ACTA ARITHMETICA
173.1 (2016)

Shimura lifting on weak Maass forms

by

Youngju Choie (Pohang) and Subong Lim (Seoul)

1. Introduction. In his famous paper [12], Shimura constructed a map between modular forms of a half-integral weight and modular forms of an even integral weight, which is now called the *Shimura lifting*. This lifting has played an important role in many areas of modern number theory (for example, see [11, 14]). An important turning point was understanding it as a theta lifting. Shintani [13] showed how to construct a lifting from the theta series associated with an indefinite quadratic form; subsequently, Niwa [10] studied the Shimura lifting using the theta lifting, and Cipra [7] extended the Shimura lifting to all positive half-integral weights.

Since the theta lifting is given by integration against a two-variable theta series, only cusp forms have been used because the integral is not well-defined for forms which permit bad behavior at the cusps (called "weak forms"). To solve this problem, Borcherds [1] used the regularized integral introduced by Harvey and Moore [8] to construct modular forms of integral weight on a higher-dimensional Hermitian homogeneous domain. After the breakthrough of Borcherds, many researchers began to study the theta lifting on weak forms and found interesting applications. Two examples are the work of Bruinier and Funke [5] on traces of CM values of modular functions and the recent work of Bruinier and Ono [6] on central values and derivatives of quadratic twists of weight 2 modular L-functions.

In this note, we use the regularized integral to treat the Shimura lifting for weak Maass forms of arbitrary positive half-integral weight with arbitrary eigenvalue that satisfy a certain growth condition at the cusps. The resulting functions are automorphic with possible singularities on the upper half-plane $\mathbb{H}$. We also study their convergence regions and singularity types. Finally, we compute the value of the lifted function on the imaginary axis.

2010 *Mathematics Subject Classification*: Primary 11F27.
Key words and phrases: Shimura lifting, weak Maass forms, regularized integral.
Received 4 September 2014; revised 13 October 2015.
Published online 25 March 2016.

DOI: 10.4064/aa7916-12-2015

Let Γ be a congruence subgroup of level N, let $k \in \mathbb{Z}$ and let χ be a character modulo N. Assume that $4 \mid N$ if k is not an even integer. A *weak Maass form* of weight $k/2$ on Γ is a real analytic function on $\mathbb{H}$ with possible poles at cusps that transforms like a modular form of weight $k/2$ and is an eigenfunction of the weight $k/2$ Laplace operator

$$\Delta_{k/2} := -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + \frac{k}{2} iv \left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right),$$

where $u = \operatorname{Re} z$ and $v = \operatorname{Im} z$ for $z \in \mathbb{H}$. If the eigenvalue λ is zero, then a weak Maass form is called *harmonic*. We denote the space of weak Maass forms of weight $k/2$ on Γ with eigenvalue $\lambda \in \mathbb{C}$ and character χ by $\mathrm{WMF}_{k/2,\lambda}(\Gamma, \chi)$. Furthermore, let $\mathrm{WMF}^*_{k/2,\lambda}(\Gamma, \chi)$ be the subspace consisting of forms satisfying a certain condition on Fourier coefficients (for the precise definition see Section 2).

The groups $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{O}(2,1)$ form a dual reductive pair in the sense of Howe [9]. Thus we can consider the theta lifting given by the Siegel theta function $\theta(z, w)$. If k is a positive odd integer and g is a weak Maass form of weight $k/2$ on $\Gamma_0(4N)$, then we define

$$\Phi(g)(w) := \int_{\mathcal{F}}^{\mathrm{reg}} \sum_{\alpha \in \Gamma_0(4N)\backslash \mathrm{SL}_2(\mathbb{Z})} g_\alpha(z)\overline{\theta_\alpha(z,w)} v^{k/2} \frac{du\,dv}{v^2} \tag{1.1}$$

using the regularized integral, where $\mathcal{F}$ denotes the standard fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$, and $g_\alpha(z) = (cz+d)^{-k/2} g\big(\frac{az+b}{cz+d}\big)$ and $\theta_\alpha(z,w) = (cz+d)^{-k/2}\theta\big(\frac{az+b}{cz+d}, w\big)$ for $\alpha = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. This gives an extension of the Shimura lifting to weak Maass forms.

THEOREM 1.1. *Let k be a positive odd integer, let $\lambda \in \mathbb{C}$ and let χ be a character modulo $4N$. For a weak Maass form $g \in \mathrm{WMF}^*_{k/2,\lambda}(\Gamma_0(4N), \chi)$, the lifted function $\Phi(g)$ in* (1.1) *is a singular weak Maass form of weight $k-1$ on $\Gamma_0(2N)$ with eigenvalue 4λ and character χ^2, and has singularities supported on $\big\{\frac{x_2+\sqrt{x_2^2-x_1x_3}}{2Nx_3} \mid x_1, x_2, x_3 \in \mathbb{Z}$ and $x_2^2 - x_1x_3 < 0\big\}$. In particular, $\Phi(g)$ is harmonic if g is harmonic.*

Here, a *singular weak Maass form* is a function that has all the properties of a weak Maass form except that it may have singularities on $\mathbb{H}$ (for the precise definition see Section 2). The main part of the proof consists of analyzing the constant term of the function

$$\sum_{\alpha \in \Gamma_0(4N)\backslash \mathrm{SL}_2(\mathbb{Z})} g_\alpha(z)\overline{\theta_\alpha(z,w)}.$$

Furthermore, we determine the singularity type of $\Phi(g)$. Let $U \subset \mathbb{H}$ be an open subset, and let f and g be functions on a dense open subset of U. Then we say that f *has singularity type* g if $f - g$ can be continued to a real analytic function on U. For an open subset U of $\mathbb{H}$ define

$$\mathcal{S}(U) := \left\{ (x_1, x_2, x_3) \in \mathbb{Z}^3 \;\middle|\; x_2^2 - x_1x_3 < 0 \text{ and } \frac{x_2 + \sqrt{x_2^2 - x_1x_3}}{2Nx_3} \in U \right\}.$$

Then we have the following theorem about the singularity type of $\Phi(g)$.

THEOREM 1.2. *Suppose that k, λ, χ and g are as in Theorem 1.1. If $U \subset \mathbb{H}$ is an open subset with compact closure, then the lifted function $\Phi(g)$ has singularity type*

$$\sum_{\alpha \in \Gamma_0(4N)\backslash \mathrm{SL}_2(\mathbb{Z})} \sum_{(x_1,x_2,x_3)\in\mathcal{S}(U)} \tilde{a}_\alpha(x_2^2 - x_1x_3, s_\lambda)\overline{h_\alpha(x,z;k)}(-1)^{-k/4}$$
$$\times \Bigg\{ \sum_{n=0}^{(k-3)/2} \frac{1}{(4\pi(x_2^2 - x_1x_3))^n n!} \left(\frac{\sqrt{\pi}\,|\Lambda(x,z)|}{2N}\right)^{2n-k+1} \Gamma\left(\frac{k-1}{2} - n\right)$$
$$\times \prod_{j=1}^{n} \left[\left(s_\lambda - \frac{1}{2}\right)^2 - \left(\frac{k}{4} - j + \frac{1}{2}\right)^2 \right]$$
$$+ \frac{-1}{(4\pi(x_2^2 - x_1x_3))^{(k-1)/2}\left(\frac{k-1}{2}\right)!} \prod_{j=1}^{(k-1)/2} \left[\left(s_\lambda - \frac{1}{2}\right)^2 - \left(\frac{k}{4} - j + \frac{1}{2}\right)^2 \right]$$
$$\times \log\left(\left(\frac{\sqrt{\pi}}{2N} |\Lambda(x,z)| \right)^2 \right) \Bigg\},$$

where $\tilde{a}_\alpha(n, s_\lambda)$ is a Fourier coefficient of g_α as in (2.1) *and s_λ is given by the relation $\lambda = s_\lambda(1 - s_\lambda) + (k^2 - 2k)/4$ (for the definition of $h_\alpha(x,z;k)$ and $\Lambda(x,z)$ see Section* 3).

