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A NEW REGULARITY CRITERION FOR STRONG SOLUTIONS TO THE ERICKSEN–LESLIE SYSTEM

Abstract. A regularity criterion for strong solutions of the Ericksen–Leslie equations is established in terms of both the pressure and orientation field in homogeneous multiplier spaces.

1. Introduction and main result. In this paper, we are mainly interested to establish the regularity criteria for the following simplified version of the Ericksen–Leslie system [14, 15] in $\mathbb{R}^3$:

$$\begin{cases} \partial_t v + (v\cdot\nabla)v - \Delta v + \nabla\pi = -\nabla\cdot(\nabla u \odot \nabla u), \\ \partial_t u + (v\cdot\nabla)u - \Delta u = |\nabla u|^2 u, \\ \nabla\cdot v = 0 \quad \text{and} \quad |u| = 1, \end{cases} \tag{1.1}$$

where $v = v(x,t)$ and $u = u(x,t)$ denote the unknown velocity vector field and the orientation field, while v_0, u_0 with $\nabla\cdot v_0 = 0$ in the sense of distributions are given initial data, and $\pi = \pi(x,t)$ is the pressure of the fluid at the point $(x,t)\in\mathbb{R}^3\times(0,\infty)$. The notation $\nabla u\odot\nabla u$ denotes the 3×3 matrix whose (i,j)th entry is given by $\partial_i u\cdot\partial_j u$ $(1\le i,j\le 3)$.

Throughout this paper, we always assume that (v,u) satisfies the following boundary and initial conditions:

$$\begin{gathered} \lim_{|x|\to\infty} v(x,t) = 0, \qquad \lim_{|x|\to\infty} u(x,t) = a, \\ v(x,0) = v_0(x), \qquad u(x,0) = u_0(x), \qquad \text{for } x\in\mathbb{R}^3. \end{gathered} \tag{1.2}$$

for some constant unit vector a.

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Ericksen–Leslie theory is one of the most successful models for nematic liquid crystals. It was formulated by Ericksen [2] and Leslie [13], who derived suitable constitutive equations. Due to its complexity, the earlier attempts at a mathematical study of the Ericksen–Leslie system were focused on simplified models, first considered by Lin and Liu [15, 16].

Very recently, there have been some important advances on (1.1). It has been established that the three-dimensional Ericksen–Leslie system has only local (strong) solutions [8, 19], and global (strong) solutions were obtained under some smallness conditions on the initial data [17]. We also refer the reader to [5] and the references therein for other related work on this system.

Since there is no general global existence result for strong solutions to the 3D Ericksen–Leslie system, to understand the possible singularities, it is of interest to find whether the solution $(v(t), u(t))$ really loses its regularity at $t = T$ and to establish some new regularity criteria.

Wen and Ding [19] have established the local existence and uniqueness of strong solutions to system (1.1)–(1.2), while the existence of regular solutions is still an open problem; there are many interesting sufficient conditions which guarantee that a given weak solution is smooth. A well-known condition states that if

$$\begin{aligned} &v \in L^q(0,T;L^p), \qquad \nabla u \in L^r(0,T;L^s) \quad \text{with} \\ &\frac{2}{q}+\frac{3}{p}=1, \quad \frac{2}{r}+\frac{3}{s}=1, \quad 3<p,s\le\infty, \end{aligned} \tag{1.3}$$

then the solution u is actually regular [1, 4, 11]. A similar condition

$$\begin{aligned} &\operatorname{rot} v = \nabla \times v \in L^q(0,T;L^p), \quad \nabla u \in L^r(0,T;L^s) \quad \text{with} \\ &\frac{2}{q}+\frac{3}{p}=2, \quad \frac{2}{r}+\frac{3}{s}=1, \quad \frac{3}{2}<p\le\infty, \quad 3<s\le\infty \end{aligned} \tag{1.4}$$

also implies the regularity, as shown by Fan and Guo [4].

