ACTA ARITHMETICA
173.1 (2016)

On the representation of numbers by quaternary and quinary cubic forms: I

by

C. Hooley (Cardiff)

1. Introduction. Let $f(\boldsymbol{x}) = f(x_1, \ldots, x_n)$ be a cubic form with rational integral coefficients. Then, of the indeterminate equations

$$f(\boldsymbol{x}) = 0$$
$$\text{and } f(\boldsymbol{x}) = N \qquad (N \neq 0), \tag{0}$$

the former has been studied more than the latter even though the reverse is so when $f(\boldsymbol{x})$ is replaced by a quadratic form. At first sight this is surprising because in some ways the second equation may seem the more interesting owing to its connection with the much examined Waring's problem for cubes. Yet, on reflection, we see the explanation for the emphasis on the first equation lies in its homogeneity, which makes it the more tractable of the two and which also associates it with the study of projective hypersurfaces. Indeed, in elucidation of the first point, we should compare the known results for the two equations, confining ourselves temporarily for simplicity to the cases where $f(\boldsymbol{x})$ has non-zero discriminant and also supposing where necessary that the equations are non-trivially soluble in every p-adic field $\mathbb{Q}_p$. For the first equation, Heath-Brown [3] established solubility for $n = 10$, a bound we ([5], [6], later denoted by J_a, J_b, respectively) improved to $n = 9$ by substantial modifications to Heath-Brown's method; in fact, by assuming a type of Riemann hypothesis for certain Hasse–Weil L-functions (Hypothesis HW) and by using Heath-Brown's new version of the circle method ([4], denoted by J_c), we recently shewed that the value $n = 8$ sufficed ([7], [8], denoted by J_d, J_e). On the other hand, for the second equation the least value 10 found for n stems, as a special example, from work by Browning

2010 *Mathematics Subject Classification*: Primary 11D72.
Key words and phrases: quaternary cubic forms, quinary cubic forms.
Received 9 June 2015; revised 4 December 2015.
Published online 30 March 2016.

DOI: 10.4064/aa8189-1-2016

and Heath-Brown [1] on the more general equation

$$f(\boldsymbol{x}) + g(\boldsymbol{x}) = 0$$

containing a polynomial $g(\boldsymbol{x})$ of degree less than 3. Also, more generally, the results for the second equation are still inferior to those for the first when the stipulation that $f(\boldsymbol{x})$ have a non-zero discriminant is dropped, as is evident from the Introduction to the paper by Browning and Heath-Brown already cited.

One way of further studying the second equation is to follow the precedent set in connection with Waring's problem and to examine the values of n for which N is representable by $f(\boldsymbol{x})$ for almost all values of N that are compatible with the appropriate p-adic conditions. This we now do, following in the footsteps of Davenport and others, who shewed that almost all positive integers are a sum of four non-negative cubes. As a consequence we shall prove (i) that a quinary cubic form with non-vanishing discriminant represents almost all integers N of either sign that are p-adically consistent with such a representation and (ii) the corresponding proposition for quaternary cubics with non-vanishing discriminant provided that we assume the truth of the already mentioned Hypothesis HW for certain octonary forms. In their structure the proofs involve a procedure akin to that used previously in such investigations but also involve other ideas such as the employment of Fourier transforms and certain properties of cubic forms. Clearly, also, although we make no further mention of this, the method for quinary forms will apply, with a lessening degree of difficulty, to the cases $n = 6$ to $n = 9$, for which unconditional results for individual values of N are still unavailable.

These conclusions certainly suggest that (0) is soluble for all compatible integers N when $n = 5$ and even when $n = 4$. But we await the enunciation of the theorems stating them before enlarging on such speculations, at which point there will be a brief discussion about the limitations of our methods.

We shall draw heavily from the papers J_a–J_e and therefore assume the reader has them to hand for consultation. These are all based on the circle method, the form of which depends on the year they were compiled. The earlier papers used the classical Hardy–Littlewood method with Kloosterman refinement and the later ones used Heath-Brown's version of the method after it became available (J_c). However, although we can be fully satisfied by the way we have derived our conclusions through the use of these papers, we must mention that what we need in the circumstances of our application can be otherwise derived by a mere simplification of the classical circle method that invokes neither a Kloosterman refinement nor the ideas behind Heath-Brown's difficult method. Furthermore, we then see through this revision of the circle method that there are actually two different structures of proof through which such propositions can be substantiated. But to give an account of these developments here would be to outstrip the limits set for the present paper and we therefore intend to return to them on a later occasion.

The proofs of propositions (i) and (ii) are very similar. We therefore concentrate solely on the latter and more interesting proposition (ii) concerning quaternary forms until it is established and then indicate briefly how the former is to be handled.

2. Notation. We continue to use much of the notation that was defined in our paper J_e (i.e. [8] as stated above). In particular X denotes a positive real variable that is to be regarded as tending to infinity, all inequalities that are valid for sufficiently large values of X being assumed to hold; also p is a (positive) prime number.

Ordered n-tuples or vectors are indicated by bold type, where eventually n will take the value 4 or 8. In the latter instance an asterisk will be attached to these symbols; the zero vector is denoted by 0 even when $n = 8$.

The positive constants A_i depend at most on the coefficients of the given quaternary form $f(\boldsymbol{x})$, the constants implied by the O-notation being of this type.

3. Preparatory treatment. As stated in the Introduction, we confine ourselves to the treatment of proposition (ii) until it is proved and therefore now consider the representations of integers N by the non-singular quaternary cubic form $f(\boldsymbol{x}) = f(x_1, \ldots, x_4)$, where unless otherwise indicated we do not assume that $N \neq 0$ until §10. To do this, knowing that the theory of §3 and the beginning of §5 of J_a is still valid when $n = 4$ and $f(x_1, \ldots, x_n)$ therein becomes our given form $f(\boldsymbol{x})$, we continue to adopt the previous notation for the entities associated with $f(\boldsymbol{x})$ such as the Hessian matrix $\boldsymbol{M}(\boldsymbol{x})$. With this understood, we choose a fixed real point $\boldsymbol{a}$ on the hypersurface $f(\boldsymbol{x}) = 0$ that is a sufficiently large scalar multiple $\lambda\boldsymbol{a}'$ of a given real point $\boldsymbol{a}'$ on the hypersurface such that the Hessian $H(\boldsymbol{a}') = |\boldsymbol{M}(\boldsymbol{a}')|$ is non-zero. The basis of our investigation is then to be the counting function

$$r(N, X) = \sum_{f(\boldsymbol{l})=N} \Gamma_4\left(\frac{\boldsymbol{l}}{X} - \boldsymbol{a}\right) \tag{1}$$

where X is large and where

$$\Gamma_4(\boldsymbol{t}) = \prod_{1\le i\le 4} \gamma(t_i)$$

is defined through $J_a(10)$ or the statement at the beginning of §4 in J_d.

