is a nonzero integer because it is the resultant of P and Q_j.

Therefore, if α is not a root of Q_j, we have

$$\mathrm{r}_2(\alpha) \geq e^m.$$

It is possible to reduce the domain of study of the function f. If we consider the function $g(x) = \frac{1}{2}[f(x) + f(1/x)]$, we get a minimum greater than or equal to that of f. But g is invariant under the change $x \mapsto 1/x$, so it is sufficient to study g on $(0, 1)$. Thus, without loss of generality, we can limit our study to auxiliary functions invariant under this transformation. This implies that we can take for Q_j reciprocal polynomials, i.e., $Q_j(x) = x^{\deg Q_j} Q_j(1/x)$. The condition $f(x) = f(1/x)$ gives $2c_0 + \sum_{1\leq j\leq J} c_j \deg(Q_j) = 1$.

We denote $\deg(Q_j) = 2d_j$ for $1 \leq j \leq J$.

On $(0, 1)$, the auxiliary function f can be written

$$f(x) = \frac{1}{2}\log x + \frac{1}{2}\log(x + 1/x) - c_0 \log x - \sum_{1\leq j\leq J} c_j \log\left|\frac{Q_j(x)}{x^{d_j}}\right| - \sum_{1\leq j\leq J} c_j \log x^{d_j} \geq m.$$

Thus, if we set $y = x + 1/x - 2$, $f(x)$ becomes

$$g(y) = \frac{1}{2}\log(y + 2) - \sum_{1\leq j\leq J} c_j \log|U_j(y)| \geq m \quad \text{for } y > 0,$$

where $\deg(U_j) = d_j$.

The main problem is to find a good list of polynomials Q_j which gives a value of m as large as possible. Thus, we link the auxiliary function with the integer transfinite diameter in order to find the polynomials by means of our recursive algorithm.

3.2. Auxiliary functions and integer transfinite diameter. In this section, we shall need the following definition. Let K be a compact subset of $\mathbb{C}$. If φ is a positive function defined on K, the *φ-integer transfinite diameter of K* is defined as

$$t_{\mathbb{Z},\varphi}(K) = \liminf_{\substack{n\geq 1\\ n\to\infty}} \inf_{\substack{P\in\mathbb{Z}[Y]\\ \deg(P)=n}} \sup_{y\in K} |P(y)|^{1/n}\varphi(y).$$

This weighted version of the integer transfinite diameter was introduced by F. Amoroso [A] and is an important tool in the study of rational approximation of logarithms of rational numbers.

In the auxiliary function (3.1), we replace the numbers c_j by rational numbers. Then we can write

$$f(y) = \frac{1}{2}\log(y+2) - \frac{t}{r}\log|Q(y)| \geq m \quad \text{for } y > 0, \tag{3.2}$$

where $Q \in \mathbb{Z}[Y]$ is of degree r and t is a positive real number. We want to get a function whose minimum m is as large as possible. Thus we search for a polynomial $Q \in \mathbb{Z}[Y]$ such that

$$\sup_{y>0} |Q(y)|^{t/r}(y+2)^{-1/2} \leq e^{-m}.$$

If we suppose that t is fixed, it is clear that we need an effective upper bound for the quantity

$$t_{\mathbb{Z},\varphi}((0,\infty)) = \liminf_{\substack{r\geq 1\\ r\to\infty}} \inf_{\substack{P\in\mathbb{Z}[Y]\\ \deg(P)=r}} \sup_{y>0} |P(y)|^{t/r}\varphi(y)$$

where we use the weight $\varphi(y) = (y+2)^{-1/2}$.

Even if we replace the compact subset K by the infinite interval $(0,\infty)$, the weight φ ensures that the quantity $t_{\mathbb{Z},\varphi}((0,\infty))$ is finite.