As regards (1.3) and (1.4) for $p = s = \infty$, Fan et al. [3] made an improvement to

$$v, \nabla u \in L^2(0,T;\dot{B}^0_{\infty,\infty}),$$

or

$$\nabla \times v \in L^1(0,T;\dot{B}^0_{\infty,\infty}) \quad \text{and} \quad \nabla u \in L^2(0,T;\dot{B}^0_{\infty,\infty}),$$

where $\dot{B}^0_{\infty,\infty}$ is the homogeneous Besov space. On the other hand, Fan and Li [5] proposed another regularity criterion in terms of the pressure. They showed that if the pressure π satisfies

$$\begin{aligned} &\pi \in L^{2/(1+\alpha)}(0,T;\dot{B}^\alpha_{\infty,\infty}), \qquad \nabla u \in L^r(0,T;L^s) \quad \text{with} \\ &-1\le\alpha\le1, \qquad \frac{2}{r}+\frac{3}{s}=1, \qquad 3<s\le\infty, \end{aligned}$$

or

$$\pi \in L^{2/(1+r)}(0,T;\dot{B}^{\alpha}_{\infty,\infty}), \qquad \nabla u \in L^2(0,T;\mathrm{BMO}) \quad \text{with} \quad -1 \le \alpha \le 1,$$

then (v,u) is smooth. Here, BMO is the space of functions of bounded mean oscillations.

For convenience, we present definitions related to global strong solutions before stating our main result. Let $T>0$ be a given fixed time.

DEFINITION 1.1. A couple (v,u) is a *strong solution* to system (1.1)–(1.2) on $\mathbb{R}^3 \times (0,T)$ if

$$v, \nabla u \in C([0,T], H^1(\mathbb{R}^3)) \cap L^2(0,T;H^2(\mathbb{R}^3)),$$
$$\partial_t v \in L^2(0,T;L^2(\mathbb{R}^3)), \qquad \partial_t u \in L^2(0,T;H^1(\mathbb{R}^3)),$$

and (v,u) satisfies (1.1) a.e. on $\mathbb{R}^3 \times (0,T)$, and the initial condition (1.2).

Our main result is the following criterion in terms of the gradient of the pressure.

THEOREM 1.2. *Let* $(v_0, \nabla u_0) \in H^1(\mathbb{R}^3)$ *be given initial data with* $\operatorname{div} v_0 = 0$ *in* $\mathbb{R}^3$ *and* $|u_0| = 1$. *Let* (v,u) *be the unique local strong solution of* (1.1) *in* $[0,T)$. *If* $\nabla\pi$ *and* ∇u *satisfy Serrin's type condition*

$$\int_0^T \big(\|\nabla\pi(s)\|_{\mathrm{BMO}}^{2/3} + \|\nabla u(s)\|_{\dot{X}_r}^{2/(1-r)}\big)\,ds < \infty \quad \textit{for some } 0 \le r \le 1, \tag{1.5}$$

then the solution (v,u) *to the problem* (1.1) *remains smooth on* $[0,T]$.

REMARK 1.1. This result extends the previous work by Fan and Guo [4].

COROLLARY 1.3. *Let* $T>0$. *Assume that* $(v_0, \nabla u_0) \in H^1(\mathbb{R}^3)$ *with* $\operatorname{div} v_0 = 0$ *in* $\mathbb{R}^3$ *and* $|u_0| = 1$. *Let* (v,u) *be the unique local strong solution of* (1.1) *in* $[0,T)$. *If* $\nabla\pi$ *and* $\nabla \times u$ *satisfy Serrin's type condition*

$$\int_0^T \big(\|\nabla\pi(s)\|_{\mathrm{BMO}}^{2/3} + \|\omega(s)\|_{\dot{X}_r}^{2/(1-r)}\big)\,ds < \infty \quad \textit{for some } 0 \le r \le 1,$$

then the solution (v,u) *to the problem* (1.1) *remains smooth on* $[0,T]$, *where* $\omega = \nabla \times u$.

REMARK 1.2. Note that

$$\|\nabla u\|_{\dot{X}_r} \le C\|\omega\|_{\dot{X}_r}, \qquad \|\nabla u\|_{\mathrm{BMO}} \le C\|\omega\|_{\mathrm{BMO}}$$

by Calderón–Zygmund estimates.