In like manner, the integral

$$H(u, \boldsymbol{v}; x) = \int_{(\boldsymbol{a}-\boldsymbol{v})}^{(\boldsymbol{a}+\boldsymbol{v})} \Gamma_4\left(\frac{\boldsymbol{t}}{X} - \boldsymbol{a}\right) e^{2\pi i(uf(\boldsymbol{t})+\boldsymbol{v}\boldsymbol{t})}\, d\boldsymbol{t} \tag{2}$$

appearing in $J_a(48)$ or J_d, Lemma 4, is to appertain to the case $n = 4$, where in line with $J_a(55)$ we still set ([1])

$$H(u, \boldsymbol{v}) = H(u, \boldsymbol{v}; 1)$$

and have now

$$H(u, \boldsymbol{v}; X) = X^4 H(uX^3, \boldsymbol{v}X); \tag{3}$$

also, for convenience, we shall write

$$H_1(u) = H(u, 0). \tag{4}$$

To gain some insight into the behaviour of $r(N, X)$ we temporarily bring in the function

$$g(\theta) = \sum_{\boldsymbol{l}} \Gamma_4\left(\frac{\boldsymbol{l}}{X} - \boldsymbol{a}\right) e^{2\pi i f(\boldsymbol{l})\theta} \qquad (n = 4) \tag{5}$$

defined in $J_a(11)$, and observe that we have

$$r(N, X) = \int_0^1 g(\theta) e^{-2\pi i N\theta}\, d\theta \tag{6}$$

as an analogue of $J_a(12)$. We may then consider in the heuristic way how the analysis of the penultimate paragraph in J_a, §19, might be affected by the change of integral from $J_a(12)$ to (6) and thereby to make the conjecture that

$$r(N, X) \sim \mathscr{S}(N, X)\mathfrak{S}(N)X = s(N, X), \qquad \text{say}, \tag{7}$$

where $\mathfrak{S}(N)$ is a surrogate for a singular series and

$$\mathscr{S}(N, X) = \int_{-\infty}^{\infty} H_1(\phi) e^{-2\pi i N\phi/X^3}\, d\phi \tag{8}$$

is a (finite) integral we shall discuss at the end of this section; also, in doing this, we should note that we may, if we wish, confine attention to the case where $|N| < AX^3$ for a suitable large A, since otherwise $r(N, X)$ is zero and does not give a meaningful portrayal of the representations of N by $f(\boldsymbol{x})$ ([2]). Emboldened by such meditations, we shall now define a certain octonary form through which we shall see that at least $s(N, X)$ is a fitting companion for $r(N, X)$ in a covariance that will shortly appear.

The octonary cubic form is

$$f^*(\boldsymbol{x}^*) = f^*(\boldsymbol{x}_1, \boldsymbol{x}_2) = f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2),$$

([1]) These functions are not to be confused with the Hessian $H(\boldsymbol{x})$.

([2]) We shall shew shortly that $s(N, X)$ is also zero for $|N| > A_1X^3$ for A_1 sufficiently large.

where, as is suggested, for any octuplet $\boldsymbol{\phi}^*$ we write $\boldsymbol{\phi}^* = (\boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$; furthermore, we add an asterisk to the relevant symbols associated with a cubic form when that form is $f^*(\boldsymbol{x}^*)$. From the non-singularity of $f(\boldsymbol{x})$ it is immediately apparent that $f^*(\boldsymbol{x}^*)$ is also non-singular, a fact that alternatively follows from the relation

$$D^* = D^{16} \tag{9}$$

between the discriminants of $f(\boldsymbol{x})$ and $f^*(\boldsymbol{x}^*)$ that we shall later need. Also, if $\boldsymbol{a}^* = (\boldsymbol{a}, \boldsymbol{a})$ where $\boldsymbol{a}$ is the point defined at the beginning of this section, then not only does $f^*(\boldsymbol{a}^*)$ vanish but also the Hessian $H^*(\boldsymbol{x}^*)$ of $f^*(\boldsymbol{x}^*)$ at $\boldsymbol{a}^*$ is non-zero, since

$$|\boldsymbol{M}^*(\boldsymbol{a}^*)| = \begin{vmatrix} \boldsymbol{M}(\boldsymbol{a}) & 0 \\ 0 & \boldsymbol{M}(\boldsymbol{a}) \end{vmatrix} = H^2(\boldsymbol{a}) \neq 0.$$

It is time to bring Hypothesis HW into play. This is stated in §12 of J_d for all non-singular octonary forms but will only be needed here for forms of the type $f^*(\boldsymbol{x}^*)$. On this assumption, starting as in J_d, §4, and ending as in the theorem in J_e, §7, we thus obtain an asymptotic formula for the sum

$$\Upsilon^*(X) = \sum_{f^*(\boldsymbol{l}^*)=0} \varGamma_8\left(\frac{\boldsymbol{l}^*}{X} - \boldsymbol{a}^*\right) = \sum_{f(\boldsymbol{l}_1)-f(\boldsymbol{l}_2)=0} \varGamma_4\left(\frac{\boldsymbol{l}_1}{X} - \boldsymbol{a}\right)\varGamma_4\left(\frac{\boldsymbol{l}_2}{X} - \boldsymbol{a}\right)$$

of the type

$$\Upsilon^*(X) = \mathscr{S}^*\mathfrak{S}^* X^5 + O(X^{5-\delta_1}) \tag{10}$$

for some (small) $\delta_1 > 0$. In this, $\mathfrak{S}^*$ is the singular series for the equation $f^*(\boldsymbol{x}^*) = 0$ and

$$\mathfrak{S}^* > 0$$

because non-zero equal determinations of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in $\mathbb{Z}_p$ provide a zero of $f^*(\boldsymbol{x}^*)$ in $\mathbb{Z}_p$ for every prime p (indeed, $f^*(\boldsymbol{x}^*)$ has non-trivial rational integral zeros). Also

$$\mathscr{S}^* = \mathscr{S}^*(0) > 0 \tag{11}$$

is the value at $y = 0$ of the function $\mathscr{S}^*(y)$ first appearing in §16 of J_d, being therefore equal to the integral

$$\int \frac{\varGamma_8(\boldsymbol{t}^* - \boldsymbol{a}^*)}{\partial f^*/\partial t_1}\, d\boldsymbol{t}'$$

taken over septuplets $\boldsymbol{t}'$ compatible with the conditions $\|\boldsymbol{t}' - \boldsymbol{a}'\| \leq 1$ and $f^*(\boldsymbol{t}^*) = 0$ (the components of $\boldsymbol{a}'$ are the last seven components of $\boldsymbol{a}^*$). But this is exactly the same as [3] $Z(0) = \mathscr{S}(1) = \mathscr{S}$ in §19 of J_a, whereupon

([3]) We cannot attach an asterisk to these two symbols, since otherwise there would be a confusion between these and those arising from J_d.

we deduce that

$$\mathscr{S}^* = \mathscr{S} = \int_{-\infty}^{\infty} H^*(\phi, 0; 1)\, d\phi = \int_{-\infty}^{\infty} H_1^*(\phi)\, d\phi, \qquad \text{say,}$$

from (173) in J_a (where $H_1^*(\phi)$ is the counterpart of $H_1(\phi)$ in (4)).

In beginning to turn back from the octonary form $f^*(\boldsymbol{x}^*)$ to the quaternary form $f(\boldsymbol{x})$ from which it was formed, we must confirm that

$$\begin{aligned} H_1^*(\phi) &= \int_{\boldsymbol{a}^*-\boldsymbol{v}^*}^{\boldsymbol{a}^*+\boldsymbol{v}^*} \Gamma_8\left(\frac{\boldsymbol{t}^*}{X} - \boldsymbol{a}^*\right) e^{2\pi i\phi f^*(\boldsymbol{t}^*)}\, d\boldsymbol{t}^* \\ &= \int_{\boldsymbol{a}-\boldsymbol{v}}^{\boldsymbol{a}+\boldsymbol{v}} \Gamma_4\left(\frac{\boldsymbol{t}_1}{X} - \boldsymbol{a}\right) e^{2\pi i\phi f(\boldsymbol{t}_1)}\, d\boldsymbol{t}_1 \int_{\boldsymbol{a}-\boldsymbol{v}}^{\boldsymbol{a}+\boldsymbol{v}} \Gamma_4\left(\frac{\boldsymbol{t}_2}{X} - \boldsymbol{a}\right) e^{-2\pi i\phi f(\boldsymbol{t}_2)}\, d\boldsymbol{t}_2 \\ &= |H_1(\phi)|^2 \end{aligned}$$