On the other hand, using the unfolding method we can compute the value of the lifted function on the imaginary axis.

THEOREM 1.3. *Suppose that k, λ, χ and g are as in Theorem 1.1. Let $a(n, s_\lambda)$ be the Fourier coefficients of g as in* (2.1) *for $\gamma = I$. If $k > 1$, then*

$$\Phi(g)(iv) = C' \sum_{\nu=0}^{(k-1)/2} \binom{(k-1)/2}{\nu} (2\pi)^{-\nu} \int_0^\infty \Bigg(a(0, s_\lambda) \left(\frac{t^2}{8\pi}\right)^{s_\lambda - k/4} H_\nu(0)$$
$$+ \sum_{n\neq 0} a(n^2, s_\lambda) \left|\frac{n^2t^2}{2}\right|^{-k/4} W_{k/4, s_\lambda - 1/2}\left(\frac{n^2t^2}{2}\right) \exp\left(-\frac{n^2t^2}{4}\right) H_\nu(tn) \Bigg) \left(\frac{v}{t}\right)^{1-\nu}$$
$$\times \sum_{m=-\infty}^{\infty} \chi_1(m) m^{(k-1)/2-\nu} \exp\left(-2\pi^2 m^2 \left(\frac{v}{t}\right)^2\right) \frac{dt}{t}$$

for $v > 0$, where $C' = (-1)^{(k-1)/2} 2^{-2k+4} N^{k/4} (2\pi)^{1/2}$, $W_{k,m}$ is the standard W-Whittaker function and H_ν is the Hermite polynomial given by $H_\nu(x) = (-1)^\nu \exp(x^2/2) \frac{d^\nu}{dx^\nu} \exp(-x^2/2)$ for $\nu \in \mathbb{Z}_{\geq 0}$.

This result is comparable to [7, Theorem 2.12], which gives the Fourier expansion of the lifted function when g is a holomorphic modular form. In our case, since the lifted function is not holomorphic and the Fourier coefficients of weak Maass forms do not give an L-function, we cannot deduce the Fourier expansion from this result. But if we assume that $\lambda = 0$, then one can see that only the holomorphic part of g appears (for the definition of holomorphic part, see [4]). This suggests that the Fourier expansion of the lifted function is completely determined by the holomorphic part of g.

The remainder of the paper is organized as follows. In Section 2, we review the basic notions of a weak Maass form and a theta kernel studied by Shintani [13] and Cipra [7], and explain how to regularize integrals. We also define theta liftings for weak Maass forms in the sense of Borcherds [1]. In Section 3 we prove Theorems 1.1–1.3.

2. Review of basic material

2.1. Weak Maass forms. Let k be an integer. For z and w in $\mathbb{H}$ we will often use without further explanation $u = \operatorname{Re} z$, $v = \operatorname{Im} z$, $\xi = \operatorname{Re} w$, and $\eta = \operatorname{Im} w$. Let

$$j(\gamma, z) = \epsilon_d^{-1} \left(\frac{c}{d}\right) (cz+d)^{1/2} \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4),$$

where $\epsilon_d = 1$ or i as $d \equiv 1$ or $3 \pmod 4$, and $\left(\frac{c}{d}\right)$ is the quadratic residue symbol as defined in [12]. We have a slash operator defined by

$$(f|_{k/2}\gamma)(z) := \begin{cases} (cz+d)^{-k/2} f(\gamma z) & \text{if } k \text{ is even and } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}), \\ j(\gamma, z)^{-k} f(\gamma z) & \text{if } k \text{ is odd and } \gamma \in \Gamma_0(4). \end{cases}$$

Now we introduce the definition of a weak Maass form.

DEFINITION 2.1. Let Γ be a congruence subgroup of level N and χ a character modulo N. A *singular weak Maass form* of weight $k/2$ on Γ with eigenvalue $\lambda \in \mathbb{C}$ and character χ is a real analytic function $f : \mathbb{H} \to \mathbb{C}$ with possible singularities satisfying

(1) $f|_{k/2}\gamma = \chi(d) f$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,
(2) $\Delta_{k/2} f = \lambda f$, and
(3) $(f|_{k/2}\gamma)(z) = O(v^\delta)$ as $v \to \infty$, uniformly in u for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, for some fixed real $\delta > 0$.

We denote by $\mathrm{WMF}^s_{k/2,\lambda}(\Gamma, \chi)$ the space of such forms. If f does not have singularities on $\mathbb{H}$, it is called a *weak Maass form*, and the space of weak Maass forms is denoted by $\mathrm{WMF}_{k/2,\lambda}(\Gamma, \chi)$. When $\Gamma = \Gamma_0(N)$ for a positive integer N, the Fourier expansion of a weak Maass form $f \in \mathrm{WMF}_{k/2,\lambda}(\Gamma, \chi)$

at the cusp corresponding to $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ is given by

$$(2.1)\quad (f_\gamma)(z) = \tilde{a}_\gamma(0, s_\lambda)v^{1-s_\lambda-k/4} + a_\gamma(0, s_\lambda)v^{s_\lambda-k/4} + \sum_{\substack{-n_\gamma \le n \le m_\gamma \\ n\in\mathbb{Z}\setminus\{0\}}} \tilde{a}_\gamma(n/N, s_\lambda)|4\pi nv/N|^{-k/4} W_{-\operatorname{sgn}(n)k/4, s_\lambda-1/2}(-4\pi|n|v/N) \times \exp(2\pi inu/N) + \sum_{n\in\mathbb{Z}\setminus\{0\}} a_\gamma(n/N, s_\lambda)|4\pi nv/N|^{-k/4} W_{\operatorname{sgn}(n)k/4, s_\lambda-1/2}(4\pi|n|v/N) \times \exp(2\pi inu/N),$$

where $m_\gamma, n_\gamma > 0$ and $\lambda = s_\lambda(1-s_\lambda) + (k^2-4k)/16$ (for example, see [2]). Here, for $t \in \mathbb{R}$ we denote by $W_{k/2,m}(t)$ and $W_{-k/2,m}(-t)$ the standard W-Whittaker functions, which can be distinguished by their asymptotic behavior

$$|W_{\pm k/2,m}(\pm t)| = \exp(\mp t/2)|t|^{\pm k/2}(1 + O(t^{-1}))$$

as $|t| \to \infty$. It is also known that $a_\gamma(n, s_\lambda) = O(\exp(\delta\sqrt{n}))$ for some $\delta > 0$ (see [3, 4]). In the case $\gamma = I$, we abbreviate $\tilde{a}(n, s_\lambda) = \tilde{a}_\gamma(n, s_\lambda)$ and $a(n, s_\lambda) = a_\gamma(n, s_\lambda)$. We denote by $\mathrm{WMF}^*_{k/2,\lambda}(\Gamma, \chi)$ the subspace of $\mathrm{WMF}_{k/2,\lambda}(\Gamma, \chi)$ which consists of f such that $\tilde{a}_\gamma(n, s_\lambda) = 0$ for $n \ge 0$.

