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AN EXTENSION OF A BOUNDEDNESS RESULT FOR SINGULAR INTEGRAL OPERATORS

BY

DENIZ KARLI (Istanbul)

Abstract. We study some operators originating from classical Littlewood–Paley theory. We consider their modification with respect to our discontinuous setup, where the underlying process is the product of a one-dimensional Brownian motion and a d-dimensional symmetric stable process. Two operators in focus are the G^* and area functionals. Using the results obtained in our previous paper, we show that these operators are bounded on L^p. Moreover, we generalize a classical multiplier theorem by weakening its conditions on the tail of the kernel of singular integrals.

1. Introduction and preliminaries. Boundedness of singular integral operators has been studied for a long time. There are some well-known results which were proved first by using classical analytic techniques. In these techniques, there are some important operators providing intermediate steps for the proof. Three often used operators are Lusin's area functional (A_f), the non-tangential maximal function (N_α^f) and the G^* functional (G_f^*). They played an important role in the development of harmonic analysis (see Stein [17] and [18]).

With the introduction of probabilistic techniques, alternative proofs have surfaced. In these classical techniques, Brownian motion plays a central role. One such approach is to consider a $(d+1)$-dimensional Brownian motion on the upper half-space and provide a probabilistic definition of harmonic functions in terms of martingales. By means of martingales, one can define Littlewood–Paley functions and hence provide probabilistic proofs of boundedness of some operators. (See, for example, Varopoulos [20], Burkholder and Gundy [7], Burkholder, Gundy and Silverstein [8], Durrett [9] and Bass [2]. For a more detailed literature overview on square functions and these operators, see Bañuelos and Davis [4].)

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In [12], we studied a more general process in the $(d+1)$-dimensional half-space $\mathbb{R}^d \times \mathbb{R}^+$. We would like to obtain generalizations of some theorems using probabilistic techniques and the weaker conditions imposed on the process we start with. This paper can be considered as a continuation of [12].

The main results of this paper include (i) boundedness of two important operators, namely the area functional and the G^* functional, and (ii) an extension of a classical multiplier theorem for singular integrals with kernels $\kappa : \mathbb{R}^d \to \mathbb{R}$ satisfying the cancelation property

$$\int_{r<|x|<R} \kappa(x)\, dx = 0 \quad \text{for all } 0 < r < R. \tag{1.1}$$

Together with a smoothing condition and some control on the tail, it is known that the corresponding convolution operator is bounded. The classical version is stated as follows. (The proof of the case $d = 1$ is given in [2, Theorem 5.3, p. 270]. For $d > 1$, the same argument applies easily with a slight modification. See also [1, Theorem 1.1].)

Theorem 1.1. *Suppose κ is the kernel of a convolution operator T. If $\kappa \in \mathcal{C}^1$ and κ satisfies the cancelation condition* (1.1), *and if*

$$|\kappa(x)| \le c|x|^{-d} \quad \text{and} \quad |\nabla\kappa(x)| \le c|x|^{-d-1}, \qquad x \neq 0, \tag{1.2}$$

then for any $1 < p < \infty$ there is a finite constant c_p depending only on p such that

$$\|T\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} < c_p.$$

Our goal is to weaken the condition (1.2) by replacing d in the exponent with $d - 1 + \alpha/2$ for some $\alpha \in (1, 2)$ when $|x| > 1$ (Theorem 2.8). We note that for $\alpha = 2$, we obtain (1.2).

First we introduce our notation and some preliminary results. Throughout the paper, c will denote a positive constant whose value may change from line to line.

We consider a d-dimensional right continuous rotationally symmetric α-stable process $(Y_t)_{t\ge0}$ for $\alpha \in (0, 2)$, that is, $(Y_t)_{t\ge0}$ is a right continuous Markov process with independent and stationary increments whose characteristic function is $\mathbb{E}(e^{i\xi Y_s}) = e^{-s|\xi|^\alpha}$, $\xi \in \mathbb{R}^d$, $s > 0$. By $p(s, x, y)$, we denote its (symmetric) transition density such that

$$\mathbb{P}^x(Y_s \in A) = \int_A p(s, x, y)\, dy,$$

and P_s is the corresponding semigroup $P_s(f)(x) = \mathbb{E}^x(f(Y_s))$. Here $\mathbb{P}^x$ is the probability measure for the process started at $x \in \mathbb{R}^d$, and $\mathbb{E}^x$ is the expectation with respect to $\mathbb{P}^x$. The transition density $p(s, x, 0)$ satisfies the scaling property

$$p(s, x, 0) = s^{-d/\alpha} p(1, x/s^{1/\alpha}, 0), \qquad x \in \mathbb{R}^d,\ s > 0. \tag{1.3}$$

Similarly, we denote by Z_s a one-dimensional Brownian motion (independent of Y_s) and by $\mathbb{P}^t$ the probability measure for the process started at $t > 0$. The process of interest is the product $X_s = (Y_s, Z_s)$ started at $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$; the corresponding probability measure and expectation are $\mathbb{P}^{(x,t)}$ and $\mathbb{E}^{(x,t)}$, respectively. Define the stopping time $T_0 = \inf\{s \geq 0 : Z_s = 0\}$ which is the first time X_t hits the boundary of $\mathbb{R}^d \times \mathbb{R}^+$. It is clear that T_0 and Y are independent since T_0 is expressed in terms of Z only.

To provide a connection between probabilistic and deterministic integrals, we will use two tools: a new measure $\mathbb{P}^{m_a}$ and the vertical Green function. Denoting the Lebesgue measure on $\mathbb{R}^d$ by $m(\cdot)$, we define

$$\mathbb{P}^{m_a} = \int_{\mathbb{R}^d} \mathbb{P}^{(x,a)}\, m(dx), \qquad a > 0.$$

Let $\mathbb{E}^{m_a}$ denote the expectation with respect to this measure. We note that the law of X_{T_0} under this measure is $m(\cdot)$. Moreover, the semigroup P_t is invariant under the Lebesgue measure, that is,

$$\int_{\mathbb{R}^d} P_t f(x)\, m(dx) = \int_{\mathbb{R}^d} f(x)\, m(dx). \tag{1.4}$$

This follows from the symmetry of the kernel and the conservativeness of Y.