so that

$$\mathscr{S}^* = \int_{-\infty}^{\infty} |H_1(\phi)|^2\, d\phi. \tag{12}$$

Also, to elicit an important feature of $\mathscr{S}(N, X)$ in (8), we once again look at the analysis in the final paragraph of §19 in J_a for $n = 4$ and deduce, as before, that

$$H_1(\phi) = \int_{-\infty}^{\infty} e^{2\pi i\phi y} Z(y)\, dy,$$

where now

$$Z(y) = \int \frac{\Gamma_4(\boldsymbol{x} - \boldsymbol{a})}{|\partial f\, \partial x_1|}\, dx_2 \cdots dx_4$$

is an integral over the domain of $(x_2, \ldots, x_4)$ taken over

$$\|\boldsymbol{x} - \boldsymbol{a}\| \le 1, \qquad f(\boldsymbol{x}) = y.$$

Consequently, by (8) and Fourier's integral theorem,

$$\mathscr{S}(N, X) = Z(N/X^3)$$

and we incidentally confirm that the right side of (7), just as the left side, is zero when $|N|$ exceeds a bound of type $A_1 X^3$. Yet here, as in (173) of J_a, $Z(0) > 0$ and, by continuity, we deduce that

$$\mathscr{S}(N, X) \sim Z(0) = \mathscr{S}, \qquad \text{say,} \tag{13}$$

when $N = o(X^3)$.

As has been the situation so far, some of the following work involves the sums $H(u, \boldsymbol{v})$ in (3) and we therefore reproduce below Lemma 7 of J_a for the case $n = 4$.

Lemma 1. *We have*

$$H(u,\boldsymbol{v}) = \begin{cases} O(1) & \textit{always}, \\ O(|u|^{-2}\log^4 2u) & \textit{if } |u| \ge 1, \\ O(e^{-A_2\|\boldsymbol{v}\|^{1/2}}) & \textit{if } \|\boldsymbol{v}\| > A_3|u|. \end{cases}$$

4. Formation of the covariance. The covariance we shall introduce contains the difference between the constituents in (7) when $\mathfrak{S}(N)$ is suitably defined. To this end, we let $\nu(a,k)$ be the multiplicative function of k that counts the number of incongruent solutions, mod k, of

$$f(\boldsymbol{l}) \equiv a, \ \bmod k,$$

then agreeing that $\nu^*(k)$ is to be the number of incongruent solutions, mod k, of

$$f^*(\boldsymbol{l}) \equiv f^*(\boldsymbol{l}_1) - f^*(\boldsymbol{l}_2) \equiv 0, \ \bmod k,$$

in concordance with our notational practices; here, since $\nu^2(a,k)$ is the number of incogruent solutions, mod k, of $f(\boldsymbol{l}_1) \equiv f(\boldsymbol{l}_2) \equiv a, \ \bmod k$, we have

$$\nu^*(k) = \sum_{0<a\le k} \nu^2(a,k). \tag{14}$$

Then, for

$$\xi = [\sqrt{\delta_2 \log X}] \tag{15}$$

with a small positive value of δ_2, we set

$$K = \Big(\prod_{p\le\xi} p\Big)^{\xi} \le (e^{2\xi})^{\xi} = e^{2\xi^2} \le X^{2\delta_2} \tag{16}$$

and then decide to take

$$\mathfrak{S}(N) = \mathfrak{S}(N,K) = \frac{\nu(N,K)}{K^3}$$

so that through (8) we confirm that now

$$s(N,X) = \frac{X\nu(N,K)}{K^3} \int_{-\infty}^{\infty} H_1(\phi) e^{-2\pi i N\phi/X^3}\, d\phi. \tag{17}$$

Alongside these definitions, we must note the trivial inequalities

$$\frac{\nu(N,K)}{K^3}, \frac{\nu^*(N)}{K^7} \le K \le X^{2\delta_2}, \tag{18}$$

which will suffice for the time being but which will be replaced by keener versions in due course.

The constituents in the covariance

$$C(X) = \sum_{N=-\infty}^{\infty} \{r(N,X) - s(N,X)\}^2 \tag{19}$$

having been fully designated and giving rise to a finite sum because they vanish for $|N| > A_1X^3$ by an earlier observation, we therefore see that

$$(20)\quad C(X) = \sum_{N=-\infty}^{\infty} r^2(N,X) - 2\sum_{N=-\infty}^{\infty} r(N,X)s(N,X) + \sum_{N=-\infty}^{\infty} s^2(N,X)$$
$$= C_1(X) - 2C_2(X) + C_3(X), \quad \text{say.}$$

The estimation of $C_1(X)$ is easy and can be dismissed at once before we go on to the harder $C_2(X)$ and $C_3(X)$ in the succeeding sections. In fact, by (10),

$$C_1(X) = \sum_{N=-\infty}^{\infty}\Big(\sum_{f(\boldsymbol{l})=N} \varGamma_4\Big(\frac{\boldsymbol{l}}{X} - \boldsymbol{a}\Big)\Big)^2 = \sum_{f(\boldsymbol{l}_1)-f(\boldsymbol{l}_2)=0} \varGamma_4\Big(\frac{\boldsymbol{l}_1}{X} - \boldsymbol{a}\Big)\varGamma_4\Big(\frac{\boldsymbol{l}_2}{X} - \boldsymbol{a}\Big),$$

whence through (10) we gain the asymptotic formula

$$(21)\qquad C_1(X) = \mathscr{S}^*\mathfrak{S}^*X^5 - X^{5-\delta_1}.$$

5. Estimation of $C_2(X)$. First, by (1) and (17), the constituent $C_2(X)$ in (20) equals

$$(22)\quad X\sum_{N=-\infty}^{\infty} \frac{\nu(N,K)}{K^3}\Big(\int_{-\infty}^{\infty} H_1(\phi)e^{-2\pi iN\phi/X^3}\,d\phi\Big)\sum_{f(\boldsymbol{l})=N}\varGamma_4\Big(\frac{\boldsymbol{l}}{X} - \boldsymbol{a}\Big)$$
$$= X\sum_{N=-\infty}^{\infty} \frac{\nu(N,K)}{K^3}\Big\{\Big(\int_{-Y}^{Y} + \int_{|\phi|>Y}\Big)H_1(\phi)e^{-2\pi iN\phi/X^3}\,d\phi\Big\}\sum_{f(\boldsymbol{l})=N}\varGamma_4\Big(\frac{\boldsymbol{l}}{X} - \boldsymbol{a}\Big)$$
$$= X(C_2'(X) + C_2''(X)), \quad \text{say,}$$

for a suitable value of Y that it is convenient to choose here as

$$(23)\qquad Y = X^{1/2+3\delta_2}.$$

The second sum $C_2''(X)$ is quickly sent off because within it the integral is

$$O\Big(\int_Y^{\infty} |H_1(\phi)|\,d\phi\Big) = O\Big(\int_Y^{\infty} \frac{\log^4\phi}{\phi^2}\,d\phi\Big) = O\Big(\frac{\log^4 Y}{Y}\Big)$$

by Lemma 1, the consequential estimate

$$(24)\qquad C''(X) = O\Big(\frac{X^{2\delta_2}\log^4 Y}{Y}\sum_{\|\boldsymbol{l}\|\le X(1+\|\boldsymbol{a}\|)} 1\Big) = O(X^{7/2})$$

then flowing from (23) and the trivial estimate (18).