2.2. Indefinite theta series. In this subsection, we recall the definition of the Weil representation following the discussion in [7], and explain how to construct a theta series using the Weil representation. Let Q be a rational symmetric matrix of signature (p, q) with $p+q = n$. For $x, y \in \mathbb{R}^n$ we have an inner product defined by $\langle x, y\rangle := {}^t xQy$. Let $S(\mathbb{R}^n)$ be the space of Schwartz functions on $\mathbb{R}^n$. For a matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and a Schwartz function $f \in S(\mathbb{R}^n)$, the Weil representation is defined by

$$(r(\gamma, Q)f)(x) = \begin{cases} |a|^{n/2} e\left(\frac{ab}{2}\langle x, x\rangle\right) f(ax) & \text{if } c = 0, \\ |\det Q|^{1/2}|c|^{-n/2} \int\limits_{\mathbb{R}^n} e\left(\frac{a\langle x,x\rangle - 2\langle x,y\rangle + d\langle y,y\rangle}{2c}\right) f(y)\, dy & \text{if } c \ne 0, \end{cases}$$

where $e(x) := \exp(2\pi ix)$. The Weil representation has the following properties.

PROPOSITION 2.2 ([7]). *Let* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ *and*

$$\sigma_z = \begin{pmatrix} v^{1/2} & uv^{-1/2} \\ 0 & v^{-1/2} \end{pmatrix} \quad \textit{for } z = u + iv \in \mathbb{H}.$$

Define $\phi \pmod{2\pi}$ *by* $\exp(-i\phi) = \frac{cz+d}{|cz+d|}$ *and let* $\kappa(\phi) = \left(\begin{smallmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{smallmatrix}\right)$. *Then*

(1) $\gamma\sigma_z = \sigma_{\gamma z}\kappa(\phi)$,

(2) $r(\gamma,Q)r(\sigma_z,Q)=r(\sigma_{\gamma z},Q)r(\kappa(\phi),Q)$, *and*
(3) *if we let*

$$\gamma_t=\begin{pmatrix} a & bt^2\\ ct^{-2} & d\end{pmatrix}=\begin{pmatrix} t & \\ & t^{-1}\end{pmatrix}\gamma\begin{pmatrix} t^{-1} & \\ & t\end{pmatrix}\qquad \textit{for } t\in\mathbb{R}^*,$$

then

$$\gamma_t\sigma_{t^2z}=\sigma_{t^2(\gamma z)}\kappa(\phi),$$
$$r(\gamma_t,Q)r(\sigma_{t^2z},Q)=r(\sigma_{t^2(\gamma z)},Q)r(\kappa(\phi),Q).$$

Let L be an even lattice in $\mathbb{R}^n$ such that $\langle x,x\rangle\in 2\mathbb{Z}$ for all $x\in L$, and let L^* be its dual lattice defined by $L^*:=\{x\in\mathbb{R}^n\mid \langle x,y\rangle\in\mathbb{Z}$ for all $y\in L\}$. The volume $v(L)$ of a fundamental parallelotope of L in $\mathbb{R}^n$ is defined by $v(L)=\int_{\mathbb{R}^n/L}dx$.

Definition 2.3 (First permutation and spherical property). We say that a function $\omega: L^*/L\to\mathbb{C}$ has the *first permutation property* for $\Gamma_0(4N)$ with character χ modulo $4N$ if

(a) $\omega(l)=0$ if $\langle l,l\rangle\notin 2\mathbb{Z}$,
(b) $\omega(dl)=\chi(d)\omega(l)$ for $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_0(4N)$.

We say that a function $f\in S(\mathbb{R}^n)$ has the *first spherical property* for weight $k/2$, $k\in\mathbb{Z}$, if

$$r(\kappa(\phi),Q)f=\epsilon(\kappa(\phi))^{p-q}\exp(i\phi k/2)f$$

for all $\kappa(\phi)$, where for $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{R})$,

$$\epsilon\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right):=\begin{cases}\sqrt{i} & \text{if } c>0,\\ i^{(1-\operatorname{sgn} d)/2} & \text{if } c=0,\\ \sqrt{i}^{-1} & \text{if } c<0.\end{cases}$$

With this we construct a theta series as follows. For $f\in S(\mathbb{R}^n)$ and $h\in L^*/L$ let

$$\theta(f,h):=\sum_{x\in L}f(x+h).$$

Take f and ω having the first spherical property for weight $k/2$ and the first permutation property for $\Gamma_0(4N)$ with character χ, respectively. Consider

$$\theta(z,f,h):=v^{-k/4}\theta(r(\sigma_z,Q)f,h)\qquad\text{for } h\in L^*/L,$$
$$\theta(z,f;\omega):=\sum_{h\in L^*/L}\omega(h)\theta(z,f,h).$$

The following was proved by Shintani.

Theorem 2.4 ([13]). *Let* $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$. *Then*

$$(cz+d)^{-k/2}\theta(\gamma z,f,h)=\sqrt{i}^{-(p-q)\operatorname{sgn} c}\sum_{j\in L^*/L}c(h,j)_\gamma\theta(z,f,j),$$

where

$$c(h,j)_\gamma = \begin{cases} \delta_{h,aj}\, e\left(\dfrac{ab}{2}\langle h,j\rangle\right) & \text{if } c=0, \\ |\det Q|^{-1/2} v(L)^{-1} |c|^{-n/2} \displaystyle\sum_{r\in L/cL} e\left(\dfrac{a\langle h+r,h+r\rangle - 2\langle j,h+r\rangle + d\langle j,j\rangle}{2c}\right) & \text{if } c\neq 0, \end{cases}$$

where $\delta_{a,b}$ *is the Kronecker delta. In particular, if* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4N)$*, then*

$$j(\gamma,z)^{-k}\theta(\gamma z, f;\omega) = \chi'(d)\theta(z,f;\omega),$$

where

$$\chi'(d) = \left(\frac{-1}{d}\right)^{(k-n)/2}\left(\frac{2}{d}\right)^n\left(\frac{B}{d}\right)((-1)^q B, d)_\infty \chi(d)$$

with the Hilbert symbol

$$(x,y)_\infty := \begin{cases} -1 & \text{if } x,y<0, \\ 1 & \text{otherwise.} \end{cases}$$

Here, $B := \det(\langle \lambda_i, \lambda_j\rangle)$*, where* $\{\lambda_1,\dots,\lambda_n\}$ *is a* $\mathbb{Z}$*-basis for* L*.*

2.3. Niwa's theta kernel. In this subsection we recall the theta series given by Niwa [10]. Let $\mathrm{O}(Q)$ be the orthogonal group of Q defined by $\mathrm{O}(Q) := \{g \mid {}^t g Q g = Q\}$. Let $\mathrm{SO}(Q)$ denote the connected component of the identity in $\mathrm{O}(Q)$ consisting of those matrices g with $\det g = 1$. Define a unitary representation $p : \mathrm{SO}(Q) \to \mathrm{GL}(L^2(\mathbb{R}^n))$ by letting

$$(p(g)f)(x) = f(g^{-1}x).$$

By definition of $\mathrm{SO}(Q)$, $p(g)$ commutes with the Weil representation (see [7, p. 64]), i.e.

$$p(g)(r(\gamma,Q)f) = r(\gamma,Q)(p(g)f). \tag{2.2}$$

Take a special

$$Q = \frac{2}{N}\begin{pmatrix} & & -2 \\ & 1 & \\ -2 & & \end{pmatrix}$$

with signature $(2,1)$ and let $L = 4N\mathbb{Z} \oplus N\mathbb{Z} \oplus \frac{N}{4}\mathbb{Z}$. One can check that $v(L) = N^3$, $L^* = \mathbb{Z} \oplus \frac{1}{2}\mathbb{Z} \oplus \frac{1}{16}\mathbb{Z}$ and $B = -32N^3$. As a quadratic form, Q is given by the determinant of a matrix as follows. For $x = (x_1,x_2,x_3)$ we have

$$Q(x) = {}^t x Q x = \frac{2}{N}(x_2^2 - 4x_1x_3) = \frac{-8}{N}\begin{vmatrix} x_1 & x_2/2 \\ x_2/2 & x_3 \end{vmatrix}.$$

It is known that there is an isomorphism from $\mathrm{SL}_2(\mathbb{R})/\pm I$ to $\mathrm{SO}(Q)$ given by (for more details, see [7])

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}. \tag{2.3}$$

Definition 2.5 (Second permutation and spherical property). Let Γ_Q be a discrete subgroup of $\mathrm{SO}(Q)$ which leaves L invariant, and let Γ_Q^* be the (normal) subgroup of Γ_Q which fixes L^*/L.