Second, for a positive Borel function f, the *vertical Green function*, which is the Green function for the one-dimensional Brownian motion, is given by

$$\mathbb{E}^a\Big[\int_0^{T_0} f(Z_s)\, ds\Big] = \int_0^\infty (s \wedge a) f(s)\, ds. \tag{1.5}$$

Harmonic functions play a key role in showing boundedness of Littlewood–Paley operators. Here we take the probabilistic interpretation of a harmonic function (with respect to the process X). A continuous function $u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$ is said to be *harmonic* (or *α-harmonic*) if $u(X_{s\wedge T_0})$ is a martingale with respect to the filtration $\mathcal{F}_s = \sigma(X_{r\wedge T_0} : r \leq s)$ and the probability measure $\mathbb{P}^{(x,t)}$ for any starting point $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$. One way to obtain such a harmonic function is to start with a bounded Borel function $f : \mathbb{R}^d \to \mathbb{R}$ and define its extension u by

$$u(x,t) := \mathbb{E}^{(x,t)} f(Y_{T_0}) = \int_0^\infty \mathbb{E}^x f(Y_s)\, \mathbb{P}^t(T_0 \in ds),$$

where $\mathbb{P}^t(T_0 \in ds)$ is the exit distribution of the one-dimensional Brownian motion from $(0,\infty)$, which is given by (see [14])

$$\mu_t(ds) := \mathbb{P}^t(T_0 \in ds) = \frac{t}{2\sqrt{\pi}} e^{-t^2/4s} s^{-3/2}\, ds.$$

By abuse of notation, we will denote both the function on $\mathbb{R}^d$ and its extension to the upper half-space by the same letter: $f_t(x) := f(x,t) = \mathbb{E}^{(x,t)} f(Y_{T_0})$.

Next, we define the semigroup $Q_t = \int_0^\infty P_s \, \mu_t(ds)$. It provides a representation

$$f_t(x) = f(x,t) = Q_t f(x) = \int_{\mathbb{R}^d} f(y) \int_0^\infty p(s,x,y) \, \mu_t(ds) \, dy.$$

We note that this is a convolution with the probability kernel

$$q_t(x) = \int_0^\infty p(s,x,0) \, \mu_t(ds),$$

whose Fourier transform is $e^{-t|\cdot|^{\alpha/2}}$. So $q_t(x)$ can be identified with the density of a symmetric $\alpha/2$-stable process, which will allow us to write the estimate (1.8) below. Moreover, $q_t(x)$ is radially decreasing in x. To see this, it is enough to write

$$p(1,x,0) = \int_{\mathbb{R}^d} \frac{1}{(4\pi s)^{d/2}} e^{-|x|^2/(4s)} g_{\alpha/2}(1,s) \, ds, \tag{1.6}$$

where $g_{\alpha/2}$ is the density of an $\alpha/2$-stable subordinator whose Laplace transform is $\int_0^\infty e^{-\lambda v} g(s,v) \, dv = e^{-s\lambda^{\alpha/2}}$. (See [16, p. 261] for details.)

One of the key tools in proving certain inequalities is the density estimates on $p(s,x,0)$. Although there is an infinite series expansion, it is not easy to work with. For this purpose, we will use the well-known two-sided estimate

$$c_1\left(s^{-d/\alpha} \wedge \frac{s}{|x-y|^{d+\alpha}}\right) \le p(s,x,y) \le c_2\left(s^{-d/\alpha} \wedge \frac{s}{|x-y|^{d+\alpha}}\right) \tag{1.7}$$

for $(s,x,y) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$, which allows us to control the tail of the transition density. (See [6, Theorem 2.1].) This estimate leads to an estimate on $q_t(x)$ due to the observation that $q_t(x)$ coincides with the density of a symmetric $\alpha/2$-stable process. We have

$$c_1\left(t^{-2d/\alpha} \wedge \frac{t}{|x|^{d+\alpha/2}}\right) \le q_t(x) \le c_2\left(t^{-2d/\alpha} \wedge \frac{t}{|x|^{d+\alpha/2}}\right). \tag{1.8}$$

In addition, we will need to control the derivative of $p(s,x,0)$. The following lemma provides this control. Let $\partial_{x_j}^k$ denote the kth partial derivative with respect to the jth coordinate.

LEMMA 1.2. *For $k = 1,2$ and $j = 1,\dots,d$, we have*

(i) $|\partial_{x_j}^k p(1,x,0)| \le c\left(1 \wedge \frac{1}{|x|^k}\right) p(1,x,0)$ *and*

(ii) $|\partial_{x_j}^k p(t,x,0)| \le c\left(t^{-k/\alpha} \wedge \frac{1}{|x|^k}\right) p(t,x,0)$ *whenever $t > 0$.*

This lemma is a direct consequence of [3, Proposition 3.3] and the inequality (1.7) above.

For the rest of the paper, we will need some results and definitions from [12]. To keep this paper as self-contained as possible, we repeat some of them here. For details, we refer to [12]. One of the main results of [12] is that harmonic functions, as defined above, satisfy the Harnack inequality. We will use this result to show boundedness of some operators in the next section. Let D_r be the open rectangular box with center $(y,s)\in\mathbb{R}^d\times\mathbb{R}^+$,

$$D_r=\{(x,t)\in\mathbb{R}^d\times\mathbb{R}^+ : x=(x_1,\dots,x_d),\ |x_i-y_i|<r^{2/\alpha}/2,\ i=1,\dots,d,\\ |s-t|<r/2\}.$$

When using these rectangular boxes, we will consider nested boxes with the same center. That is why we do not include the center point in the notation, and just write D_r for simplicity.

THEOREM 1.3 ([12, Theorem 3]). *There exists $c>0$ such that if u is non-negative and bounded on $\mathbb{R}^d\times\mathbb{R}^+$, harmonic in D_{16} and in D_{32}, then*

$$u(x,t)\le cu(x',t'),\qquad (x,t),(x',t')\in D_1.$$

Using this inequality, we proved a Littlewood–Paley theorem in [12]. We define a new operator with respect to our product process $X_s=(Y_s,Z_s)$. The horizontal component of the classical operator is replaced by the one corresponding to the symmetric stable process. The two components are defined as

$$\overrightarrow{G}_f(x)=\left[\int_0^\infty t\int_{\mathbb{R}^d}\frac{[f_t(x+h)-f_t(x)]^2}{|h|^{d+\alpha}}\,dh\,dt\right]^{1/2},$$

$$G_f^{\uparrow}(x)=\left[\int_0^\infty t\left[\frac{\partial}{\partial t}f(x,t)\right]^2dt\right]^{1/2},$$

and hence the Littlewood–Paley operator G_f is defined as

$$G_f=[(\overrightarrow{G}_f)^2+(G_f^{\uparrow})^2]^{1/2}.$$

Unlike the Brownian motion case, the Littlewood–Paley Theorem (Theorem 1.4(i)) cannot be extended to $p\in(1,2)$. This seems to be due to the large jump terms of the horizontal process. Therefore, we truncate the part of the horizontal component which corresponds to the large jumps. We denote the new operator obtained after truncation by $\overrightarrow{G}_{f,\alpha}$,

$$\overrightarrow{G}_{f,\alpha}(x)=\left[\int_0^\infty t\,\Gamma_\alpha(f_t,f_t)(x)\,dt\right]^{1/2},$$

where

$$\Gamma_\alpha(f_t, f_t)(x) = \int\limits_{|h|<t^{2/\alpha}} [f_t(x+h) - f_t(x)]^2 \frac{dh}{|h|^{d+\alpha}}, \tag{1.9}$$

and the new restricted Littlewood–Paley operator is

$$G_{f,\alpha}(x) = [(\overrightarrow{G}_{f,\alpha}(x))^2 + (G_f^{\uparrow}(x))^2]^{1/2}.$$