2. The proof of Theorem 1.2. In this section, we shall give the proof of Theorem 1.2. Let us first recall the definition of the multiplier space.

Definition 2.1. For $0 \le r < 3/2$, define $\dot{X}_r(\mathbb{R}^3)$ as the space of functions $f \in L^2_{\rm loc}(\mathbb{R}^3)$ such that

$$\|f\|_{\dot{X}_r} = \sup_{\|g\|_{\dot{H}^r} \le 1} \|fg\|_{L^2} < \infty,$$

where we denote by $\dot{H}^r(\mathbb{R}^3)$ the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|_{\dot{H}^r} := \|(-\Delta)^{r/2}u\|_{L^2}$.

The multiplier space $\dot{X}_r(\mathbb{R}^3)$ has the following homogeneity properties: for all $x_0 \in \mathbb{R}^3$ and $\lambda > 0$,

$$f(\cdot + x_0)_{\dot{X}_r} = \|f(\cdot)\|_{\dot{X}_r}, \quad \|f(\lambda\cdot)\|_{\dot{X}_r} = \frac{1}{\lambda^r}\|f(\cdot)\|_{\dot{X}_r}.$$

Moreover, the following imbedding holds:

$$L^{3/r}(\mathbb{R}^3) \hookrightarrow L^{3/r,\infty}(\mathbb{R}^3) \hookrightarrow \dot{X}_r(\mathbb{R}^3) \quad \text{for } 0 \le r < 3/2.$$

It is easy to verify that

$$v := \left(\frac{x_2}{|x|^{1+r}}, -\frac{x_1}{|x|^{1+r}}, 0\right) \in \dot{X}_r(\mathbb{R}^3) \quad \text{for } 0 \le r < 3/2$$

and $\operatorname{div} v = 0$, but $v \notin L^{3/r}(\mathbb{R}^3)$. Thus $\dot{X}_r(\mathbb{R}^3)$ with $0 \le r < 3/2$ is much wider than the Lebesgue space $L^{3/r}(\mathbb{R}^3)$. For more detailed properties of $\dot{X}_r(\mathbb{R}^3)$, we refer to [12].

Remark 2.1. Since $\dot{X}_r(\mathbb{R}^3)$ with $0 \le r \le 1$ is wider than $L^{3/r}(\mathbb{R}^3)$, our result (1.5) shows that condition (1.3) still implies that the weak solution (v, u) is regular on $\mathbb{R}^3 \times (0, T)$. We also notice that the result of Theorem 1.2 is still valid for the direction regularity problem for the 3D Navier–Stokes equations. Hence it is an improvement of the recent result obtained by Fan and Ozawa [6].

Remark 2.2. When $r=0$, we notice that $\dot{X}_0(\mathbb{R}^3) \cong \mathrm{BMO}(\mathbb{R}^3)$ (see [12]). Hence, Theorem 1.2 shows that the condition

$$\nabla\pi \in L^{2/3}(0, T; \mathrm{BMO}(\mathbb{R}^3)) \quad \text{and} \quad \nabla u \in L^2(0, T; \mathrm{BMO}(\mathbb{R}^3))$$

still implies that the weak solution (v, u) is regular on $\mathbb{R}^3 \times (0, T)$.

We will use the elementary interpolation inequalities (see e.g. [10])

$$\|f\|^2_{L^{2q}} \le C\|f\|_{\mathrm{BMO}}\|f\|_{L^q}, \quad 1 \le q < \infty, \tag{2.1}$$

and

$$\|f\|_{\dot{H}^r} \le C\|f\|^{1-r}_{L^2}\|f\|^r_{\dot{H}^1}, \quad 0 \le r \le 1.$$

To prove Theorem 1.2, we will use the following result of Giga [7] (see also [9]).