As we shall see, our method of estimating $C_2'(X)$ owes its success to the truncation of the integral which is contained within it. From (2) we have

$$(25)\quad C_2'(X) = \frac{1}{K^3}\sum_{0<a\le K}\nu(a,K) \times \sum_{\substack{N=-\infty \\ N\equiv a,\,\mathrm{mod}\,K}}^{\infty}\Big(\int_{-Y}^{Y} H_1(\phi)e^{-2\pi iN\phi/X^3}\,d\phi\Big)\sum_{f(\boldsymbol{l})=N}\Gamma_4\Big(\frac{\boldsymbol{l}}{X}-a\Big)$$
$$= \frac{1}{K^3}\sum_{0<a\le K}\nu(a,K) \times \int_{-Y}^{Y} H_1(\phi)\sum_{f(\boldsymbol{l})\equiv a,\,\mathrm{mod}\,K}\Gamma_4\Big(\frac{\boldsymbol{l}}{X}-a\Big)e^{-2\pi if(\boldsymbol{l})\phi/X^3}\,d\phi$$
$$= \frac{1}{K^3}\sum_{0<a\le K}\nu(a,K)\int_{-Y}^{Y} H_1(\phi)L_1(\phi;X,a,K)\,d\phi, \qquad \text{say,}$$

in which $L_1(\phi;X,a,K)$ is a sum that is analogous to an integral of the type $H(u,\boldsymbol{v};X)$ in (3) and that will be expressed in terms of such integrals by the Poisson summation formula.

The incongruent zeros $\boldsymbol{b}$, mod K, of $f(\boldsymbol{l})-a$ being $\nu(a,K)$ in number, we have

$$(26)\quad L_1(\phi;X,a,K) = \sum_{\substack{0<\boldsymbol{b}\le K \\ f(\boldsymbol{b})-a\equiv 0,\,\mathrm{mod}\,K}}\ \sum_{\boldsymbol{l}\equiv\boldsymbol{b},\,\mathrm{mod}\,K}\Gamma_4\Big(\frac{\boldsymbol{l}}{X}-\boldsymbol{a}\Big)e^{-2\pi if(\boldsymbol{l})\phi/X^3}$$
$$= \sum_{\substack{0<\boldsymbol{b}\le K \\ f(\boldsymbol{b})-a\equiv 0,\,\mathrm{mod}\,K}}\ \sum_{\boldsymbol{q}}\Gamma_4\Big(\frac{\boldsymbol{b}+K\boldsymbol{q}}{X}-\boldsymbol{a}\Big)e^{-2\pi if(\boldsymbol{b}+K\boldsymbol{q})\phi/X^3}$$
$$= \sum_{\substack{0<\boldsymbol{b}\le K \\ f(\boldsymbol{b})-a\equiv 0,\,\mathrm{mod}\,K}} L_2(\phi;X,\boldsymbol{b},K), \qquad \text{say.}$$

Then, by the substitution $\boldsymbol{s}=(\boldsymbol{b}+K\boldsymbol{t})/X$,

$$(27)\quad L_2(\phi;X,\boldsymbol{b},K) = \sum_{\boldsymbol{m}}\int_{-\infty}^{\infty}\Gamma_4\Big(\frac{\boldsymbol{b}+K\boldsymbol{t}}{X}-\boldsymbol{a}\Big)e^{-2\pi if(\boldsymbol{b}+K\boldsymbol{t})\phi/X^3+2\pi i\boldsymbol{m}\boldsymbol{t}}\,d\boldsymbol{t}$$
$$= \frac{X^4}{K^4}\sum_{\boldsymbol{m}}e^{-2\pi i\boldsymbol{m}\boldsymbol{b}/K}\int_{\boldsymbol{a}-\boldsymbol{v}}^{\boldsymbol{a}+\boldsymbol{v}}\Gamma_4(\boldsymbol{s}-\boldsymbol{a})e^{2\pi i\{-\phi f(\boldsymbol{s})+X\boldsymbol{m}\boldsymbol{s}/K\}}\,d\boldsymbol{s}$$
$$= \frac{X^4}{K^4}H_1(-\phi) + \frac{X^4}{K^4}\sum_{\boldsymbol{m}\ne 0}e^{-2\pi i\boldsymbol{m}\boldsymbol{b}/K}H(-\phi,X\boldsymbol{m}/K)$$

because of (3) and (4).

The effect $E^{\dagger}(X)$ on $C_2'(X)$ via (26) and (25) of the first term in the last line of (27) is in the first place

$$\frac{X^4}{K^7}\sum_{0<a\le K}\nu^2(a,K)\int_{-Y}^{Y}|H_1(\phi)|^2\,d\phi=\frac{X^4\nu^*(K)}{K^7}\int_{-Y}^{Y}|H_1(\phi)|^2\,d\phi$$

by (14). But, owing to Lemma 1 and (12),

$$\int_{-Y}^{Y}|H_1(\phi)|^2\,d\phi=\mathscr{S}^*+O\Big(\int_{Y}^{\infty}\frac{\log^8\phi}{\phi^4}\,d\phi\Big)=\mathscr{S}^*+O\Big(\frac{1}{Y^2}\Big),$$

from which, (23), and (18) we acquire the estimate

$$E^{\dagger}(X)=\mathscr{S}^*\frac{\nu^*(K)}{K^7}X^4+O(X^{7/2}). \tag{28}$$

To gauge the influence $E^{\dagger\dagger}(X)$ of the second portion of $L_2(\phi;X,\boldsymbol{b},K)$ we observe that for the terms of the infinite series therein we have

$$\Big\|\frac{X\boldsymbol{m}}{K}\Big\|\ge X^{1-2\delta_2}>A_3Y\ge A_3|\phi|$$

by (18) and (23). Hence, by Lemma 1, we have

$$H(-\phi,X\boldsymbol{m}/K)=O(e^{-A_2(\|\boldsymbol{m}\|X/K)^{1/2}})$$

and, having deduced that the second part of $L_2(\phi;X,\boldsymbol{b},K)$ is

$$O\Big(\frac{X^4}{K^4}e^{-A_2(X/K)^{1/2}}\Big)=O\Big(\frac{X^2}{K^4}\Big),$$

we find that

$$E^{\dagger\dagger}(X)=O\Big(\frac{X^2\nu^*(K)}{K^7}\int_{-\infty}^{\infty}|H_1(\phi)|\,d\phi\Big)=O(X^{7/2}) \tag{29}$$

by Lemma 1 and other previously cited estimates.

Finally, summarising the import of (22), (28), and (29), we derive the formula

$$C_2(X)=\mathscr{S}^*\frac{\nu^*(K)}{K^7}X^5+O(X^{9/2}). \tag{30}$$

6. Estimation of $C_3(X)$. Moving to $C_3(X)$, we find from (20) and (17) that

$$C_3(X) = X^2 \sum_{N=-\infty}^{\infty} \frac{\nu^2(N,K)}{K^6}\Big(\int_{-\infty}^{\infty} H_1(\phi)e^{-2\pi iN\phi/X^3}\,d\phi\Big)^2 \tag{31}$$
$$= \frac{X^2}{K^6}\sum_{0<a\le K}\nu^2(a,K)\sum_{q=-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} H_1(\phi)e^{-2\pi i(a+Kq)\phi/X^3}\,d\phi\Big)^2$$
$$= \frac{X^2}{K^6}\sum_{0<a\le K}\nu^2(a,K)J(a,K), \quad \text{say,}$$

in which $J(a,K)$ is to be transformed by the Poisson summation formula and the substitution $s=-(a+Kt)$ to get

$$J(a,K) = \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} H_1(\phi)e^{-2\pi i(a+Kt)\phi/X^3}\,d\phi\Big)^2 e^{2\pi imt}\,dt \tag{32}$$
$$= \frac{1}{K}\sum_{m=-\infty}^{\infty} e^{-2\pi iam/K}\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty} H_1(\phi)e^{2\pi is\phi/X^3}\,d\phi\Big)^2 e^{-2\pi ims/K}\,ds$$
$$= \frac{1}{K}\sum_{m=-\infty}^{\infty} e^{-2\pi iam/K}D_m(K), \quad \text{say.}$$

To calculate the items $D_m(K)$ in the above sum we deploy the Fourier transform

$$\widehat{H}_1(u) = \int_{-\infty}^{\infty} H_1(\phi)e^{2\pi iu\phi}\,d\phi,$$

which is real when u is real (compare with $\mathscr{S}(N,x)$). Hence first we have

$$D_0(K) = \int_{-\infty}^{\infty}\Big|\widehat{H}_1\Big(\frac{s}{X^3}\Big)\Big|^2\,ds = X^3\int_{-\infty}^{\infty}|\widehat{H}_1(u)|^2\,du \tag{33}$$
$$= X^3\int_{-\infty}^{\infty}|H_1(\phi)|^2\,d\phi = X^3\mathscr{S}^*$$

by Parseval's theorem and (12).