THEOREM 1.4. *If $f \in L^p(\mathbb{R}^d)$, then for some constant $c > 0$:*

(i) $\|G_f\|_p \le c\|f\|_p$ *for* $p \ge 2$,
(ii) $\|G_f^{\uparrow}\|_p \le c\|f\|_p$ *for* $p > 1$ *and*
(iii) $\|\overrightarrow{G}_{f,\alpha}\|_p \le c\|f\|_p$ *for* $p > 1$.

Part (i) is due to P. A. Meyer [14]. This is a special case of his study of symmetric Markov processes. Part (ii) is obtained by E. M. Stein in [19, Chapter V] in the case of symmetric semigroups. The proof of the third part is given in [12, Theorem 7].

There are also some recent results based on an analytic approach to a differential equation involving the fractional Laplacian. I. Kim and K. Kim [13] discussed another operator by applying the fractional Laplacian to $P_t f(x)$ where P_t is defined as above. This operator plays the role of the classical Littlewood–Paley operator, where the Laplacian is the generator when $\alpha = 2$ (that is, when the process is a Brownian motion) and hence the authors obtain an analogue of the classical inequality in the fractional Laplacian case. However, as in Meyer's result (Theorem 1.4(i)), this inequality holds for $p \ge 2$. One of our main results in [12] (Theorem 1.4(iii)) allows us to generalize this inequality first by considering the harmonic extension $Q_t f$ and then writing the integrand as the singular integral (1.9) instead of the differential ∂_x^α on a restricted domain to provide some control over the large jump terms. Without this restriction, it is not possible to extend this result to $p \in (1,2)$. In this paper, we will make use of this inequality for $p > 1$.

In addition to the theorem above, it is also not difficult to see that part (ii) can be written as a two sided-inequality. Here we provide a short proof by a well-known duality argument.

LEMMA 1.5. *If $p > 1$ and $f \in L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ then $\|f\|_p \le c\|G_f^{\uparrow}\|_p$.*

Proof. First note that by the Plancherel identity,

$$\|G_f^{\uparrow}\|_2^2 = c\int_0^\infty t \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 |\xi|^\alpha e^{-2t|\xi|^{\alpha/2}}\, d\xi\, dt = c\|f\|_2, \tag{1.10}$$

since $(Q_t f)\widehat{\ }(\cdot) = e^{-t|\cdot|^{\alpha/2}} \widehat{f}(\cdot)$.

Second, if $h \in L^q(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, where $1/p + 1/q = 1$, then using polarization identity and (1.10), we get

$$\int_{\mathbb{R}^d} f(x)h(x)\,dx = \frac{1}{4}(\|f+h\|_2^2 - \|f-h\|_2^2) = c(\|G^{\uparrow}_{f+h}\|_2^2 - \|G^{\uparrow}_{f-h}\|_2^2)$$
$$= c\int_{\mathbb{R}^d}\int_0^\infty t\frac{\partial f}{\partial t}(x,t)\frac{\partial h}{\partial t}(x,t)\,dt\,dx.$$

Using the Cauchy–Schwarz inequality and then the Hölder inequality, we obtain

$$\int_{\mathbb{R}^d} f(x)h(x)\,dx \le c\int_{\mathbb{R}^d} G^{\uparrow}_f(x)G^{\uparrow}_h(x)\,dx \le c\|G^{\uparrow}_f\|_p\|G^{\uparrow}_h\|_q \le c\|G^{\uparrow}_f\|_p\|h\|_q,$$

where the last inequality follows from Theorem 1.4.

Finally, the result follows if we take the supremum over all such h with $\|h\|_q \le 1$. ∎

In the classical Littlewood–Paley theory, there are some operators which are often used to prove intermediate steps of boundedness arguments. We believe that they should also be studied in the present context, and analogous results to the classical theory should be provided in order to obtain a complete picture. In the next section we will discuss some of these operators and prove their boundedness in $L^p(\mathbb{R}^d)$. Among these operators, two important ones are the area functional and the G^* functional. The area functional in our setup is given by

$$A_f(x) = \Big[\int_0^\infty \int_{|y|<t^{2/\alpha}} t^{1-2d/\alpha}\Gamma_\alpha(f_t,f_t)(x-y)\,dy\,dt\Big]^{1/2}.$$

The reason for this name is that it represents the area of $f(D)$ in the classical setup (for $\alpha = 2$ and Γ_α replaced by $|\nabla|^2$) where D is the cone $\{(y,t) : |y-x| < t\}$ and $d = 2$.

Second, we define the new G^* functional by means of its horizontal and vertical components. But first we set

$$K_t^\lambda(x) = t^{-2d/\alpha}\Big[\frac{t^{2/\alpha}}{t^{2/\alpha}+|x|}\Big]^{\lambda d}, \qquad t > 0.$$

We will take $\lambda > 1$. Note that $\|K_t^\lambda\|_1 = \|K_1^\lambda\|_1 = c_d$. Hence the normalized function $c_d^{-1}K_t^\lambda$ is a bounded approximate identity. Using this kernel we define two components by

$$\overrightarrow{G}^*_{\lambda,f}(x) = \Big[\int_0^\infty t\cdot K_t^\lambda * \Gamma_\alpha(f_t,f_t)(x)\,dt\Big]^{1/2},$$
$$G^{*,\uparrow}_{\lambda,f}(x) = \Big[\int_0^\infty t\cdot K_t^\lambda * \Big(\frac{\partial}{\partial t}f_t(\cdot)\Big)^2(x)\,dt\Big]^{1/2},$$

and the G^* functional is

$$G^*_{\lambda,f}(x) = [(\overrightarrow{G}^*_{\lambda,f}(x))^2 + (G^{*,\uparrow}_{\lambda,f}(x))^2]^{1/2}.$$

2. Singular integral operators and boundedness results. As we can see from the definitions of the operators, we mostly restrict our domain of integration to a parabolic-like domain in the upper half-space. By taking the scaling factor into account, we focus on the set $\{(y,t) \in \mathbb{R}^d \times \mathbb{R}^+ : |y-x| < t^{2/\alpha}\}$ with vertex at $x \in \mathbb{R}^d$. Our first observation is that the growth of an extension function is controlled by the Hardy–Littlewood maximal function $\mathcal{M}(\cdot)$, given by

$$\mathcal{M}(f)(x) = \sup_{r>0} \frac{1}{|B(0,1)| \cdot r^d} \int_{|y|<r} |f(x-y)|\, dy.$$

To see this, we define

$$N^f_\alpha(x) := \sup\{|f_t(y)| : t > 0, |x-y| < t^{2/\alpha}\}.$$

The classical version of this function is sometimes referred to as the (*non-tangential*) *maximal function* (see [19, Chapter II]). In that case, the growth of this function is studied at a single point $x \in \mathbb{R}^d$. In our setup, we should consider the terms corresponding to jumps of the horizontal process. However, we still need to restrict our function to small jumps so that comparison of the points at any given "height" is possible by Harnack's inequality. For this purpose, the domain is considered to be the parabolic-like region given above.