(1) Let χ be a character of Γ_Q which is trivial on Γ_Q^*. We say that $\omega : L^*/L \to \mathbb{C}$ has the *second permutation property* for Γ_Q with character χ if $\omega(\gamma l) = \chi(\gamma)\omega(l)$ for $\gamma \in \Gamma_Q$ and $l \in L^*$.

(2) Let $m \in \mathbb{Z}$. We say $f \in S(\mathbb{R}^3)$ has the *second spherical property* for weight $2m$ if $p(\kappa(\phi))f = \exp(-2im\phi)f$ for all $0 \le \phi < 2\pi$, where we identify $\kappa(\phi)$ as an element of a maximal compact subgroup of $\mathrm{SO}(Q)$ via (2.3).

Now we consider the Hermite polynomials H_ν defined by

$$H_\nu(x) = (-1)^\nu \exp\left(\frac{x^2}{2}\right)\frac{d^\nu}{dx^\nu}\exp\left(-\frac{x^2}{2}\right)$$

for $0 \le \nu \in \mathbb{Z}$. For example, $H_0(x) = 1$ and $H_1(x) = x$.

Theorem 2.6 ([7]). *Let m and λ be integers. For every positive integer μ such that $|m| \le \lambda + \mu$ there is a function $L_{m,\lambda,\mu}$ on $\mathbb{R}^3$ such that the function*

$$\begin{aligned} f_{m,\lambda,\mu}(x) \\ = L_{m,\lambda,\mu}(x) H_\mu\left(\frac{\sqrt{8\pi}}{N}(x_1+x_3)\right)\exp\left(\frac{-2\pi}{N}(2x_1^2+x_2^2+2x_3^2)\right) \end{aligned} \tag{2.4}$$

has the first spherical property for weight $k/2 = \lambda + 1/2$ and the second spherical property for weight $2m$. The function $L_{m,\lambda,\mu}(x)$ is defined by

$$L_{m,\lambda,\mu}(x) = \frac{1}{2\pi}\int_0^{2\pi} \exp(2mi\phi) L_{\lambda,\mu}(\kappa(\phi)^{-1}x)\,d\phi,$$

where $L_{\lambda,\mu}(x) = H_{\nu_1}(\sqrt{8\pi/N}\,(x_1-x_3))H_{\nu_2}(\sqrt{8\pi/N}\,x_2)$ with any choice of ν_1 and ν_2 such that $\nu_1+\nu_2-\mu = \lambda$. In particular, we may take $L_{\lambda,\lambda,0}(x) = (x_1 - ix_2 - x_3)^\lambda$.

Let $f_{k,m} = f_{m,\lambda,\mu}$ for some fixed μ as in Theorem 2.6 with $k/2 = \lambda + 1/2$. For $z, w \in \mathbb{H}$ and a given character χ on $\Gamma_0(4N)$ consider a theta kernel of weight $k/2$ given by

$$(2.5)\quad \theta(z,w;f_{k,m}) := (32N^3)^{-1/2} i^{\lambda} v^{-k/4} (4\eta)^{-m} \times \sum_{x=(x_1,x_2,x_3)\in L_N^*} \check{\bar{\chi}}_1(4x_1)\{r(\sigma_{4Nz},Q)p(\sigma_{2Nw})f_{k,m}\}(x),$$

where

$$\chi_1(d) = \left(\frac{-1}{d}\right)^{\lambda}\chi(d), \qquad \check{\bar{\chi}}_1(l) = \sum_{h=1}^{4N} \bar{\chi}_1(h)\exp\left(2\pi i\left(\frac{lh}{4N}\right)\right)$$

and $L_N^* = \frac{1}{4}\mathbb{Z}\oplus\frac{1}{2}\mathbb{Z}\oplus\frac{1}{4}\mathbb{Z}$ (the dual lattice of $L_N = N\mathbb{Z}\oplus N\mathbb{Z}\oplus N\mathbb{Z}$). In particular, if we take

$$(2.6)\quad f_{k,(k-1)/2}(x) = (x_1 - ix_2 - x_3)^{(k-1)/2}\exp\left(-\frac{2\pi}{N}(2x_1^2 + x_2^2 + 2x_3^2)\right),$$

then $\theta(z,w) := \theta(z,w;f_{k,(k-1)/2})$ is the theta function studied by Cipra [7] and Niwa [10].

THEOREM 2.7. *Let $A_{k/2}(\Gamma,\chi)$ be the space of functions $f : \mathbb{H}\to\mathbb{C}$ such that $f|_{k/2}\gamma = \chi(d)f$ for all $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \Gamma$. Then $\theta(z,w;f_{k,m}) \in A_{k/2}(\Gamma_0(4N),\chi)$ as a function of z, and $\overline{\theta(z,w;f_{k,m})} \in A_{2m}(\Gamma_0(2N),\chi^2)$ as a function of w.*

Proof. The *Fricke involution* $W(N)$ is defined by

$$(f|_{k/2}W(N))(z) := \begin{cases} N^{-k/4}(-iz)^{-k/2}f(-1/(Nz)) & \text{if } k \text{ is odd},\\ N^{-k/4}z^{-k/2}f(-1/(Nz)) & \text{if } k \text{ is even}.\end{cases}$$

Let $|_{k/2}W(4N)$ act on the variable z, and $|_{2m}W(2N)$ act on w. Now we consider the theta series given by

$$\Theta(z,w;f_{k,m}) := (4\eta)^{-m}v^{-k/4}\sum_{x\in L'}\bar{\chi}_1(x_1)\{r(\sigma_z,Q)p(\sigma_{4w})f_{k,m}\}(x),$$

where $L' = \mathbb{Z}\oplus N\mathbb{Z}\oplus(N/4)\mathbb{Z}$. Then one can see that

$$\theta(z,w;f_{k,m}) = \left(\Theta|_{k/2}W(4N)\overline{|_{2m}W(2N)}\right)(z,w;f_{k,m}),$$

where $f\overline{|_{2m}W(2N)}(z,w) = \overline{N^{-m}w^{-2m}}f(z,-1/(Nw))$. So it is enough to prove that

$$\Theta(z,w;f_{k,m}) \in A_{k/2}\left(\Gamma_0(4N),\bar{\chi}\left(\frac{N}{\cdot}\right)\right)$$

as a function of z, and

$$\overline{\Theta(z,w;f_{k,m})} \in A_{2m}(\Gamma_0(2N),\bar{\chi}^2)$$

as a function of w.