Proposition 2.2. *Suppose* $(v_0, u_0) \in L^q(\mathbb{R}^3) \times \dot{W}^{1,q}(\mathbb{R}^3)$ *for some* $q > 3$ *and* $\operatorname{div} v_0 = 0$. *Then there exist a constant* T_0 *and a unique classical solution*

$$(v, u) \in BC([0, T_0]; L^q(\mathbb{R}^3) \times \dot{W}^{1,q}(\mathbb{R}^3)).$$

Moreover, let $(0, T_*)$ *be the maximal interval such that* (v, u) *solves system* (1.1)–(1.2) *in* $C((0, T_*); L^q(\mathbb{R}^3) \times \dot{W}^{1,q}(\mathbb{R}^3))$. *Then, for any* $t \in (0, T_*)$,

$$\|v(\cdot, t)\|_{L^q} \geq \frac{C}{(T_* - t)^{\frac{q-3}{2q}}} \quad \text{and} \quad \|\nabla u(\cdot, t)\|_{L^q} \geq \frac{C}{(T_* - t)^{\frac{q-3}{2q}}},$$

with the constant C *independent of* T_* *and* q.

Proof of Theorem 1.2. To establish blow up criteria for the Ericksen–Leslie system, following the argument in [5], we derive a priori estimates for smooth solutions of (1.1)–(1.2). To this end, multiplying both sides of the first equation of $(1.1)_1$ by v, after integration by parts we get

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 = -\int_{\mathbb{R}^3} (v \cdot \nabla) u \cdot \Delta u \, dx. \tag{2.2}$$

Testing $(1.1)_2$ with $-(\Delta u + |\nabla u|^2 u)$, and using the fact that $\nabla \cdot v = 0$, shows that

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \big\|\Delta u + |\nabla u|^2 u\big\|_{L^2}^2 = \int_{\mathbb{R}^3} (v \cdot \nabla) u \cdot \Delta u \, dx. \tag{2.3}$$

Here, we have used the fact that $u \cdot u_t = 0$ and $(v \cdot \nabla u) \cdot u = 0$ guaranteed by $|u| = 1$. Summing up (2.2) and (2.3), we get

$$\frac{1}{2}\frac{d}{dt}(\|v\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) + \|\nabla v\|_{L^2}^2 + \big\|\Delta u + |\nabla u|^2 u\big\|_{L^2}^2 = 0.$$

Taking the inner product of the second equation of $(1.1)_1$ with $|v|^2 v$ and integrating by parts yields

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}\|v\|_{L^4}^4 + \big\||v|\,|\nabla v|\big\|_{L^2}^2 + \frac{1}{2}\big\|\nabla |v|^2\big\|_{L^2}^2 &= -\int_{\mathbb{R}^3} v|v|^2 \operatorname{div}(\nabla u \odot \nabla u)\, dx - \int_{\mathbb{R}^3} v \cdot \nabla \pi \, |v|^2 \, dx \\ &= \int_{\mathbb{R}^3} \nabla (v|v|^2)(\nabla u \odot \nabla u)\, dx - \int_{\mathbb{R}^3} v \cdot \nabla \pi \, |v|^2 \, dx.\end{aligned} \tag{2.4}$$

In a similar way, applying the operator ∇ to $(1.1)_2$, taking the inner product of the result with $|\nabla u|^2 \nabla u$ and integrating by parts yields

$$\frac{1}{4}\frac{d}{dt}\|\nabla u\|_{L^4}^4 + \big\||\nabla u|\,|\nabla^2 u|\big\|_{L^2}^2 + \frac{1}{2}\big\|\nabla|\nabla u|^2\big\|_{L^2}^2 \tag{2.5}$$
$$= \int_{\mathbb{R}^3} \nabla(|\nabla u|^2 u - v\cdot\nabla u) : |\nabla u|^2\nabla u\,dx$$
$$= \int_{\mathbb{R}^3} (v\cdot\nabla u - |\nabla u|^2 u)\cdot \operatorname{div}(|\nabla u|^2\nabla u)\,dx$$
$$\le C\int_{\mathbb{R}^3} (|v|\,|\nabla u| + |\nabla u|^2)|\nabla u|^2|\nabla^2 u|\,dx,$$

where in the last step we have used the initial condition $|u| = 1$. Summing (2.4) and (2.5), we get