LEMMA 2.1. *Let $p > 1$ and $f \in L^p(\mathbb{R}^d)$. Then*

(i) $N^f_\alpha(x) \le c\mathcal{M}(f)(x)$ *for* $x \in \mathbb{R}^d$,
(ii) $N^f_\alpha \in L^p(\mathbb{R}^d)$ *and* $\|N^f_\alpha\|_p \le c\|f\|_p$.

Proof. We first show that it is enough to consider positive functions to prove (i). Indeed, if f is not positive, then we can consider the decomposition $f = f^+ - f^-$, where $f^+, f^- \ge 0$. Then we can use linearity of the semigroup Q_t, the inequalities

$$N^f_\alpha \le N^{f^+}_\alpha + N^{f^-}_\alpha \quad \text{and} \quad \mathcal{M}(f^+) + \mathcal{M}(f^-) \le 2\mathcal{M}(f)$$

and the fact that both $Q_t f^+$ and $Q_t f^-$ are positive harmonic to prove the result for f.

So suppose $f > 0$. Then for a fixed $t > 0$ and $y \in B(x, t^{2/\alpha})$, Theorem 1.3 applied several times implies that $f_t(y) \le c f_t(x)$. Here we emphasize that the constant c does not depend on t, since these balls scale as t varies and so the same number of applications of the Harnack inequality suffices at each t for fixed x.

Moreover, $f_t(x) = f * q_t(x)$ where q_t is radially decreasing and its L^1-norm equals one. To see this we note that the transition density $p(s,x,0)$ is obtained from the characteristic function $e^{-s|x|^\alpha}$ by the inverse Fourier transform. Hence we can write $p(s,x,0)$ as in (1.6). Thus $p(s,x,0)$ is radially decreasing in x, and so is $q_t(x)$. Then $f_t(x) \le c\mathcal{M}(f)(x)$ for any $t > 0$ [10, Section 2.1] and $N_\alpha^f(x) \le c\mathcal{M}(f)(x)$. Finally, using the fact

$$\|\mathcal{M}(f)\|_p \le c\|f\|_p, \qquad p > 1,$$

one can obtain the result. ∎

Before we study the area functional, we define an auxiliary operator L_f^*. This operator is closely related to $\overrightarrow{G}^*_{\lambda,f}$ for a particular value of λ, and hence it provides an intermediate step to prove boundedness of the area functional. Moreover, the classical version L_f^* is used to give a probabilistic proof of boundedness of the Littlewood–Paley function.

For a given $f \in L^p(\mathbb{R}^d)$, we define

$$L_f^*(x) = \Big[\int_0^\infty t \cdot Q_t \Gamma_\alpha(f_t, f_t)(x)\, dt\Big]^{1/2},$$

where Γ_α is as in (1.9). This operator is bounded on $L^p(\mathbb{R}^d)$ whenever $p > 2$.

THEOREM 2.2. *Let $p > 2$ and $f \in L^p(\mathbb{R}^d)$. Then*

$$\|L_f^*\|_p \le c\|f\|_p.$$

Proof. Let $f \in L^p(\mathbb{R}^d)$, $r = 2p$ and q be the conjugate of r, that is, $1/r + 1/q = 1$. Let h be a continuously differentiable function with compact support. Then

$$\begin{aligned}
\mathbb{E}^{(x,a)}\Big[\int_0^{T_0} &\Gamma_\alpha(f_{Z_s}, f_{Z_s})(Y_s)\, ds \cdot h(X_{T_0})\Big] \\
&= \int_0^\infty \mathbb{E}^{(x,a)}\big[\mathbb{E}^{(x,a)}[\mathbb{1}_{\{s<T_0\}}\Gamma_\alpha(f_{Z_s}, f_{Z_s})(Y_s) h(X_{T_0}) \mid \mathcal{F}_s]\big]\, ds \\
&= \mathbb{E}^{(x,a)}\Big[\int_0^\infty \mathbb{1}_{\{s<T_0\}}\Gamma_\alpha(f_{Z_s}, f_{Z_s})(Y_s)\mathbb{E}^{(x,a)}[h(X_{T_0}) \mid \mathcal{F}_s]\, ds\Big] \\
&= \mathbb{E}^{(x,a)}\Big[\int_0^\infty \mathbb{1}_{\{s<T_0\}}\Gamma_\alpha(f_{Z_s}, f_{Z_s})(Y_s)\mathbb{E}^{X_s}[h(X_{T_0})]\, ds\Big],
\end{aligned}$$

by the Markov property. Then using invariance of the semigroup P_t under the Lebesgue measure (equation (1.4)) and the vertical Green function (equation (1.5)), we obtain

$$\mathbb{E}^{m_a}\Big[\int_0^{T_0} \Gamma_\alpha(f_{Z_s}, f_{Z_s})(Y_s)\,ds \cdot h(X_{T_0})\Big]$$
$$= \int_{\mathbb{R}^d} \mathbb{E}^a\Big[\int_0^{T_0} \Gamma_\alpha(f_{Z_s}, f_{Z_s})(x) \cdot \mathbb{E}^{(x,Z_s)}[h(X_{T_0})]\,ds\Big]\,dx$$
$$= \int_{\mathbb{R}^d} \int_0^\infty (a \wedge t)\Gamma_\alpha(f_t, f_t)(x) \cdot \mathbb{E}^{(x,t)}[h(X_{T_0})]\,dt\,dx.$$

Now as $a \to \infty$, the last expression above approaches

$$\int_{\mathbb{R}^d} \int_0^\infty t \cdot \Gamma_\alpha(f_t, f_t)(x) \cdot h_t(x)\,dt\,dx.$$