3.3. Construction of the auxiliary function. The improvement compared with Wu's algorithm is that our polynomials are obtained by induction. Suppose that we have $Q_1, \ldots, Q_J$. Then we use semi-infinite linear programming (introduced in number theory by C. J. Smyth [Sm2]) to optimize f for this set of polynomials (i.e., to get the greatest possible m). We obtain the numbers $c_1, \ldots, c_J$ and f in the form (3.2) with $t = \sum_{i=1}^{J} c_j \deg(Q_j)$.

For several values of k, we seek a polynomial $R(y) = \sum_{l=0}^{k} a_l y^l \in \mathbb{Z}[y]$ such that

$$\sup_{y>0} |Q(y)R(y)|^{t/(r+k)}(y+2)^{-1/2} \leq e^{-m},$$

i.e., such that

$$\sup_{y>0} |Q(y)R(y)|(y+2)^{-(r+k)/2t}$$

is as small as possible.

We apply the LLL algorithm to the linear forms in $a_0, \ldots, a_k$

$$Q(y_i)R(y_i)(y_i+2)^{-(r+k)/2t}$$

where y_i are control points uniformly distributed in the interval $[0, 70]$, including the points where f has its least local minima. We get a polynomial R whose factors R_j are good candidates to enlarge the set of polynomials $(Q_1, \ldots, Q_J)$. We only keep the polynomials R_j which have a nonzero coefficient c_j in the new optimized auxiliary function f. After optimization, some previous polynomials Q_j may have a zero coefficient and are removed.

Table 1

j	c_j	d_j	Highest half coefficients of Q_j
1	0.097723	2	1 −2
2	0.051674	2	1 −3
3	0.000533	2	1 −4 1
4	0.017814	4	1 −7 13
5	0.000985	4	1 −8 15
6	0.003163	6	1 −11 41 −63
7	0.000202	6	1 −12 48 −77
8	0.000371	6	1 −12 44 −67
9	0.001273	8	1 −15 84 −225 311
10	0.000221	8	1 −16 91 −244 337
11	0.000131	8	1 −16 92 −249 345
12	0.000060	8	1 −16 92 −248 343
13	0.006621	8	1 −15 83 −220 303
14	0.000284	10	1 −19 143 −557 1231 −1599
15	0.000069	10	1 −19 142 −548 1202 −1557
16	0.000418	12	1 −23 218 −1118 3438 −6651 8271
17	0.000145	12	1 −23 218 −1119 3446 −6675 8305
18	0.000044	14	1 −27 308 −1964 7800 −20348 35853 −43247 35853
19	0.000017	14	1 −27 309 −1979 7893 −20661 36484 −44041 36484
20	0.000023	14	1 −26 289 −1812 7124 −18484 32488 −39161
21	0.000202	14	1 −27 308 −1963 7790 −20307 35763 −43131
22	0.000496	14	1 −26 290 −1826 7205 −18741 32986 −39779
23	0.000278	14	1 −27 308 −1965 7812 −20404 35986 −43423
24	0.000376	14	1 −27 309 −1979 7894 −20668 36503 −44067
25	0.000232	16	1 −31 415 −3177 15538 −51389 118680 −194903 229733
26	0.000290	16	1 −30 391 −2932 14123 −46215 106000 −173418 204161
27	0.000092	16	1 −31 414 −3160 15414 −50875 117330 −192534 226883
28	0.000043	16	1 −31 415 −3179 15566 −51554 119216 −195961 231055
29	0.001203	16	1 −31 413 −3141 15261 −50187 115410 −189036 222621
30	0.000084	16	2 −58 732 −5330 25023 −80175 181020 −293277 344127
31	0.000045	18	1 −35 541 −4891 28887 −117982 344282 −731869 1146235 −1330340
32	0.000160	20	1 −38 645 −6492 43388 −204358 702800 −1804604 3509324 −5213890 5946449

In order to get the constant of Theorem 3, we take k from 4 to 15 successively.

The polynomials Q_j of degree d_j and the coefficients c_j involved in the auxiliary function of Theorem 3 are listed in Table 1. Only polynomials numbered 1, 2, 4, 6, 9 and 13 from the list have r_2 measure less than the constant in the theorem.

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