Next suppose that $m\ne 0$. Then the integrand in the outer integral defining $D_m(K)$ is the product of the integrals

$$\int_{-\infty}^{\infty} H_1(\phi)e^{2\pi is\phi/X^3}\,d\phi \quad\text{and}\quad \int_{-\infty}^{\infty} H_1(\phi)e^{2\pi is(\phi/X^3-m/K)}\,d\phi$$

that are, respectively, the real $\widehat{H}_1(s/X^3)$ and $\widehat{H}_\dagger(s/X^3)$, where $\widehat{H}_\dagger(u)$ is the

Fourier transform of $H_1(\phi + mX^3/K)$. Consequently,

$$D_m(K) = \int_{-\infty}^{\infty} \widehat{H}_1\Big(\frac{s}{X^3}\Big)\widehat{H}_\dagger\Big(\frac{s}{X^3}\Big)\,ds = X^3 \int_{-\infty}^{\infty} \overline{\widehat{H}_1(u)}\widehat{H}_\dagger(u)\,du$$

$$= X^3 \int_{-\infty}^{\infty} H_1(\phi)\overline{H_1\Big(\phi + \frac{mX^3}{K}\Big)}\,d\phi$$

and therefore

$$|D_m(K)| \le X^3 \int_{-\infty}^{\infty} |H_1(\phi)| \,\Big|H_1\Big(\phi + \frac{mX^3}{K}\Big)\Big|\,d\phi$$

$$= 2X^3 \int_0^{\infty} \Big|H_1\Big(\phi + \frac{|m|X^3}{2K}\Big)\Big|\,\Big|H_1\Big(\phi - \frac{|m|X^3}{2K}\Big)\Big|\,d\phi.$$

It then follows from Lemma 1 that

$$D_m(K) = O\bigg(\frac{X^3\log^4(|m|X^3/K)}{(|m|X^3/K)^2}\int_{-\infty}^{\infty}|H_1(\phi)|\,d\phi\bigg) \tag{34}$$

$$= O\bigg(\frac{X^{5\delta_2}}{|m|^{3/2}}\bigg) \qquad (m \ne 0)$$

when (16) is taken into account. Going back to (32) along with (33) and (34), we find at once that

$$J(a,K) = \frac{X^3\mathscr{S}^*}{K} + O\bigg(\frac{X^{5\delta_2}}{K}\sum_{m=1}^{\infty}\frac{1}{m^{3/2}}\bigg) = \frac{X^3\mathscr{S}^*}{K} + O\bigg(\frac{X^{5\delta_2}}{K}\bigg)$$

and deduce through (31) and (18) the formula

$$C_3(X) = \mathscr{S}^*\frac{\nu^*(K)}{K^7}X^5 + O\bigg(\frac{X^{2+5\delta_2}\nu^*(K)}{K^7}\bigg) = \mathscr{S}^*\frac{\nu^*(K)}{K^7}X^5 + O(X^{9/2}) \tag{35}$$

that we sought.

7. Further estimates associated with the covariance. There are three things to be done before we insert the formulae for $C_1(X)$, $C_2(X)$, and $C_3(X)$ in the expression (20) for $C(X)$ and then interpret the resultant effect. The first is to compare the singular series $\mathfrak{S}^*$ in (10) or (21) with $\nu^*(K)/K^7$, the second is to find lower bounds for $\mathfrak{S}(N,K) = \nu(N,K)/K^3$ when N adheres to the Supposition P stated below, and the third is to discuss the distribution of such numbers N. All these tasks are undertaken by familiar procedures with the aid, where helpful, of references to our paper J_a. We now define the

SUPPOSITION P. *The number N is p-adically representable by $f(\boldsymbol{x})$ for all primes p through vectors $\boldsymbol{x}$ with components in $\mathbb{Z}_p$.*

We begin by stating a lemma that is proved by using the method that established (86) in J_a.

LEMMA 2. *Let $g(\boldsymbol{x})$ be a non-singular cubic form in $n \geq 3$ variables having rational integral coefficients and non-zero discriminant Δ, where for each prime p the number $\gamma' = \gamma'_p$ is defined by $p^{\gamma'} \parallel \Delta$. Then, if $\rho'(N_1, p^\theta)$ be the number of incongruent (primitive) solutions of*

$$g(\boldsymbol{l}) \equiv N_1, \ \bmod p^\theta, \qquad (\boldsymbol{l}, p) = 1,$$

we have

$$\frac{\rho'(N_1, p^\theta)}{p^{(n-1)\theta}} = \frac{\rho'(N_1, p^{2\gamma'+1})}{p^{(n-1)(2\gamma'+1)}}$$

for $\theta \geq 2\gamma_1 + 1$.

We should stress that the truth of the result depends on the fact that only primitive solutions of the congruences are being counted.

Also required is the following surely familiar lemma that extends equation (86) in J_a.

LEMMA 3. *In the notation of Lemma 2, we have*

$$\rho'(N_1, p) = p^{n-1} + O(p^{\frac{1}{2}n})$$

for $p > A_4$ and bounded discriminant Δ.

Since $p \nmid \Delta$ for a suitable choice of A_4, the form $g(\boldsymbol{x})$ is non-singular, mod p, as is the form $g(\boldsymbol{x}) - N_1 x_{n+1}^3$, mod p, when $p \nmid N_1$. Hence first, if $p \mid N_1$, then $\rho'(N_1, p)$ equals $p-1$ times the number of points with coefficients in $\mathbb{F}_p$ on the non-singular projective hypersurface $g(\boldsymbol{x}) = 0$ of dimension $n-2$ and is therefore

$$(p-1)\left(\frac{p^{n-1}-1}{p-1} + O(p^{\frac{1}{2}(n-2)})\right) = p^{n-1} + O(p^{\frac{1}{2}n})$$

by a celebrated theorem of Deligne's [2]. But, if $p \nmid N_1$, the sum $\rho'(N_1, p)$ equals the number of points with coordinates in $\mathbb{F}_p$ on the projective hypersurface $g(\boldsymbol{x}) - N_1 x_{n+1}^3 = 0$ for which $x_{n+1} \neq 0$. Therefore, counting all the points on the latter hypersurface that do not lie on $g(\boldsymbol{x}) = 0$, we deduce that

$$\begin{aligned}(36) \qquad \rho'(N_1, p) &= \frac{p^n - 1}{p-1} + O(p^{\frac{1}{2}(n-1)}) - \frac{p^{n-1}-1}{p-1} + O(p^{\frac{1}{2}(n-2)}) \\ &= p^{n-1} + +O(p^{\frac{1}{2}(n-1)})\end{aligned}$$

by again using Deligne's theorem. This concludes the proof; we note, however, that we shall not avail ourselves ([4]) of the better remainder term in (36).

([4]) Except implicitly in §10 in connection with (52).