By the symmetry of the kernel $q_t(\cdot)$, this limit equals

$$\int_{\mathbb{R}^d} \int_0^\infty t \cdot \Gamma_\alpha(f_t, f_t)(x) \cdot h * q_t(x)\,dt\,dx$$
$$= \int_{\mathbb{R}^d} \int_0^\infty t \cdot \Gamma_\alpha(f_t, f_t) * q_t(x) \cdot h(x)\,dt\,dx$$
$$= \int_{\mathbb{R}^d} h(x)(L_f^*(x))^2\,dx.$$

Next, by the Hölder inequality with exponents q and r,

$$\mathbb{E}^{m_a}\Big[\int_0^{T_0} \Gamma_\alpha(f_{Z_s}, f_{Z_s})(Y_s)\,ds \cdot h(X_{T_0})\Big]$$
$$\le (\mathbb{E}^{m_a}|h(X_{T_0})|^q)^{1/q}\Big(\mathbb{E}^{m_a}\Big[\int_0^{T_0} \Gamma_\alpha(f_{Z_s}, f_{Z_s})(Y_s)\,ds\Big]^r\Big)^{1/r}.$$

Now denote the martingale $f(X_{t\wedge T_0})$ by M_t^f. By [14, p. 158] or [12, Section 2],

$$\mathbb{E}^{m_a}\Big[\int_0^{T_0} \Gamma_\alpha(f_{Z_s}, f_{Z_s})(Y_s)\,ds\Big]^r \le c\mathbb{E}^{m_a}\Big[\int_0^{T_0} g(Y_s, Z_s)\,ds\Big]^r \le c\mathbb{E}^{m_a}[\langle M^f\rangle_{T_0}]^r,$$

where

$$g(x,t) = \int_{\mathbb{R}^d} [f_t(x+h) - f_t(x)]^2 \frac{dh}{|h|^{d+\alpha}} + \Big[\frac{\partial}{\partial t} f(x,t)\Big]^2. \tag{2.1}$$

By the Burkholder–Gundy–Davis inequality, the last term is bounded by a constant multiple of $\mathbb{E}^{m_a}[\sup_{s\le T_0} |M_s^f|]^{2r}$, which is bounded by $c\mathbb{E}^{m_a}|M_{T_0}^f|^{2r}$ by Doob's inequality. Hence

$$\lim_{a\to\infty} \mathbb{E}^{m_a}\Big[\int_0^{T_0} \Gamma_\alpha(f_{Z_s}, f_{Z_s})(Y_s)\,ds \cdot h(X_{T_0})\Big]$$
$$\le c \lim_{a\to\infty} (\mathbb{E}^{m_a}|h(X_{T_0})|^q)^{1/q}(\mathbb{E}^{m_a}|f(X_{T_0})|^{2r})^{1/r} \le c\|h\|_q\|f\|_{2r}^2.$$

Using the first part gives

$$\int_{\mathbb{R}^d} h(x)(L_f^*(x))^2\,dx \le c\|h\|_q\|f\|_{2r}^2.$$

Finally, if we take the supremum over all such h with $\|h\|_q \le 1$, then

$$\Big[\int_{\mathbb{R}^d} (L_f^*(x))^{2r}\,dx\Big]^{1/r} \le c\|f\|_{2r}^2,$$

which gives the result if we replace r with $p/2$. ∎

Now, if we consider $\lambda_0 = (2d+\alpha)/(2d)$ then we find a relation between L_f^* and $\overrightarrow{G}_{\lambda_0,f}^*$. Hence we can show boundedness of the area functional A_f.

THEOREM 2.3. *Suppose $p > 2$ and $f \in L^p(\mathbb{R}^d)$. Then*

(i) *For $\lambda > 0$, $A_f \le c_\lambda \overrightarrow{G}_{\lambda,f}^*$.*
(ii) *If $\lambda_0 = (2d+\alpha)/(2d)$, then*

$$\|\overrightarrow{G}_{\lambda_0,f}^*\|_p \le c\|f\|_p.$$

(iii) $\|A_f\|_p \le c\|f\|_p$.

Proof. Part (i) is easy when we observe

$$\left[\frac{t^{2/\alpha}}{t^{2/\alpha}+|y|}\right]^{\lambda d} \ge 2^{-\lambda d}$$

for $|y| < t^{2/\alpha}$. Part (iii) is a corollary of (i) and (ii).

So it is enough to prove (ii). First we recall that

$$K_t^{\lambda_0}(x) = \frac{t}{(t^{2/\alpha}+|x|)^{d+\alpha/2}} = t^{-2d/\alpha}\left(\frac{1}{1+|x|/t^{2/\alpha}}\right)^{d+\alpha/2}.$$

We also know that $q_t(x)$ is comparable to

$$t^{-2d/\alpha} \wedge \frac{t}{|x|^{d+\alpha/2}} = t^{-2d/\alpha}\left(1 \wedge \frac{1}{(|x|/t^{2/\alpha})^{d+\alpha/2}}\right),$$

by (1.8). Hence q_t is comparable to $K_t^{\lambda_0}$ and we have

$$K_t^{\lambda_0} \le cq_t(x).$$

This leads to

$$\overrightarrow{G}_{\lambda_0,f}^*(x) \le cL_f^*(x).$$

Then the result follows from Theorem 2.2. ∎

The result of the previous theorem is not restricted to the horizontal component with parameter λ_0. We can generalize it to the case including the vertical component and any parameter $\lambda > 1$.

THEOREM 2.4. *If* $\lambda > 1$, $p \geq 2$ *and* $f \in L^p(\mathbb{R}^d)$ *then*

$$\|G^*_{\lambda,f}\|_p \leq c\|f\|_p.$$

Proof. Set

$$g_\alpha(y,t) = \Gamma_\alpha(f_t, f_t)(y) + \left(\frac{\partial}{\partial t} f(y,t)\right)^2.$$

Assume $h \in \mathcal{C}^1_K(\mathbb{R}^d)$. Then by the symmetry of $K^\lambda_t(x)$ in x,

$$\begin{aligned}\int_{\mathbb{R}^d} h(x)(G^*_{\lambda,f}(x))^2\,dx &= \int_0^\infty t \int_{\mathbb{R}^d} h(x) \int_{\mathbb{R}^d} K^\lambda_t(x-y) g_\alpha(y,t)\,dy\,dx\,dt \\ &= \int_0^\infty t \int_{\mathbb{R}^d} g_\alpha(y,t) \cdot h * K^\lambda_t(y)\,dy\,dt.\end{aligned}$$

Since K^λ_t is radially decreasing and integrable, $h*K^\lambda_t(y) \leq c\mathcal{M}(h)(y)$. Hence

$$\int_{\mathbb{R}^d} h(x)(G^*_{\lambda,f}(x))^2\,dx \leq c \int_{\mathbb{R}^d} \mathcal{M}(h)(x)(G_{f,\alpha}(x))^2\,dx. \tag{2.2}$$