With the notation of Theorem 2.4 we see that

$$\Theta(z,w;f_{k,m}) = (4\eta)^{-m}\theta(z,p(\sigma_{4w})f_{k,m};\omega)$$

with $\omega : L^*/L \to \mathbb{C}$ defined by

$$\omega(l) = \begin{cases} 0 & \text{if } l \notin L', \\ \bar{\chi}_1(l_1) & \text{if } l = (l_1, l_2, l_3) \in L', \end{cases}$$

where $L = 4N\mathbb{Z} \oplus N\mathbb{Z} \oplus \frac{N}{4}\mathbb{Z}$ and its dual is $L^* = \mathbb{Z} \oplus \frac{1}{2}\mathbb{Z} \oplus \frac{1}{16}\mathbb{Z}$. Note that $L \subset L' \subset L^*$ and ω has the first permutation property for $\Gamma_0(4N)$ with character $\bar{\chi}_1$. Moreover, we can observe that if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2N)$, then the map

$$\gamma_2 = \begin{pmatrix} 2 & \\ & 1/2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1/2 & \\ & 2 \end{pmatrix} = \begin{pmatrix} a & 4b \\ c/4 & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & 4ab & 16b^2 \\ ac/2 & ad+bc & 8bd \\ c^2/16 & cd/4 & d^2 \end{pmatrix}$$

leaves L and L' invariant and that

$$\omega(\gamma_2 l) = \bar{\chi}_1(a^2 l_1 + 4ab l_2 + 16b^2 l_3) = \chi^2(d)\bar{\chi}_1(l_1) = \chi^2(d)\omega(l)$$

for $(l_1, l_2, l_3) \in L'$ because χ has modulus $4N$. So ω has the second permutation property for $\Gamma_Q = \left(\begin{smallmatrix} 2 & \\ & 1/2 \end{smallmatrix}\right)\Gamma_0(2N)\left(\begin{smallmatrix} 1/2 & \\ & 2 \end{smallmatrix}\right)$ with character χ^2 (for more details, see [7, Proposition 2.2]). Also note that $p(\sigma_{4w})f_{k,m}$ has the first spherical property of weight $k/2$ since p commutes with the Weil representation as in (2.2). By Theorem 2.4, $\Theta(z, w; f_{k,m})$ is a non-holomorphic modular form of weight $k/2$ on $\Gamma_0(4N)$ with character $\left(\frac{N}{\cdot}\right)\bar{\chi}$ as a function of z.

Next, using Proposition 2.2(3) we see that

$$p(\sigma_{4(\gamma w)}) = p(\gamma_2 \sigma_{4w} \kappa(\phi)^{-1})$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2N)$ and $\exp(-i\phi) = \frac{cw+d}{|cw+d|}$. Since $\gamma_2 \in \Gamma_Q$, if we use the second spherical property of $f_{k,m}$ and the second permutation property of ω, we find that

$$\begin{aligned}\Theta(z, \gamma w; f_{k,m}) &= (4\eta)^{-m}|cw+d|^{2m} v^{-k/4} \\ &\quad \times \sum_{x \in L'} \omega(x)\left\{ r(\sigma_z, Q)\left(\frac{c\bar{w}+d}{|cw+d|}\right)^{2m} p(\sigma_{4w}) f_{k,m} \right\}(\gamma_2^{-1}x) \\ &= (c\bar{w}+d)^{2m}\chi^2(d)(4\eta)^{-m} v^{-k/4} \sum_{x \in L'} \omega(x)\{r(\sigma_z, Q)p(\sigma_{4w})f_{k,m}\}(x) \\ &= (c\bar{w}+d)^{2m}\chi^2(d)\Theta(z, w; f_{k,m}).\end{aligned}$$

This completes the proof. ∎

Furthermore, this theta function satisfies the following differential equation.

Proposition 2.8 ([7]). *The theta function $\theta(z, w; f_{k,m})$ defined in (2.5) satisfies the partial differential equation*

$$4\left[v^2\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)-i\frac{k}{2}v\left(\frac{\partial}{\partial u}+i\frac{\partial}{\partial v}\right)+\frac{k}{4}\left(\frac{k}{4}-1\right)\right]\theta(z,w;f_{k,m})$$
$$=\left[\eta^2\left(\frac{\partial^2}{\partial \xi^2}+\frac{\partial^2}{\partial \eta^2}\right)+2mi\eta\left(\frac{\partial}{\partial \xi}-i\frac{\partial}{\partial \eta}\right)+m(m-1)-\frac{3}{4}\right]\theta(z,w;f_{k,m}).$$

2.4. Regularized theta lifting. In the case of weak Maass forms, because of singularities at the cusps the integral in a theta lifting may not converge. In this subsection we describe how to regularize the integral to get a theta lifting for weak Maass forms. We use a regularized integral introduced by Borcherds [1] as follows. We integrate over the region $\mathcal{F}_t$, where $\mathcal{F}=\{z\in\mathbb{H}\mid |z|\geq 1, |\mathrm{Re}(z)|\leq 1/2\}$ is the usual fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$ and $\mathcal{F}_t$ is the subset of $\mathcal{F}$ of points z with $\mathrm{Im}(z)\leq t$. Suppose that

$$\lim_{t\to\infty}\int_{\mathcal{F}_t}F(z)v^{-s}\,\frac{du\,dv}{v^2} \tag{2.7}$$

exists for $\mathrm{Re}(s)\gg 0$ and can be continued to a meromorphic function on $\mathbb{C}$. Borcherds [1] defined a regularized integral $\int_{\mathcal{F}}^{\mathrm{reg}}F(z)\,\frac{du\,dv}{v^2}$ to be the constant term of the Laurent expansion of the function (2.7) at $s=0$. This is a regularized integral for $\mathrm{SL}_2(\mathbb{Z})$. Furthermore, in the case when $g(z)\in\mathrm{WMF}^*_{k/2,\lambda}(\Gamma_0(4N),\chi)$, we define

$$\Phi_m(g)(w)=\int_{\mathcal{F}}^{\mathrm{reg}}v^{k/2}\sum_{\alpha\in\Gamma_0(4N)\backslash\mathrm{SL}_2(\mathbb{Z})}g_\alpha(z)\overline{\theta_\alpha(z,w;f_{k,m})}\,\frac{du\,dv}{v^2}. \tag{2.8}$$

We use the notation $\Phi(g)$ instead of $\Phi_m(g)$ if we take $f_{k,m}=f_{k,(k-1)/2}$ as in (2.6).

3. Proof of the main theorems. First we prove the following more general result which implies Theorem 1.1 as a corollary.

THEOREM 3.1. *Assume that k is a positive odd integer and m is an integer. Let χ be a character modulo $4N$ and g be a weak Maass form in $\mathrm{WMF}^*_{k/2,\lambda}(\Gamma_0(4N),\chi)$. Then $\Phi_m(g)$ has the following properties:*

(1) *$\Phi_m(g)|_{2m}\gamma=\chi^2(\gamma)\Phi_m(g)$ for all $\gamma\in\Gamma_0(2N)$,*
(2) *$\Delta_{2m}\Phi_m(g)=(m(m-1)-3/4-k(k/4-1)+4\lambda)\Phi_m(g)$, and*
(3) *the singularities of $\Phi_m(g)(w)$ are supported on the set of Heegner points of the form $w=(b\pm\sqrt{b^2-ac})/(2Nc)$ in $\mathbb{H}$ for $a,b,c\in\mathbb{Z}$.*

Proof. First we show the convergence of the regularized integral $\Phi_m(g)$ for $g \in \mathrm{WMF}^*_{k/2,\lambda}(\Gamma_0(4N),\chi)$. It suffices to prove the convergence of

$$\lim_{t\to\infty}\int_1^t\int_{-1/2}^{1/2}\sum_{\alpha\in\Gamma_0(4N)\backslash\mathrm{SL}_2(\mathbb{Z})} g_\alpha(z)\overline{\theta_\alpha(z,w;f_{k,m})}\,v^{k/2-s}\,\frac{du\,dv}{v^2}. \tag{3.1}$$

By the translation invariance of $\sum_{\alpha\in\Gamma_0(4N)\backslash\mathrm{SL}_2(\mathbb{Z})} g_\alpha(z)\overline{\theta_\alpha(z,w;f_{k,m})}\,v^{k/2}$, this function has a Fourier expansion of the form

$$\sum_{n\in\mathbb{Z}} a(n,v,w)\exp(2\pi inu).$$

Then

$$\int_1^t\int_{-1/2}^{1/2}\sum_{\alpha\in\Gamma_0(4N)\backslash\mathrm{SL}_2(\mathbb{Z})} g_\alpha(z)\overline{\theta_\alpha(z,w;f_{k,m})}\,v^{k/2-s}\,\frac{du\,dv}{v^2} = \int_1^t a(0,v,w)v^{-s-2}\,dv.$$

To ensure the convergence of the regularized theta lifting, we need to find the constant term of $g_\alpha(z)\overline{\theta_\alpha(z,w;f_{k,m})}$.