$$\frac{1}{4}\frac{d}{dt}(\|v\|_{L^4}^4 + \|\nabla u\|_{L^4}^4) + \big\||v|\,|\nabla v|\big\|_{L^2}^2 + \big\||\nabla u|\,|\nabla^2 u|\big\|_{L^2}^2 \tag{2.6}$$
$$+ \frac{1}{2}\big(\big\|\nabla|v|^2\big\|_{L^2}^2 + \big\|\nabla|\nabla u|^2\big\|_{L^2}^2\big)$$
$$\le \int_{\mathbb{R}^3} \nabla(v|v|^2)(\nabla u\odot\nabla u)\,dx - \int_{\mathbb{R}^3} v\cdot\nabla\pi\,|v|^2\,dx$$
$$+ C\int_{\mathbb{R}^3} \big(|v|\,|\nabla u| + |\nabla u|^2\big)\,|\nabla u|^2|\nabla^2 u|\,dx.$$

With the use of Young's inequality, we bound the first terms on the right-hand side of (2.6):

$$\int_{\mathbb{R}^3} \nabla(v|v|^2)(\nabla u\odot\nabla u)\,dx \tag{2.7}$$
$$\le C\int_{\mathbb{R}^3} |\nabla v|\,|v|^2|\nabla u|^2\,dx \le C\int_{\mathbb{R}^3} |\nabla v|\,|v|(|v|^2 + |\nabla u|^2)|\nabla u|\,dx$$
$$\le C\big\||v|\,|\nabla v|\big\|_{L^2}\big\|(|v|^2 + |\nabla u|^2)|\nabla u|\big\|_{L^2}$$
$$\le \tfrac{1}{4}\big\||v|\,|\nabla v|\big\|_{L^2}^2 + C\big\|(|v|^2 + |\nabla u|^2)|\nabla u|\big\|_{L^2}^2$$
$$\le \tfrac{1}{4}\big\||v|\,|\nabla v|\big\|_{L^2}^2 + C\|\nabla u\|_{\dot X_r}^2\big\||v|^2 + |\nabla u|^2\big\|_{\dot H^r}^2$$
$$\le \tfrac{1}{4}\big\||v|\,|\nabla v|\big\|_{L^2}^2 + C\|\nabla u\|_{\dot X_r}^2\big\||v|^2 + |\nabla u|^2\big\|_{L^2}^{2(1-r)}\big\||v|^2 + |\nabla u|^2\big\|_{\dot H^1}^{2r}$$
$$\le \tfrac{1}{4}\big\||v|\,|\nabla v|\big\|_{L^2}^2 + C\|\nabla u\|_{\dot X_r}^{2/(1-r)}\big\||v|^2 + |\nabla u|^2\big\|_{L^2}^2$$
$$+ \tfrac{1}{8}\big\|\nabla|v|^2 + \nabla|\nabla u|^2\big\|_{L^2}^2$$
$$\le \tfrac{1}{4}\big(\big\||v|\nabla v|\big\|_{L^2}^2 + \big\||\nabla u|\nabla^2 u\big\|_{L^2}^2\big) + C\|\nabla u\|_{\dot X_r}^{2/(1-r)}(\|v\|_{L^4}^4 + \|\nabla u\|_{L^4}^4).$$

In order to estimate $-\int_{\mathbb{R}^3} v\cdot\nabla\pi\,|v|^2\,dx$, we first establish some estimates between the pressure and the velocity. Taking the operator div on both sides of the second equation of $(1.1)_1$ gives

$$-\Delta\pi = \operatorname{div}\operatorname{div}(v\otimes v + \nabla u\odot\nabla u).$$

Applying L^q $(1<q<\infty)$ boundedness of singular integral operators yields

$$\|\pi\|_{L^q} \le C(\|v\|_{L^{2q}}^2 + \|\nabla u\|_{L^{2q}}^2),$$
$$\|\nabla\pi\|_{L^q} \le C\big(\big\||v|\nabla v\big\|_{L^q} + \big\|\nabla u\,|\nabla^2 u|\big\|_{L^q}\big).$$