Apart from its positivity stated earlier, what we need to know about $\mathfrak{S}^*$ is to be found in §19 of J_a and §16 of J_d and consists of the equations

$$\mathfrak{S}^* = \prod_p \sum_{\alpha=0}^{\infty} \frac{Q(0,p^\alpha)}{p^{8\alpha}},$$

$$\frac{Q(0,p^\alpha)}{p^{8\alpha}} = \frac{\nu^*(p^\alpha)}{p^{7\alpha}} - \frac{\nu^*(p^{\alpha-1})}{p^{7(\alpha-1)}} = O\Big(\frac{1}{p^{5\alpha/3}}\Big) \quad (\alpha > 0),$$

and

$$Q(0,1) = 1 \ (^5).$$

From these we infer that

$$\begin{aligned}\mathfrak{S}^* &= \prod_{p\le\xi}\Big(\frac{\nu^*(p^\xi)}{p^{7\xi}} + \sum_{\alpha>\xi}\frac{Q(0,p^\alpha)}{p^{8\alpha}}\Big)\prod_{p>\xi}\sum_{\alpha=0}^{\infty}\frac{Q(0,p^\alpha)}{p^{8\alpha}}\\ &= \prod_{p\le\xi}\Big\{\frac{\nu^*(p^\xi)}{p^{7\xi}} + O\Big(\sum_{\alpha>\xi}\frac{1}{p^{5\alpha/3}}\Big)\Big\}\prod_{p>\xi}\Big\{1 + O\Big(\sum_{\alpha=1}^{\infty}\frac{1}{p^{5\alpha/3}}\Big)\Big\}\\ &= \prod_{p\le\xi}\Big\{\frac{\nu^*(p^\xi)}{p^{7\xi}} + O\Big(\frac{1}{p^{5\xi/3}}\Big)\Big\}\prod_{p>\xi}\Big\{1 + O\Big(\frac{1}{p^{5/3}}\Big)\Big\}\\ &= \Big\{1 + O\Big(\frac{1}{\xi^{2/3}}\Big)\Big\}\prod_{p\le\xi}\Big\{\frac{\nu^*(p^\xi)}{p^{7\xi}} + O\Big(\frac{1}{p^{5\xi/3}}\Big)\Big\},\end{aligned}$$

to proceed from which we introduce the modification $\nu^{*\prime}(k)$ of $\nu^*(k)$ that is the number of (primitive) solutions $\boldsymbol{l}^*$, mod k, of

$$f^*(\boldsymbol{l}^*) \equiv 0, \ \mathrm{mod}\ k, \qquad (\boldsymbol{l}^*, k) = 1.$$

Here, because of the configuration of $f^*(\boldsymbol{l}^*)$ as a difference $f(\boldsymbol{l}_1) - f(\boldsymbol{l}_2)$, $\nu^{*\prime}(k)$ is always positive; moreover, if $p^\gamma \parallel D$ so that $p^{16\gamma} \parallel D^*$ by (9), then for $p \le A_5$, Lemma 2 gives

$$\frac{\nu^*(p^\xi)}{p^{7\xi}} \ge \frac{\nu^{*\prime}(p^\xi)}{p^{7\xi}} = \frac{\nu^{*\prime}(p^{(32\gamma+1)})}{p^{7(32\gamma+1)}} > A_6,$$

which inequality subsists for $p > A_5$ by Lemma 3 when A_5 is chosen suitably. Then

$$\mathfrak{S}^* = \frac{\nu^*(K)}{K^7}\Big\{1+O\Big(\frac{1}{\xi^{2/3}}\Big)\Big\}\prod_{p\le\xi}\Big\{1+O\Big(\frac{1}{p^{5\xi/3}}\Big)\Big\} = \frac{\nu^*(K)}{K^7}\Big\{1+O\Big(\frac{1}{\xi^{2/3}}\Big)\Big\},$$

whence obviously we gain the equation

$$\mathfrak{S}^* = \frac{\nu^*(K)}{K^7} + O\Big(\frac{1}{\xi^{2/3}}\Big) \tag{37}$$

(5) The subscript on $Q(0,p^\alpha)$ in J_a is now superfluous and is therefore omitted as in J_d and J_e.

and the improvement

$$\frac{\nu^*(K)}{K^7} = O(1) \tag{38}$$

of the earlier (18).

When turning to

$$\mathfrak{S}(N,K) = \prod_{p\le\xi} \frac{\nu(N,p^\xi)}{p^{3\xi}} \tag{39}$$

on the assumption of Supposition P, we shall find that the main difficulty in its estimation arises when N is divisible by a high power of a small prime p not exceeding a suitably chosen number p_0. Consequently, in reaction to this state of affairs and introducing the large integer ([6])

$$\eta = [\log\log\xi] \tag{40}$$

defined in terms of ξ in (15), we shall restrict attention to the scene where N is indivisible by powers of primes up to p_0 with exponents exceeding η, the remaining numbers N forming an exceptional set of which we take account later; thus, in particular, the number $N = 0$ is excluded, a fact of little significance because at this stage our emphasis is on large values of N.

Yet, first taking the easy case where the prime p in the product (39) exceeds p_0, let $\nu'(N,k)$ denote the contribution to $\nu(N,k)$ of its primitive solutions, mod k, and deduce from Lemmata 2 and 3 that

$$\frac{\nu(N,p^\xi)}{p^{3\xi}} \ge \frac{\nu'(N,p^\xi)}{p^{3\xi}} = \frac{\nu'(N,p)}{p^3} > 1 - \frac{A_7}{p},$$

whence follows the bound

$$\prod_{p_0<p\le\xi} \frac{\nu(N,p^\xi)}{p^{3\xi}} > \prod_{p_0<p\le\xi}\left(1-\frac{A_7}{p}\right) > \frac{A_8}{\log^{A_7}\xi} \tag{41}$$

for one part of the product contained in $\mathfrak{S}(N,K)$.

In finding the lower bound for the complementary factor in $\mathfrak{S}(N,K)$, we adopt the hitherto unused Supposition P and in considering the congruence

$$f(\boldsymbol{l}) - N \equiv 0, \ \bmod p^\alpha, \tag*{$(42)_\alpha$}$$

assume as agreed that for each p not exceeding p_0 we have $p^\beta \parallel N$ where $\beta \le \eta$. Certainly, if $\alpha = \alpha_1$ where $\alpha_1 = \beta + 1 + 2\gamma$, there are solutions $\boldsymbol{l}$ of $(42)_\alpha$ belonging to an exponent δ defined by $p^\delta \parallel \boldsymbol{l}$ so that $\boldsymbol{l} = p^\delta\boldsymbol{l}'$ and $p \nmid \boldsymbol{l}'$. Then, corresponding to *all* such solutions appertaining to this exponent δ, the defining congruence is tantamount to

$$p^{3\delta} f(\boldsymbol{l}') - N \equiv 0, \ \bmod p^{\alpha_1}, \quad 0 < \boldsymbol{l}' \le p^{\alpha_1-\delta}, \quad (\boldsymbol{l}',p) = 1$$

([6]) This choice of η is not optimal but suffices. A larger value of η is possible but would complicate our calculations.

and hence to

$$(43)_{\alpha_1} \qquad f(\boldsymbol{l}') - Np^{-3\delta} \equiv 0, \text{ mod } p^{\alpha_1 - 3\delta}, \qquad 0 < \boldsymbol{l}' \le p^{\alpha_1-\delta}, \qquad (\boldsymbol{l}', p) = 1$$

because the former displayed congruence implies that $3\delta \le \beta$. Moreover, with $\alpha \ge \alpha_1$ replacing α_1, the solutions $\boldsymbol{l}'$ of $(43)_\alpha$ stem from the solutions of $(42)_{\alpha_1}$ that belong to the exponent δ already defined.