Let $Q_4 := \frac{1}{2}\begin{pmatrix} & & -2\\ & 1 & \\ -2 & & \end{pmatrix}$ and

$$f_4(x_1,x_2,x_3) := (x_1-ix_2-x_3)^\lambda\exp\biggl(-\frac{\pi}{2}(2x_1^2+x_2^2+2x_3^2)\biggr).$$

Since these are just the original Q and f with $N=4$, f_4 has the first and second spherical properties for the weights $k/2$ and $2m$, respectively. Now, let $L = 4N\mathbb{Z}\oplus 2\mathbb{Z}\oplus\mathbb{Z}$, $L' := \mathbb{Z}\oplus 2\mathbb{Z}\oplus\mathbb{Z}$ and

$$\omega(l) := \begin{cases} 0 & \text{if } l\notin L',\\ \check{\bar{\chi}}(l_1) & \text{if } l=(l_1,l_2,l_3)\in L'.\end{cases}$$

Then $L^* = \mathbb{Z}\oplus\mathbb{Z}\oplus\frac{1}{4N}\mathbb{Z}$, and ω has the first permutation property for $\Gamma_0(4N)$ with character χ_1 and the second permutation property for $\Gamma_Q = \begin{pmatrix}2 & \\ & 1/2\end{pmatrix}\Gamma_0(2N)\begin{pmatrix}1/2 & \\ & 2\end{pmatrix}$ with character $\bar{\chi}^2$. One can see that $\theta(z,w;f_{k,m})$ is the same as $\eta^{-m}\theta(z,p(\sigma_{2Nw})f_4;\omega)$ up to a constant multiple. By Theorem 2.4, $\theta_\alpha(z,w;f_{k,m})$ can be written as

$$v^{1/2}\sum_{x\in\mathbb{Z}^3} h_\alpha(x,w;k)\exp\biggl(\frac{-\pi v}{4N^2}|\Lambda(x,w)|^2 + 2\pi i\bar{z}(x_2^2-x_1x_3)\biggr)$$

for each $\alpha\in\Gamma_0(4N)\setminus\mathrm{SL}_2(\mathbb{Z})$, where $\Lambda(x,w) := \frac{1}{\eta}(x_1-4Nwx_2+4N^2w^2x_3)$ and $h_\alpha(x,w;k)$ is a polynomial in x_1, x_2, x_3 and w.

If g_α has the Fourier expansion $g_\alpha(z) = \sum_{n\in\mathbb{Q}} b_\alpha(n,v)\exp(2\pi inu)$, then the constant term of $g_\alpha(z)\overline{\theta_\alpha(z,w;f_{k,m})}$ is a sum of terms of the form

$$b_\alpha(x_2^2-x_1x_3,v)\exp(2\pi(x_2^2-x_1x_3)v)v^{1/2}\,\overline{h_\alpha(x,w;k)}\exp\biggl(\frac{-\pi v}{4N^2}|\Lambda(x,w)|^2\biggr),$$

where $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$. Since $g \in \mathrm{WMF}^*_{k/2,\lambda}(\Gamma_0(4N), \chi)$, the function $b_\alpha(x_2^2 - x_1x_3, v)\exp(2\pi(x_2^2 - x_1x_3)v)$ has polynomial growth when v tends to infinity, and thus every term with $\Lambda(x,w) \neq 0$ is exponentially decreasing. Therefore, the integral in (3.1) converges and $\Phi_m(g)$ is well-defined if $\Lambda(x,w) \neq 0$ for all $x \in \mathbb{Z}^3$, and singularities may only occur when $w = \frac{x_2+\sqrt{x_2^2-x_1x_3}}{2Nx_3} \in \mathbb{H}$ for some $(x_1, x_2, x_3) \in \mathbb{Z}^3$.

The transformation properties of $\Phi_m(g)$ come easily from the fact that the function $\overline{\theta(z, w; f_{k,m})}$ is in $A_{2m}(\Gamma_0(2N), \chi^2)$ as a function of w. Finally, to prove the property of $\Phi_m(g)$ involving the weight $2m$ Laplace operator, consider the Maass differential operators on smooth functions defined on $\mathbb{H}$ (see [3, p. 97]) by

$$R_k = 2i\frac{\partial}{\partial z} + kv^{-1} \quad \text{and} \quad L_k = 2iv^2\frac{\partial}{\partial \bar{z}}.$$

For any smooth function $f : \mathbb{H} \to \mathbb{C}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, it is well known that $(R_kf)|_{k+2}\gamma = R_k(f|_k\gamma)$ and $(L_kf)|_{k-2}\gamma = L_k(f|_k\gamma)$. The operator Δ_k can be expressed in terms of R_k and L_k by

$$\Delta_k = L_{k+2}R_k - k = R_{k-2}L_k = -4v^2\frac{\partial^2}{\partial z\partial\bar{z}} + 2ivk\frac{\partial}{\partial\bar{z}}. \tag{3.2}$$

We write Δ_{2m} for the Laplace operator with respect to w, and $\Delta_{k/2}$ for the Laplace operator with respect to z.

Lemma 3.2. *If $f, g \in A_k(\Gamma_0(4N), \chi)$ are smooth, then*

$$\begin{aligned}\int_{\mathcal{F}_t} \sum_{\alpha\in\Gamma_0(4N)\backslash\mathrm{SL}_2(\mathbb{Z})} & \Delta_k f_\alpha(z)\overline{g_\alpha(z)}v^{k-2}\,du\,dv \\ & - \int_{\mathcal{F}_t} \sum_{\alpha\in\Gamma_0(4N)\backslash\mathrm{SL}_2(\mathbb{Z})} f_\alpha(z)\overline{\Delta_k g_\alpha(z)}v^{k-2}\,du\,dv \\ & = \int_{-1/2}^{1/2}\Big[\sum_{\alpha\in\Gamma_0(4N)\backslash\mathrm{SL}_2(\mathbb{Z})} v^{k-2}f_\alpha(z)\overline{L_kg_\alpha(z)}\Big]_{v=t}du \\ & - \int_{-1/2}^{1/2}\Big[\sum_{\alpha\in\Gamma_0(4N)\backslash\mathrm{SL}_2(\mathbb{Z})} v^{k-2}L_kf_\alpha(z)\overline{g_\alpha(z)}\Big]_{v=t}du.\end{aligned}$$

Proof. The proof is based on the argument in [3]. Note that by the definition of the differential d,

$$\begin{aligned} & d\Big(v^{k-2}\sum_{\alpha\in\Gamma_0(4N)\backslash\mathrm{SL}_2(\mathbb{Z})} \overline{f_\alpha(z)}L_kg_\alpha(z)\,d\bar{z}\Big) \\ & = v^{k-2}\Big[\frac{1}{v^2}\sum_{\alpha\in\Gamma_0(4N)\backslash\mathrm{SL}_2(\mathbb{Z})}\big(\overline{L_kf_\alpha(z)}L_kg_\alpha(z) - \overline{f_\alpha(z)}R_{k-2}L_kg_\alpha(z)\big)\Big]\,du\,dv.\end{aligned}$$