For $p = 2$, it is enough to consider $h = 1$. Then by parts (ii) and (iii) of Theorem 1.4,

$$\|G^*_{\lambda,f}\|_2 \leq c\|G_{f,\alpha}\|_2 \leq c\|f\|_2.$$

Now suppose $p > 2$. We take $r = p/2$ and $q > 0$ such that $1/r + 1/q = 1$. Using Hölder's inequality in (2.2) gives

$$\begin{aligned}\int_{\mathbb{R}^d} h(x)(G^*_{\lambda,f}(x))^2\,dx &\leq c\Big[\int_{\mathbb{R}^d} (\mathcal{M}(h)(x))^q\,dx\Big]^{1/q} \cdot \Big[\int_{\mathbb{R}^d} (G_{f,\alpha}(x))^{2r}\,dx\Big]^{1/r} \\ &\leq c\|h\|_q \|G_{f,\alpha}\|^2_p.\end{aligned}$$

If we take the supremum over all such h with $\|h\|_q \leq 1$, we obtain

$$\|G^*_{\lambda,f}\|^2_p = \|(G^*_{\lambda,f})^2\|_r \leq c\|G_{f,\alpha}\|^2_p.$$

Finally, using the boundedness of the operator $G_{f,\alpha}$ when $p > 2$ (Theorem 1.4), we deduce the desired result. ■

In the final part of the paper, we discuss an application of the previous theorem. We will provide a result on boundedness of singular integrals which is a generalization of Theorem 1.1. We show that the result holds under a weaker condition on the tail of the kernel. For this purpose, we impose a boundedness condition in terms of the semigroup Q_t.

THEOREM 2.5. *Let $p > 1$. Suppose T is a convolution operator on $L^p(\mathbb{R}^d)$ with kernel κ, that is, $Tf(x) = f * \kappa(x)$. Suppose further that there exists $\lambda > 1$ such that*

$$|\partial_t Q_t \kappa(x)| \le c t^{-1-2d/\alpha} \left(\frac{t^{2/\alpha}}{t^{2/\alpha} + |x|} \right)^{\lambda d} = c t^{-1} K_t^\lambda(x). \tag{2.3}$$

Then for $f \in \mathcal{C}_K^1$ (that is, $f \in \mathcal{C}^1$ with compact support)

$$\|Tf\|_p \le c\|f\|_p.$$

The condition (2.3) above may not seem very useful in applications. Hence we will provide a sufficient and more useful condition later in Theorem 2.8.

Proof of Theorem 2.5. First suppose $p > 2$. We note that by the semigroup property, we have $Q_t = Q_{t/2} Q_{t/2}$ and $q_t = q_{t/2} * q_{t/2}$, which leads to $\partial_t q_t = 2 q_{t/2} * \partial_t q_{t/2}$. Next, we observe that $\partial_t Q_t T f(x) = 2 Q_{t/2} T(\partial_t Q_{t/2} f)(x)$, since their Fourier transforms are equal,

$$\begin{aligned} [2Q_{t/2}\widehat{T(\partial_t Q_{t/2} f)}] &= 2\widehat{q_{t/2}}\, \widehat{\kappa}\, (\widehat{\partial_t q_{t/2}}) \widehat{f} = (\widehat{2q_{t/2} * \partial_t q_{t/2}})\, \widehat{\kappa}\, \widehat{f} \\ &= \widehat{\partial_t q_t}\, \widehat{\kappa}\, \widehat{f} = \widehat{\partial_t Q_t T f}. \end{aligned}$$

Then

$$(G_{Tf}^{\uparrow}(x))^2 = \int_0^\infty t |\partial_t Q_t T f(x)|^2\, dt = 4 \int_0^\infty t |Q_{t/2} T(\partial_t Q_{t/2} f)(x)|^2\, dt.$$

Using our assumption (2.3), we see that

$$Q_{t/2} T(\partial_t Q_{t/2} f) = (1/2) \partial_t Q_t T f(x) \to 0 \quad \text{as } t \to \infty.$$

Hence the last line above equals

$$4 \int_0^\infty t \left| \int_t^\infty \frac{s}{s} \partial_s Q_{s/2} T(\partial_s Q_{s/2} f)(x)\, ds \right|^2 dt.$$

If we apply the Cauchy–Schwarz inequality first, and then change the order of the integrals, we get

$$\begin{aligned} (G_{Tf}^{\uparrow}(x))^2 &\le c \int_0^\infty t \Big[\int_t^\infty s^{-2}\, ds \Big] \cdot \Big[\int_t^\infty s^2 (\partial_s Q_{s/2} T(\partial_s Q_{s/2} f)(x))^2\, ds \Big]\, dt \\ &= c \int_0^\infty \int_t^\infty s^2 (\partial_s Q_{s/2} T(\partial_s Q_{s/2} f)(x))^2\, ds\, dt \\ &= c \int_0^\infty s^3 (\partial_s Q_{s/2} T(\partial_s Q_{s/2} f)(x))^2\, ds. \end{aligned}$$

Using the bound in (2.3) and Jensen's inequality yields

$$(G^{\uparrow}_{Tf}(x))^2 \le c\int_0^\infty s^3[(s^{-1}K^{\lambda}_{s/2}) * (\partial_s Q_{s/2}f)(x)]^2\,ds$$
$$\le c\int_0^\infty sK^{\lambda}_{s/2} * (\partial_s Q_{s/2}f)^2(x)\,ds \le c(G^{*}_{\lambda,f}(x))^2.$$

Hence for $p > 2$,

$$\|Tf\|_p \le c\|G^{\uparrow}_{Tf}\|_p \le c\|G^{*}_{\lambda,f}\|_p \le c\|f\|_p,$$

by Lemma 1.5 and Theorem 2.4.

For $p \in (1,2)$ we use a duality argument. Let q be such that $1/p+1/q = 1$. First we observe that if $\kappa^*(x) = \kappa(-x)$ and T^* is the convolution operator corresponding to κ^*, then (2.3) holds for κ^*. Thus for $h \in L^q(\mathbb{R}^d)$,

$$\Big|\int_{\mathbb{R}^d} h(x)Tf(x)\,dx\Big| = \Big|\int_{\mathbb{R}^d} T^*h(x)f(x)\,dx\Big| \le c\|T^*h\|_q\|f\|_p \le \|h\|_q\|f\|_p$$

by the first part of the proof. Finally, if we take the supremum over all such h with $\|h\|_q \le 1$, the result follows. ∎

Before any further discussion, we recall the definition of the measure

$$\mu_t(ds) = \frac{t}{2\sqrt{\pi}}e^{-t^2/4s}s^{-3/2}\,ds$$

and show the following estimates.