We can complete our calculation by applying Lemma 2 to the case where $g(\boldsymbol{l}) = f(\boldsymbol{l})$, $N_1 = Np^{-3\delta}$, and $\theta = \alpha_1 - 3\delta$ or $\alpha - 3\delta$, since the numbers of solutions of $(42)_{\alpha_1}$ and $(42)_\alpha$ belonging to the exponent δ are, respectively,

$$p^{8\delta}\rho'(N_1, p^{\alpha_1-3\delta}) \quad \text{and} \quad p^{8\delta}\rho'(N_1, p^{\alpha-3\delta})$$

where obviously $\rho'(N_1, p^{\alpha_1-3\delta}) \ge 1$. Hence, by the lemma and the inequality $\alpha_1 - 3\delta = \beta - 3\delta + 1 + 2\gamma \ge 1 + 2\gamma$, we see that $\rho'(N_1, p^{2\gamma+1})$ is also positive and then that

$$(44) \qquad \frac{\nu(N, p^\xi)}{p^{3\xi}} \ge \frac{\rho'(N_1, p^{\xi-3\delta})}{p^\delta p^{3(\xi-3\delta)}} = \frac{\rho'(N_1, p^{2\gamma+1})}{p^\delta p^{3(2\gamma+1)}} > \frac{A_9}{p^\delta} \ge \frac{A_9}{p^{\eta/3}}$$

for some (small) positive constant A_9 depending on p_0.

Consequently, combining (41) and (44) in the product (39) defining $\mathfrak{S}(N, K)$, we arrive at the lower bound

$$(45) \qquad \mathfrak{S}(N, K) > \frac{1}{A_{10}^\eta \log^{A_7} \xi} > \frac{1}{\log^{A_{11}} \xi} \qquad (\beta \le \eta)$$

in the light of (40). We should, however, add the remark that in the special excluded case where $N = 0$ this inequality could have been deduced more easily in another manner.

Although this bound suffices for the establishment of our final result, we shall now shew for interest that the restriction on β is unnecessary when $f(\boldsymbol{x})$ has a non-trivial zero in every p-adic field and hence when it has a primitive zero in $\mathbb{Z}_p$. Since the inequality (41) is still valid, we need only take the case where $p \le p_0$ in the factors within the product for $\mathfrak{S}(N, K)$, dividing our attention between the situations in which either $\beta \ge 2\gamma + 1$ or $\beta < 2\gamma + 1$. In the former instance the new hypothesis implies that the congruence

$$f(\boldsymbol{l}') - N \equiv f(\boldsymbol{l}') \equiv 0, \text{ mod } p^{2\gamma+1},$$

has a primitive solution $\boldsymbol{l}'$, mod $p^{2\gamma+1}$, wherefore

$$(46) \qquad \frac{\nu(N, p^\xi)}{p^{3\xi}} \ge \frac{\nu'(N, p^\xi)}{p^{3\xi}} = \frac{\nu'(N, p^{2\gamma+1})}{p^{3(2\gamma+1)}} \ge \frac{1}{p^{3(2\gamma+1)}}$$

by Lemma 2. But in the latter instance, the reasoning that led to (44) is yet applicable so that, with a value of δ less than $\frac{1}{3}(2\gamma + 1)$, we obtain

$$\frac{\nu(N, p^\xi)}{p^{3\xi}} \ge \frac{A_9}{p^{2\gamma/3}},$$

which when taken with (46) and (41) yields (45) without the condition on β.

Our last undertaking in this section is to use the ideas behind Hensel's Lemma to discuss the numbers N for which Supposition P is vindicated. First, for $p > p_1$ and therefore $p \nmid D$, we have shewn in Lemma 3 that the congruence

$$f(\boldsymbol{l}^{(1)}) - N \equiv 0, \bmod p, \qquad \boldsymbol{l}^{(1)} \not\equiv 0, \bmod p,$$

is soluble for every integer N. Furthermore it is then implicit in the proof of Lemma 2, which was based on the ideas of §10 of J_a, that for each N there is a sequence of vectors $\boldsymbol{l}^{(\alpha)}$ for which

$$\boldsymbol{l}^{(\alpha+1)} \equiv \boldsymbol{l}^{(\alpha)}, \bmod p^{\alpha}, \qquad f(\boldsymbol{l}^{(\alpha)}) - N \equiv 0, \bmod p^{\alpha},$$

whence $f(\boldsymbol{l}) = N$ is p-adically soluble.

Secondly, in the complementary case, take any vector designated by $\boldsymbol{l}^{(2\gamma+1)}$ for which $(\boldsymbol{l}^{(2\gamma+1)}, p) = 1$ for all $p \le p_1$ and choose any integer M that is congruent to $f(\boldsymbol{l}^{(2\gamma+1)})$, mod $p^{2\gamma+1}$, under the same stipulation. Then, once again by a construction similar to that used in §10 of J_a, there is a sequence of vectors $\boldsymbol{l}^{(\alpha)}$ together with a non-negative integer $\delta \le \gamma$ such that, for each $p \le p_1$ and $\alpha \ge 2\gamma + 1$,

$$\boldsymbol{l}^{(\alpha+1)} \equiv \boldsymbol{l}^{(\alpha)}, \bmod p^{\alpha-\delta}, \qquad M \equiv f(\boldsymbol{l}^{(\alpha)}), \bmod p^{\alpha};$$

hence, in this case also, the equation $f(\boldsymbol{l}) = M$ is p-adically soluble, the same being so for all numbers N that are congruent to M to the modulus

$$\prod_{p \le p_1} p^{2\gamma+1}.$$

We therefore deduce

LEMMA 4. *The integers N that are p-adically representable by the form $f(\boldsymbol{x})$ for all primes p have positive density.*

This lemma is susceptible to improvement but suffices for our present needs.

8. The estimation of the covariance and the deduction of the first theorem. We are ready to treat the covariance $C(X)$ and to use the ensuing estimate to obtain our first theorem. Indeed, combining the assessments (21), (30), and (35) for the constituents in the formula (20) for $C(X)$, we first find immediately that

$$\begin{aligned} C(X) &= \mathscr{S}^* \mathfrak{S}^* X^5 - 2\mathscr{S}^* \frac{\nu^*(K)}{K^7} X^5 + \mathscr{S}^* \frac{\nu^*(K)}{K^7} X^5 + O(X^{5-\delta_1}) \\ &= \mathscr{S}^* \mathfrak{S}^* X^5 - \mathscr{S}^* \frac{\nu^*(K)}{K^7} X^5 + O(X^{5-\delta_1}), \end{aligned}$$

which equality becomes

$$C(X) = O\left(\frac{X^5}{\xi^{2/3}}\right) + O(X^{5-\delta_1}) = O\left(\frac{X^5}{\xi^{2/3}}\right) \tag{47}$$

when (37) and (15) are taken into account.