Using this and Stokes' theorem one can see that

$$(3.3)\qquad \int_{\partial\mathcal{F}_t} v^{k-2} \sum_{\alpha\in\Gamma_0(4N)\backslash \mathrm{SL}_2(\mathbb{Z})} \overline{f_\alpha(z)} L_k g_\alpha(z)\, d\bar{z}$$
$$= \int_{\mathcal{F}_t} v^{k-2}\Big[\frac{1}{v^2} \sum_{\alpha\in\Gamma_0(4N)\backslash \mathrm{SL}_2(\mathbb{Z})} \big(\overline{L_k f_\alpha(z)} L_k g_\alpha(z) - \overline{f_\alpha(z)} R_{k-2} L_k g_\alpha(z)\big)\Big]\, du\, dv.$$

Since $f|_k\gamma = \chi(d) f$ and $g|_k\gamma = \chi(d) g$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(4N)$, we have

$$\begin{aligned}\mathrm{Im}(\gamma z)^{k-2}(\overline{f} L_k g)(\gamma z) d(\overline{\gamma z}) &= \left(\frac{v}{|cz+d|^2}\right)^{k-2} \overline{\chi(d)(cz+d)^k f(z)} \\ &\quad \times \chi(d)(cz+d)^{k-2} L_k g(z) \overline{(cz+d)^{-2}} d\bar{z} \\ &= v^{k-2}(\overline{f} L_k g)(z) d\bar{z}.\end{aligned}$$

From this, one can check that the integrand on the left hand side in (3.3) is $\mathrm{SL}_2(\mathbb{Z})$-invariant. Therefore, if C is the subset of $\partial\mathcal{F}_t$ consisting of the arc of the unit circle from $\exp(\pi i/3)$ to $\exp(2\pi i/3)$, then in the integral over $\partial\mathcal{F}_t$ on the left side the contributions from the lines $u = \pm 1/2$ and those from the two halves of C cancel each other, so we get

$$\int_{\partial\mathcal{F}_t} v^{k-2} \sum_{\alpha\in\Gamma_0(4N)\backslash \mathrm{SL}_2(\mathbb{Z})} \overline{f_\alpha(z)} L_k g_\alpha(z)\, d\bar{z}$$
$$= \int_{-1/2}^{1/2} \Big[\sum_{\alpha\in\Gamma_0(4N)\backslash \mathrm{SL}_2(\mathbb{Z})} v^{k-2}\, \overline{f_\alpha(z)} L_k g_\alpha(z)\Big]_{v=t} du.$$

If we continue this process and use the fact that $\Delta_k = R_{k-2} L_k$, we obtain the desired result. ∎

We now prove part (2) of Theorem 3.1. By (2.8) and (3.2), $\Delta_{2m}(\Phi_m(g))(w)$ is equal to

$$(3.4)\qquad -4\int_{\mathcal{F}}^{\mathrm{reg}} v^{k/2}$$
$$\times \sum_{\alpha\in\Gamma_0(4N)\backslash \mathrm{SL}_2(\mathbb{Z})} g_\alpha(z) \overline{\left(\eta^2 \frac{\partial^2}{\partial w \partial \bar{w}} + m i \eta \frac{\partial}{\partial w}\right) \theta_\alpha(z, w; f_{k,m})}\, \frac{du\, dv}{v^2}.$$

From this and Proposition 2.8 we see that

$$\left(-4\eta^2 \frac{\partial^2}{\partial w \partial \bar{w}} - 4 i \eta m \frac{\partial}{\partial w}\right)\theta_\alpha(z, w; f_{k,m})$$
$$= \left(-\eta^2\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right) - 2 i \eta m \left(\frac{\partial}{\partial \xi} - i \frac{\partial}{\partial \eta}\right)\right)\theta_\alpha(z, w; f_{k,m})$$

$$= \left(-4v^2\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)+2ikv\left(\frac{\partial}{\partial u}+i\frac{\partial}{\partial v}-k\left(\frac{k}{4}-1\right)\right.\right.$$
$$\left.\left.+m(m-1)-\frac{3}{4}\right)\right)\theta_\alpha(z,w;f_{k,m})$$
$$= \left(4\Delta_{k/2}-k\left(\frac{k}{4}-1\right)+m(m-1)-\frac{3}{4}\right)\theta_\alpha(z,w;f_{k,m}).$$

So the integral (3.4) is equal to

$$4\int_{\mathcal{F}}^{\mathrm{reg}} v^{k/2}\sum_{\alpha\in\Gamma_0(4N)\backslash \mathrm{SL}_2(\mathbb{Z})} g_\alpha(z)\overline{\Delta_{k/2}\theta_\alpha(z,w;f_{k,m})}\,\frac{du\,dv}{v^2}$$
$$+\left(m(m-1)-\frac{3}{4}-k\left(\frac{k}{4}-1\right)\right)\Phi_m(g)(w).$$

By Lemma 3.2 we see that

$$\int_{\mathcal{F}}^{\mathrm{reg}} v^{k/2}\sum_{\alpha\in\Gamma_0(4N)\backslash \mathrm{SL}_2(\mathbb{Z})} g_\alpha(z)\overline{\Delta_{k/2}\theta_\alpha(z,w;f_{k,m})}\,\frac{du\,dv}{v^2}-\lambda\Phi_m(g)(w)$$
$$=\lim_{t\to\infty}\int_{-1/2}^{1/2} v^{k/2-2}\Big[\sum_{\alpha\in\Gamma_0(4N)\backslash \mathrm{SL}_2(\mathbb{Z})} L_{k/2}g_\alpha(z)\overline{\theta_\alpha(z,w;f_{k,m})}\Big]_{v=t}du$$
$$-\lim_{t\to\infty}\int_{-1/2}^{1/2}\Big[v^{k/2-2}\sum_{\alpha\in\Gamma_0(4N)\backslash \mathrm{SL}_2(\mathbb{Z})} g_\alpha(z)\overline{L_{k/2}\theta_\alpha(z,w;f_{k,m})}\Big]_{v=t}du.$$

One can see that all the integrals except for the first one vanish if we apply the same argument as in the proof of convergence. So, we conclude that $\Delta_{2m}(\Phi_m(g))(w) = (4\lambda+m(m-1)-3/4-k(k/4-1))\Phi_m(g)(w)$, which completes the proof. ■

Proof of Theorem 1.2. If we substitute the Fourier expansion of $g_\alpha(z)$ as in (2.1) into (2.8), only the part involving $W_{k/4,s-1/2}(4\pi(x_2^2-x_1x_3)v)$ with $x_2^2-x_1x_3<0$ contributes to the singularity. Hence $\Phi(g)$ has singularity type

$$\sum_{\alpha\in\Gamma_0(4N)\backslash \mathrm{SL}_2(\mathbb{Z})}\ \sum_{(x_1,x_2,x_3)\in\mathcal{S}(U)} \tilde{a}_\alpha(x_2^2-x_1x_3,s_\lambda)\overline{h_\alpha(x,w;k)}$$
$$\times\int_1^\infty \exp\left(\frac{-\pi v}{4N^2}|\Lambda(x,w)|^2+2\pi v(x_2^2-x_1x_3)\right)|4\pi(x_2^2-x_1x_3)v|^{-k/4}$$
$$\times W_{k/4,s_\lambda-1/2}(4\pi(x_2^2-x_1x_3)v)v^{(k-3)/2}\,dv.$$

Now we apply the asymptotic expansion of $W_{k,m}(t)$ (see [15])

$$W_{k,m}(t)\sim\exp\left(-\frac{t}{2}\right)t^k\sum_{n=0}^\infty\left[\frac{1}{n!t^n}\prod_{j=1}^n\left(m^2-\left(k-j+\frac{1}{2}\right)^2\right)\right]$$

as $|t| \to \infty$, where we let $\prod_{j=1}^{n}(m^2 - (k-j+1/2)^2) = 1$ if $n = 0$. Then we need to compute the singularity type of the function

$$\int_1^\infty \exp\left(\frac{-\pi v}{4N^2}|\Lambda(x,w)|^2\right) v^{(k-3)/2-n}\,dv \tag{3.5}$$

for non-negative integers n. Note that the integral in (3.5) converges regardless of the value of $\Lambda(x,w)$ when $n > (k-1)/2$. By [1, Lemma 6.1], the function (3.5) has singularity type

$$\left(\frac{\sqrt{\pi}}{2N}|\Lambda(x,w)|\right)^{2n-k+1} \Gamma\left(\frac{k-1}{2} - n\right)$$

unless $(k-1)/2 - n$ is a non-positive integer, in which case it has a singularity of type

$$(-1)^{(k+1)/2-n}\left(\frac{\sqrt{\pi}}{2N}|\Lambda(x,w)|\right)^{2n-k+1} \frac{\log\left(\left(\frac{\sqrt{\pi}}{2N}|\Lambda(x,w)|\right)^2\right)}{\left(n - \frac{k-1}{2}\right)!}. \quad \blacksquare$$

Now we compute the value of the lifted function on the imaginary axis when $k > 1$ by using the unfolding method. For this we need to rewrite the theta kernel as follows.