Lemma 2.6. *For $M > 0$, we have*

(i) $\displaystyle\int_0^M |s-1/2|\,\mu_1(ds) \le \frac{1}{\sqrt{\pi}}\,M^{1/2}$,

(ii) $\displaystyle\int_M^\infty |1-1/(2s)|\,\mu_1(ds) \le \frac{1}{\sqrt{\pi}}\,M^{-1/2}$.

Proof. (i) We note that

$$|s-1/2|s^{-1/2}e^{-1/(4s)} \le 1$$

for $s > 0$. Hence the result follows.

(ii) Similarly, we also have

$$|s-1/2|e^{-1/(4s)} \le 1,$$

which results in the desired inequality. ∎

Lemma 2.7. *Suppose*

$$\psi(x) = (\partial_t q_t(x))_{t=1} = \int_0^\infty p(s,x,0)\Big(1-\frac{1}{2s}\Big)\mu_1(ds).$$

Then for some positive constants c, c_1, c_2 we have

(i) $|\psi(x)| \le c_1(1 \wedge |x|^{-d-\alpha/2}) \le c_2 q_1(x)$ *and*
(ii) $|\partial_{x_i}\psi(x)| \le c(1 \wedge |x|^{-d-1-\alpha/2})$, $i = 1, \dots, d$.

Proof. (i) First note that

$$|\psi(x)| \le c \int_0^\infty p(s,x,0)\left|1 - \frac{1}{2s}\right| s^{-3/2} e^{-1/(4s)}\, ds \le c \int_0^\infty \left|1 - \frac{1}{2s}\right| s^{-3/2-d/\alpha} e^{-1/(4s)}\, ds < \infty$$

by the estimate (1.7) on the density $p(s,x,0)$. Using the same estimate once again we obtain

$$|\psi(x)| \le c \int_0^{|x|^\alpha} \frac{s}{|x|^{d+\alpha}}\left|1 - \frac{1}{2s}\right| \mu_1(ds) + c \int_{|x|^\alpha}^\infty s^{-d/\alpha}\left|1 - \frac{1}{2s}\right| \mu_1(ds)$$
$$\le \frac{c}{|x|^{d+\alpha}} \int_0^{|x|^\alpha} \left|s - \frac{1}{2}\right| \mu_1(ds) + \frac{c}{|x|^d} \int_{|x|^\alpha}^\infty \left|1 - \frac{1}{2s}\right| \mu_1(ds).$$

By Lemma 2.6,

$$|\psi(x)| \le c_1(1 \wedge |x|^{-d-\alpha/2}).$$

The second inequality follows from the estimate (1.8) on $q_1(x)$.

(ii) Similarly, using the bound on $\partial_{x_i} p(s,x,0)$ (Lemma 1.2), we obtain

$$|\partial_{x_i}\psi(x)| \le c \int_0^\infty |\partial_{x_i} p(s,x,0)|\left|1 - \frac{1}{2s}\right| \mu_1(ds) \le c \int_0^\infty \left|1 - \frac{1}{2s}\right| s^{-3/2-(d+1)/\alpha} e^{-1/(4s)}\, ds < \infty$$

and

$$|\partial_{x_i}\psi(x)| = \left|\int_0^\infty \partial_{x_i} p(s,x,0)\left(1 - \frac{1}{2s}\right) \mu_1(ds)\right|$$
$$\le c \int_0^{|x|^\alpha} \frac{s}{|x|^{d+1+\alpha}}\left|1 - \frac{1}{2s}\right| \mu_1(ds) + c \int_{|x|^\alpha}^\infty s^{-(d+1)/\alpha}\left|1 - \frac{1}{2s}\right| \mu_1(ds)$$
$$\le \frac{c}{|x|^{d+1+\alpha}} \int_0^{|x|^\alpha} \left|s - \frac{1}{2}\right| \mu_1(ds) + \frac{c}{|x|^{d+1}} \int_{|x|^\alpha}^\infty \left|1 - \frac{1}{2s}\right| \mu_1(ds).$$

Finally, we use Lemma 2.6 to obtain the desired result. ∎

In the previous theorem, we stated a boundedness condition on the kernel of a convolution operator by means of the action of the semigroup Q_t. In order for this condition to be more useful, we state an application in purely analytic language. In Theorem 2.8, we provide two conditions under which the condition (2.3) of Theorem 2.5 holds.

THEOREM 2.8. *Suppose $\alpha \in (1,2)$ and $\kappa : \mathbb{R}^d \to \mathbb{R}$ is a function with the cancelation property* (1.1) *such that*

(i) $|\kappa(x)| \le \frac{c}{|x|^d}\mathbb{1}_{\{|x|\le 1\}} + \frac{c}{|x|^{d-1+\alpha/2}}\mathbb{1}_{\{|x|>1\}}$,

(ii) $|\nabla\kappa(x)| \le \frac{c}{|x|^{d+1}}\mathbb{1}_{\{|x|\le 1\}} + \frac{c}{|x|^{d+\alpha/2}}\mathbb{1}_{\{|x|>1\}}$.

Suppose T is a convolution operator with kernel κ. Then for $f \in \mathcal{C}^1_K$ and $p > 1$ we have

$$\|Tf\|_p \le c\|f\|_p.$$

Proof. First let ϕ be a smooth function on $\mathbb{R}$ such that $\phi(r) = 1$ whenever $|r| \le 1$ and $\phi(r) = 0$ whenever $|r| > 2$. Now let

$$\kappa_1(x) = \kappa(x)\phi(|x|^2), \qquad \kappa_2(x) = \kappa(x)(1 - \phi(|x|^2))$$

and

$$T_1 f = f * \kappa_1, \qquad T_2 f = f * \kappa_2.$$

Then $Tf = T_1 f + T_2 f$. By the classical case (Theorem 1.1), $\|T_1 f\|_p \le c\|f\|_p$. So without loss of generality we may assume $T = T_2$ and $\kappa = \kappa_2$ and ignore the indices. As before, set $\psi(x) = (\partial_t q_t(x))_{t=1}$. By scaling, we have $\partial_t q_t(x) = t^{-1-2d/\alpha}\psi(x/t^{2/\alpha})$. So by Theorem 2.5 and scaling, it is enough to show that

$$|(\partial_t Q_t \kappa(x))_{t=1}| \le \frac{c}{(1+|x|)^{\lambda d}}$$

for some $\lambda > 1$. Here we will take $\lambda = 1 + (\alpha - 1)/(2d)$.