We can go on to our principal object of study that concerns the representation by $f(\boldsymbol{x})$ of numbers that conform to Supposition P and that in magnitude range up to a large limit u. Accordingly, now supposing that X be chosen so that

$$\frac{X}{u^{1/3}} = \xi^{1/36},$$

from (19) and (47) we get

$$\sum_{|N| \le u} \{r(N,X) - s(N,X)\}^2 = O\left(\frac{X^5}{\xi^{2/3}}\right)$$

and deduce that the number of integers N for which

$$|N| \le u \tag{48}$$

and

$$|r(N,X) - s(N,X)| > \mu X$$

does not exceed

$$O\left(\frac{X^5}{\mu^2 X^2 \xi^{2/3}}\right) = O\left(\frac{X^3}{\xi^{1/3}}\right) = O\left(\frac{u}{\xi^{1/4}}\right)$$

when $\mu = \xi^{-1/6}$. Also, if as in the treatment of $\mathfrak{S}(N,K)$ in §7 we suppose that N be not a multiple of any of $2^{\eta+1}, \ldots, {p_0}^{\eta+1}$, then we exclude from the interval (48) a set of N of measure

$$O\left\{u\left(\frac{1}{2^{\eta+1}} + \cdots + \frac{1}{{p_0}^{\eta+1}}\right)\right\} = O\left(\frac{u}{2^\eta}\right) = O\left(\frac{u}{(\log \xi)^{\log 2}}\right) \tag{49}$$

by (40). Consequently, save for a set of N in (48) of measure

$$O\left(\frac{u}{\xi^{1/4}}\right) + O\left(\frac{u}{(\log \xi)^{\log 2}}\right) = O\left(\frac{u}{(\log \xi)^{\log 2}}\right)$$

we have both

$$r(N,X) = s(N,X) + O\left(\frac{X}{\xi^{1/6}}\right)$$

and also, by (13), (17), and (45),

$$s(N,X) \sim \mathscr{S}\mathfrak{S}(N,K)X > \frac{A_{12}X}{\log^{A_{11}} \xi}.$$

Then, bearing in mind Lemma 4, we arrive at

THEOREM 1. *Suppose that $f(\boldsymbol{x})$ is a quaternary cubic form with rational integral coefficients and non-zero discriminant. Then, on the appropriate form of Hypothesis HW as earlier stated, almost all numbers N p-adically representable by $f(\boldsymbol{x})$ for all p are representable by $f(\boldsymbol{x})$ through rational integral values of the components of $\boldsymbol{x}$. These numbers N have positive lower density.*

We note that the proof can be simplified if $f(\boldsymbol{x})$ have a non-trivial zero with components in $\mathbb{Z}_p$ for every p. This is because here we no longer have to consider the measure (49) in view of the irrelevance of the number β in the second estimate for $\mathfrak{S}(N, K)$ defined in §7.

9. Quinary forms. As we shall shortly see, we part company with Hypothesis HW when considering the analogue of Theorem 1 for quinary forms and shall accordingly obtain an unconditional result. Otherwise, we follow almost verbatim our treatment of the quaternary case, now taking $f(\boldsymbol{x}) = f(x_1, \ldots, x_5)$ to be a non-singular quinary form and then defining $f^*(\boldsymbol{x}^*) = f^*(\boldsymbol{x}_1, \boldsymbol{x}_2) = f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)$ to be the non-singular denary form associated with $f(\boldsymbol{x})$. Letting the previous notation relate to the new situation where $f(\boldsymbol{x})$ contains five indeterminates, we can imitate all that went before in an obvious way save for an important exception, which concerns the sum

$$\Upsilon^*(X) = \sum_{f^*(\boldsymbol{l}^*)=0} \Gamma_{10}\left(\frac{\boldsymbol{l}^*}{X} - \boldsymbol{a}^*\right) \tag{50}$$

that is parallel to the sum in (10). At this point, because there is no asymptotic formula for this sum in the literature that we can use, we find ourselves in a dilemma even though the thrust of what we know about cubic forms indicates what this formula should be. On the one hand, we could point to Heath-Brown's unconditional work [3] on denary forms where he treats a similar sum with a different weight function and then assert that this reasoning can be applied to $\Upsilon^*(x)$. But, on the other hand, we can look back at our papers J_a and J_b and see that a simplification of the argument there for nonary forms yields what we need (or, if we look at it in reverse, having seen that a method successful for ten variables failed for nine variables, we were forced to modify it very substantially indeed to be victorious in the latter situation). Either way, we can be confident that the parallel of formula (10) is (unconditionally) true for the sum (50) and we can therefore assert the truth of

THEOREM 2. *Suppose that $f(\boldsymbol{x})$ is a quinary cubic form with rational integral coefficients and non-zero discriminant. Then almost all numbers N p-adically representable by $f(\boldsymbol{x})$ for all p are representable by $f(\boldsymbol{x})$ through rational integral values of the components of $\boldsymbol{x}$. These numbers N have positive lower density.*

10. Final comments. It suffices to refer to quaternary forms in making our final comments. Our work can be regarded as partially vindicating our conjecture in (7) when we modify it slightly so that it still asserts that, with a suitable interpretation of $\mathfrak{S}(N)$,

$$r(N,X) \sim \mathscr{S}\mathfrak{S}(N)X \tag{51}$$

for large N when $X^3/|N|$ tends to infinity slowly and N is indivisible by high powers of small prime numbers. Here $\mathfrak{S}(N)$ can be, or can be associated with, the formal singular series that arises in the problem of representing N by $f(\boldsymbol{x})$ and that itself is *formally* equal to the infinite product

$$\prod_p \lim_{\alpha\to\infty} \frac{\nu(N,p^\alpha)}{p^{3\alpha}}. \tag{52}$$

This, as may already be familiar or as the methods of §7 can shew, is convergent and, for $N \neq 0$, has factors that are actually of the form $\nu(N,p^{u(p,N)})/p^{3u(p,N)}$ where $u_{p,N}$ is a non-zero integer depending on p and N. There is therefore a close correspondence between the singular series and our choice of $\mathfrak{S}(N) = \mathfrak{S}(N,K)$ in (39).

We are thus led to the proposal that $f(\boldsymbol{x})$ represents any large integer N that is p-adically compatible with such a representation and that is indivisible by high powers of small primes, the exponents of which are limited below in terms of N; moreover, as suggested by §7, we can dispense with the last condition if $f(\boldsymbol{x})$ have a non-trivial zero in every p-adic field $\mathbb{Q}_p$. Such a conclusion, however, though much to be desired, would not fully answer to our wishes because of the largeness of N. This deficiency arises because of the way u and N have been linked in the work of §8 so that, for most N, the vectors $\boldsymbol{x}$ providing representations by $f(\boldsymbol{x})$ are restricted in size by not very much more than $N^{1/3}$. However, if this restriction on $\boldsymbol{x}$ were removed, we would expect that, for any given $N \neq 0$, the equivalence (7) would hold as $X \to \infty$ with the result that $f(\boldsymbol{x})$ would represent N when $f(\boldsymbol{x}) - N = 0$ is p-adically soluble for all p.

Acknowledgements. We are greatly indebted to the Heilbronn Institute for Mathematical Research in Bristol for support during the creation of this paper.

References

[1] T. D. Browning and D. R. Heath-Brown, *Integral points on cubic hypersurfaces*, in: Analytic Number Theory, Essays in Honour of Klaus Roth, Cambridge Univ. Press, 2009, 75–90.

[2] P. Deligne, *La conjecture de Weil. I*, Inst. Hautes Études Sci. Publ. Math. 43 (1974), 273–307.

[3] D. R. Heath-Brown, *Cubic forms in ten variables*, Proc. London Math. Soc. (3) 47 (1983), 225–257.

[4] D. R. Heath-Brown, *A new form of the circle method, and its application to quadratic forms*, J. Reine Angew. Math. 481 (1996), 149–206.

[5] C. Hooley, *On nonary cubic forms*, J. Reine Angew. Math. 386 (1988), 32–98.
[6] C. Hooley, *On nonary cubic forms: III*, J. Reine Angew. Math. 456 (1994), 53–63.
[7] C. Hooley, *On octonary cubic forms*, Proc. London Math. Soc. (3) 109 (2014), 241–281.
[8] C. Hooley, *On octonary cubic forms: II*, Bull. London Math. Soc. 47 (2015), 85–94.

In the text the papers [5], [6], [4], [7], and [8] are, respectively, denoted by J_a, J_b, J_c, J_d, and J_e.

C. Hooley
School of Mathematics
Cardiff University
Senghennydd Road
Cardiff CF24 4AG, United Kingdom