Then employing Hölder's inequality, we estimate

$$\begin{aligned}\Big|\int_{\mathbb{R}^3} v\cdot\nabla\pi\,|v|^2\,dx\Big| &\le \int_{\mathbb{R}^3} |\nabla\pi|\,|v|^3\,dx \le \|\nabla\pi\|_{L^4}\|v^3\|_{L^{4/3}}\\ &\le C\|\nabla\pi\|_{\mathrm{BMO}}^{1/2}\|\nabla\pi\|_{L^2}^{1/2}\|v\|_{L^4}^3\\ &\le (C\|\nabla\pi\|_{\mathrm{BMO}}^{2/3}\|v\|_{L^4}^4)^{3/4}(\|\nabla\pi\|_{L^2}^2)^{1/4}\\ &\le C\|\nabla\pi\|_{\mathrm{BMO}}^{2/3}\|v\|_{L^4}^4 + \tfrac12\big\||v|\nabla v\big\|_{L^2}^2\\ &\le C\|\nabla\pi\|_{\mathrm{BMO}}^{2/3}\|v\|_{L^4}^4 + \tfrac12\big(\big\||v|\nabla v\big\|_{L^2}^2 + \big\||\nabla u|\nabla^2 u|\big\|_{L^2}^2\big),\end{aligned}$$

where we have used the estimate

$$\|\nabla\pi\|_{L^4}^2 \le C\|\nabla\pi\|_{\mathrm{BMO}}\|\nabla\pi\|_{L^2}.$$

Similarly, we estimate the third term on the right-hand side of (2.6). Direct calculations give

$$\begin{aligned}&\int_{\mathbb{R}^3} \big(|v|\,|\nabla u| + |\nabla u|^2\big)|\nabla u|^2|\nabla^2 u|\,dx\\ &\le C\int_{\mathbb{R}^3} |\nabla u|\big(|v|^2+|\nabla u|^2\big)|\nabla u|\,|\nabla^2 u|\,dx\\ &\le C\big\||\nabla u|\,|\nabla^2 u|\big\|_{L^2}\big\|(|v|^2+|\nabla u|^2)|\nabla u|\big\|_{L^2}\\ &\le \tfrac14\big\||\nabla u|\,|\nabla^2 u|\big\|_{L^2}^2 + C\big\|(|v|^2+|\nabla u|^2)|\nabla u|\big\|_{L^2}^2\\ &\le \tfrac14\big\||\nabla u|\,|\nabla^2 u|\big\|_{L^2}^2 + C\|\nabla u\|_{\dot X_r}^2\big\||v|^2+|\nabla u|^2\big\|_{\dot H^r}^2\\ &\le \tfrac14\big(\big\||v|\nabla v\big\|_{L^2}^2 + \big\||\nabla u|\nabla^2 u|\big\|_{L^2}^2\big) + C\|\nabla u\|_{\dot X_r}^{2/(1-r)}(\|v\|_{L^4}^4 + \|\nabla u\|_{L^4}^4).\end{aligned}\tag{2.8}$$

Substituting (2.7)–(2.8) into (2.6), and using (1.5), we arrive at

$$\sup_{0\le t\le T}(\|v\|_{L^4} + \|\nabla u\|_{L^4}) < \infty,$$

by Gronwall's inequality.

Now, we are in a position to complete the proof of Theorem 1.2. From Proposition 2.2, since $v_0\in H^1(\mathbb{R}^3)\cap L^4(\mathbb{R}^3)$ with $\operatorname{div} v_0 = 0$, and $u_0\in H^1(\mathbb{R}^3)\cap\dot W^{1,4}(\mathbb{R}^3)$, there is a maximal interval $[0,T_*)$ such that there exists a unique solution $(\widetilde v(x,t),\widetilde u(x,t))\in BC([0,T_*);L^4(\mathbb{R}^3)\times\dot W^{1,4}(\mathbb{R}^3))$. Since

$(v(x,t),u(x,t))$ is a weak solution, by using the uniqueness criterion of Serrin [18], we have

$$(v,u)\equiv(\widetilde{v},\widetilde{u})\quad \text{ on } [0,\min\{T_*,T\}).$$

By the a priori estimate (2.3) ensured by Theorem 1.2 and the standard continuation argument, we see that $T<T_*$ provided that

$$\int_0^T\big(\|\nabla\pi(s)\|_{\mathrm{BMO}}^{2/3}+\|\nabla u(s)\|_{\dot{X}_r}^{2/(1-r)}\big)\,ds<\infty\quad \text{ for some } 0\le r\le 1.$$