First assume $|x| \le 1$. Then

$$(\partial_t Q_t \kappa(x))_{t=1} = \int\limits_{|y|>1} \kappa(y)\psi(x-y)\,dy, \tag{2.4}$$

and by Lemma 2.7(i) we have

$$\begin{aligned} |(\partial_t Q_t \kappa(x))_{t=1}| &\le \int\limits_{|y|>1} |\kappa(y)|\,|\psi(x-y)|\,dy \\ &\le c \int\limits_{|y|>1} |\kappa(y)| q_1(x-y)\,dy. \end{aligned}$$

Then assumption (i) gives

$$|\kappa(y)| \le \frac{c}{|y|^{d-1+(\alpha/2)}} \le c$$

whenever $|y| > 1$. Thus

$$\int_{|y|>1} |\kappa(y)| q_1(x-y)\, dy \le c \int_{|y|>1} q_1(x-y)\, dy \le c \int_{\mathbb{R}^d} q_1(x-y)\, dy = c,$$

since q_1 is a probability kernel. Hence

$$|(\partial_t Q_t \kappa(x))_{t=1}| \le c \le \frac{c}{(1+|x|)^{\lambda d}}. \tag{2.5}$$

Now assume $|x| > 1$. Consider three subsets of $\mathbb{R}^d$: $\mathcal{D}_1 = \{y \in \mathbb{R}^d : |y| < |x|/2\}$, $\mathcal{D}_2 = \{y \in \mathbb{R}^d : |y-x| < |x|/2\}$ and $\mathcal{D}_3 = \mathbb{R}^d - (\mathcal{D}_1 \cup \mathcal{D}_2)$. Split the integral (2.4) as

$$\int_{|y|>1} \kappa(y)\psi(x-y)\, dy = \int_{\mathcal{D}_1} + \int_{\mathcal{D}_2} + \int_{\mathcal{D}_3} =: I_1 + I_2 + I_3.$$

Since κ satisfies the cancelation condition (1.1),

$$\begin{aligned} |I_1| &= \Big| \int_{\mathcal{D}_1} \kappa(y)(\psi(x-y) - \psi(x))\, dy \Big| \\ &\le \sup_{|z-x|<|x|/2} |\nabla \psi(z)| \int_{\mathcal{D}_1} \frac{c}{|y|^{d-1+\alpha/2}} |y|\, dy \le c|x|^{2-\alpha/2} \sup_{|z-x|<|x|/2} |\nabla \psi(z)|. \end{aligned}$$

By Lemma 2.7, the gradient above is bounded by $c|x|^{-d-1-\alpha/2}$. This gives

$$|I_1| \le \frac{c}{|x|^{d-1+\alpha}} \le \frac{c}{|x|^{\lambda d}}.$$

For I_3, we note that $c|y| \le |x-y| \le c'|y|$ whenever $y \in \mathcal{D}_3$. So using Lemma 2.7 again yields

$$|I_3| \le c \int_{\mathcal{D}_3} \frac{1}{|x-y|^{d+\alpha/2}} \frac{1}{|y|^{d-1+\alpha/2}}\, dy \le c \int_{|y|\ge|x|/2} |y|^{-2d+1-\alpha}\, dy \le \frac{c}{|x|^{\lambda d}}.$$

For I_2, we use assumption (ii) on $\nabla\kappa$. By a change of variables, we have

$$\begin{aligned} |I_2| &= \Big| \int_{|y-x|<|x|/2} \psi(x-y)\kappa(y)\, dy \Big| = \Big| \int_{|y|<|x|/2} \psi(y)\kappa(x-y)\, dy \Big| \\ &\le \int_{|y|<|x|/2} |\psi(y)|\, |\kappa(x-y) - \kappa(x)|\, dy + |\kappa(x)| \Big| \int_{|y|<|x|/2} \psi(y)\, dy \Big|. \end{aligned}$$

First observe that

$$\int_{\mathbb{R}^d} \psi(y)\, dy = \int_0^\infty \int_{\mathbb{R}^d} p(s,y,0)\, dy \left(1 - \frac{1}{2s}\right) \mu_1(ds) = 0.$$

Hence

$$\Big| \int_{|y|<|x|/2} \psi(y)\, dy \Big| = \Big| \int_{|y|\ge|x|/2} \psi(y)\, dy \Big| \le c \int_{|y|\ge|x|/2} |y|^{-d-\alpha/2}\, dy \le \frac{c}{|x|^{\alpha/2}}.$$

We also note that if $|y| < |x|/2$ then using (ii), we obtain

$$|\kappa(x-y)-\kappa(x)| \le c\frac{|y|}{|x|^{d+\alpha/2}} \le c\frac{|y|^{1/2}}{|x|^{d+(\alpha-1)/2}} = c\frac{|y|^{1/2}}{|x|^{\lambda d}}.$$

Then

$$|I_2| \le \frac{c}{|x|^{\lambda d}} \int_{\mathbb{R}^d} |\psi(y)|\cdot|y|^{1/2}dy + \frac{c}{|x|^{d-1+\alpha}}. \tag{2.6}$$

If we show that the integral in (2.6) is bounded by a constant, then we obtain

$$|I_2| \le \frac{c}{|x|^{\lambda d}}.$$

To show the boundedness of the integral, we consider the cases $|y| < 1$ and $|y| \ge 1$. Note that

$$\int_{\mathbb{R}^d} |\psi(y)|\cdot|y|^{1/2}\,dy \le \int_{|y|<1} |\psi(y)|\,dy + \int_{|y|\ge 1} \frac{c}{|y|^{d+\alpha/2}}\cdot|y|^{1/2}\,dy$$

by Lemma 2.7. The second term is convergent since $\alpha > 1$. The first term is bounded by

$$\begin{aligned}\int_{|y|<1}\int_0^\infty p(s,y,0)\left|1-\frac{1}{2s}\right|\mu_1(ds)\,dy &\le \int_0^\infty\int_{\mathbb{R}^d} p(s,y,0)\,dy\left|1-\frac{1}{2s}\right|\mu_1(ds)\\ &\le \int_0^{1/2}\frac{1}{2s}\,\mu_1(ds) + \int_{1/2}^\infty \mu_1(ds) \le c.\end{aligned}$$

Hence

$$|(\partial Q_t\kappa(x))_{t=1}| \le |I_1|+|I_2|+|I_3| \le \frac{c}{|x|^{\lambda d}} \le \frac{c}{(1+|x|)^{\lambda d}} \tag{2.7}$$

whenever $|x| > 1$. Finally, inequalities (2.5) and (2.7) and Theorem 2.5 imply that $\|Tf\|_p \le c\|f\|_p$ for $p > 1$ and $f \in \mathcal{C}_K^1$, which finishes the proof. ∎

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Deniz Karlı
Department of Mathematics
Işık University
AMF233, 34980 Şile, Istanbul, Turkey
E-mail: deniz.karli@isikun.edu.tr
deniz.karli@gmail.com