Lemma 3.3 ([7]). *Let $\theta(z, w; f_{k,m})$ be the theta function defined in (2.5). Then*

$$\theta(z, i\eta; f_{k,(k-1)/2})$$
$$= C \sum_{\nu=0}^{(k-1)/2} \binom{(k-1)/2}{\nu} \left(\frac{2}{\pi}\right)^{\nu/2} \eta^{1-\nu} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4N)} \frac{\chi(d)}{(\operatorname{Im}\gamma z)^{(k-1)/2-\nu/2} j(\gamma,z)^k}$$
$$\times \sum_{m,n=-\infty}^{\infty} \bar{\chi}_1(m) m^{(k-1)/2-\nu} H_\nu(2\sqrt{2\pi \operatorname{Im}\gamma z}\, n) \exp\left(2\pi i n^2 \gamma z - \frac{\pi m^2 \eta^2}{4 \operatorname{Im}\gamma z}\right),$$

where $C = (-1)^{(k-1)/2} 2^{-2(k-1)} N^{(k-1)/4-1/4}$ and $\Gamma_\infty = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4N) \mid c = 0\}$.

Proof of Theorem 1.3. By Lemma 3.3 and the Rankin–Selberg method,

$$\Phi(g)(i\eta) = C \sum_{\nu=0}^{(k-1)/2} \binom{k-1/2}{\nu} \left(\frac{2}{\pi}\right)^{\nu/2} \eta^{1-\nu} \int_0^\infty \int_0^1 g(z) v^{(\nu+1)/2} \sum_{m,n=-\infty}^{\infty} \chi_1(m)$$
$$\times m^{(k-1)/2-\nu} H_\nu(2\sqrt{2\pi v}\, n) \exp\left(-2\pi i n^2 \bar{z} - \frac{\pi \eta^2 m^2}{4v}\right) \frac{du\,dv}{v^2}.$$

If we use the Fourier expansion of g given in (2.1), we find that $\Phi(g)(i\eta)$ is the same as

$$C \sum_{\nu=0}^{(k-1)/2} \binom{(k-1)/2}{\nu} \left(\frac{2}{\pi}\right)^{\nu/2} \eta^{1-\nu}$$

$$\times \int_0^\infty v^{(\nu-1)/2} \Big(\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} a(n^2, s_\lambda) |4\pi n^2 v|^{-k/4} W_{k/4, s_\lambda - 1/2}(4\pi n^2 v) H_\nu(2\sqrt{2\pi v}\, n)$$

$$+ a(0, s_\lambda) v^{s_\lambda - k/4} H_\nu(0) \Big) \sum_{m=-\infty}^{\infty} \chi_1(m) m^{(k-1)/2-\nu} \exp\left(-\frac{\pi \eta^2 m^2}{4v} - 2\pi n^2 v \right) \frac{dv}{v}$$

$$= C' \sum_{\nu=0}^{(k-1)/2} \binom{(k-1)/2}{\nu} (2\pi)^{-\nu} \int_0^\infty \left(a(0, s_\lambda) \left(\frac{t^2}{8\pi} \right)^{s_\lambda - k/4} H_\nu(0) \right.$$

$$\left. + \sum_{n\neq 0} a(n^2, s_\lambda) \left| \frac{n^2 t^2}{2} \right|^{-k/4} W_{k/4, s_\lambda - 1/2}\left(\frac{n^2 t^2}{2} \right) \exp\left(-\frac{n^2 t^2}{4} \right) H_\nu(tn) \right) \left(\frac{\eta}{t} \right)^{1-\nu}$$

$$\times \sum_{m=-\infty}^{\infty} \chi_1(m) m^{(k-1)/2-\nu} \exp\left(-2\pi^2 m^2 \left(\frac{\eta}{t} \right)^2 \right) \frac{dt}{t},$$

where $C' = 2C(8\pi)^{1/2}$. This completes the proof. ∎

Acknowledgements. The authors would like to thank the referee for numerous helpful comments and suggestions which improved the exposition of this paper considerably. The first author was partially supported by NRF-2013R1A2A2A01068676 and NRF-2015049582.

References

[1] R. E. Borcherds, *Automorphic forms with singularities on Grassmannians*, Invent. Math. 132 (1998), 491–562.

[2] K. Bringmann and D. Penniston, *Arithmetic properties of non-harmonic weak Maass forms*, Proc. Amer. Math. Soc. 137 (2009), 825–833.

[3] J. H. Bruinier, *Borcherds Products on $O(2,l)$ and Chern Classes of Heegner Divisors*, Lecture Notes in Math. 1780, Springer, 2002.

[4] J. H. Bruinier and J. Funke, *On two geometric theta lifts*, Duke Math. J. 125 (2004), 45–90.

[5] J. H. Bruinier and J. Funke, *Traces of CM values of modular functions*, J. Reine Angew. Math. 594 (2006), 1–33.

[6] J. H. Bruinier and K. Ono, *Heegner divisors, L-functions and harmonic weak Maass forms*, Ann. of Math. (2) 172 (2010), 2135–2181.

[7] B. A. Cipra, *On the Niwa–Shintani theta-kernel lifting of modular forms*, Nagoya Math. J. 91 (1983), 49–117.

[8] J. F. Harvey and G. Moore, *Algebras, BPS states, and strings*, Nuclear Phys. B 463 (1996), 315–368.

[9] R. Howe, *θ-series and invariant theory*, in: Automorphic Forms, Representations and L-functions (Corvallis, OR, 1977), Proc. Sympos. Pure Math. 33, Part 1, Amer. Math. Soc., 1979, 275–285.

[10] S. Niwa, *Modular forms of half integral weight and the integral of certain theta-fuctions*, Nagoya Math. J. 56 (1975), 147–161.

[11] K. Ono, *Distribution of the partition function modulo m*, Ann. of Math. 151 (2000), 293–307.

[12] G. Shimura, *On modular forms of half integral weight*, Ann. of Math. 97 (1973), 440–481.

[13] T. Shintani, *On construction of holomorphic cusp forms of half integral weight*, Nagoya Math. J. 58 (1975), 83–126.

[14] J. B. Tunnell, *A classical Diophantine problem and modular forms of weight* 3/2, Invent. Math. 72 (1983), 323–334.

[15] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, reprint of the 4th ed., Cambridge Univ. Press, Cambridge, 1996.

Youngju Choie
Department of Mathematics and PMI
Pohang University of Science and Technology
Pohang, 790-784
Republic of Korea
E-mail: yjc@postech.ac.kr

Subong Lim
Department of Mathematics Education
Sungkyunkwan University
25-2, Sungkyunkwan-ro
Jongno-gu, Seoul, 03063
Republic of Korea
E-mail: subong@skku.edu