Hence, we obtain $(v,u)\in BC([0,T];L^4(\mathbb{R}^3)\times\dot{W}^{1,4}(\mathbb{R}^3))\cap C^\infty(\mathbb{R}^3\times[0,T])$. This completes the proof of Theorem 1.2. ∎

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References

[1] Q. Chen, Z. Tan and G. Wu, *LPS's criterion for incompressible nematic liquid crystal flows*, Acta Math. Sci. Ser. B 34 (2014), 1072–1080.

[2] J. Ericksen, *Conservation laws for liquid crystals*, Trans. Soc. Rheology 5 (1961), 22–34.

[3] J. Fan, H. Gao and B. Guo, *Regularity criteria for the Navier–Stokes–Landau–Lifshitz system*, J. Math. Anal. Appl. 36 (2010), 29–37.

[4] J. Fan and B. Guo, *Regularity criterion to some liquid crystal models and the Landau–Lifshitz equations in* $\mathbb{R}^3$, Sci. China Ser. A 51 (2008), 1787–1797.

[5] J. Fan and J. Li, *Regularity criteria for the strong solutions to the Ericksen–Leslie system in* $\mathbb{R}^3$, J. Math. Anal. Appl. 425 (2015), 695–703.

[6] J. Fan and T. Ozawa, *Regularity criterion for weak solutions to the Navier–Stokes equations in terms of the gradient of the pressure*, J. Inequal. Appl. 2008, art. ID 412678, 6 pp.

[7] Y. Giga, *Solutions for semilinear parabolic equations in* L^p *and regularity of weak solutions of the Navier–Stokes system*, J. Differential Equations 61 (1986), 186–212.

[8] M. Hong, J. Li and Z. Xin, *Blow-up criteria of strong solutions to the Ericksen–Leslie system in* $\mathbb{R}^3$, Comm. Partial Differential Equations 39 (2014), 1284–1328.

[9] T. Kato, *Strong* L^p*-solutions to the Navier–Stokes equations in* $\mathbb{R}^m$*, with applications to weak solutions*, Math. Z. 187 (1984), 471–480.

[10] H. Kozono and Y. Taniuchi, *Bilinear estimates in* BMO *and the Navier–Stokes equations*, Math. Z. 235 (2000), 173–194.

[11] J. Lee, *Scaling invariant blow-up criteria for simplified versions of Ericksen–Leslie system*, Discrete Contin. Dynam. Systems Ser. S 8 (2015), 381–388.

[12] P. G. Lemarié-Rieusset and S. Gala, *Multipliers between Sobolev spaces and fractional differentiation*, J. Math. Anal. Appl. 322 (2006), 1030–1054.

[13] F. M. Leslie, *Some constitutive equations for liquid crystals*, Arch. Ration. Mech. Anal. 28 (1968), 265–283.

[14] F.-H. Lin, J. Lin and C. Wang, *Liquid crystal flows in two dimensions*, Arch. Ration. Mech. Anal. 197 (2010), 297–336.

[15] F.-H. Lin and C. Liu, *Nonparabolic dissipative systems modelling the flow of liquid crystals*, Comm. Pure Appl. Math. 48 (1995), 501–537.

[16] F.-H. Lin and C. Liu, *Partial regularity of the dynamic system modeling the flow of liquid crystals*, Discrete Contin. Dynam. Systems 2 (1996), 1–22.

[17] W. Ma, H. Gong and J. Li, *Global strong solutions to incompressible Ericksen–Leslie system in* $\mathbb{R}^3$, Nonlinear Anal. 109 (2014), 230–235.

[18] J. Serrin, *On the regularity of weak solutions of the Navier–Stokes equations*, Arch. Ration. Mech. Anal. 9 (1962), 187–195.

[19] H. Wen and S. Ding, *Solutions of incompressible hydrodynamic flow of liquid crystals*, Nonlinear Anal. Real World Appl. 12 (2011), 1510–1531.

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