BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 64, No. 1, 2016

SEVERAL COMPLEX VARIABLES AND ANALYTIC SPACES

Multiplicity and Semicontinuity of the Łojasiewicz Exponent

by

Tomasz RODAK, Adam RÓŻYCKI and Stanisław SPODZIEJA

Presented by Józef SICIAK

Summary. We give an effective formula for the improper isolated multiplicity of a polynomial mapping. Using this formula we construct, for a given deformation of a holomorphic mapping with an isolated zero at zero, a stratification of the space of parameters such that the Łojasiewicz exponent is constant on each stratum.

Introduction. Let $f\colon (\mathbb{C}^n,0) \to (\mathbb{C}^m,0)$ be a germ of a holomorphic map with an isolated zero. Then a lot of numerical invariants can be associated with this map. In this note we are interested in two of them: *multiplicity* and *Łojasiewicz exponent*.

The multiplicity of f may be defined in several ways. Probably the best known is the notion of *Hilbert–Samuel multiplicity* (see [5]). Let I be the ideal generated by the components of f in the local ring $(\mathcal{O}_n,\mathfrak{m}_n)$ of germs of holomorphic functions $(\mathbb{C}^n,0)\to\mathbb{C}$. Then the Hilbert–Samuel multiplicity of I is the normalized leading coefficient of the Hilbert–Samuel polynomial of I; in our case it is given by the formula

$$e(I)=\lim_{k\to\infty}\frac{n!}{k^n}\dim \mathcal{O}_n/I^k.$$

If f is a system of parameters (i.e. $m=n$), then

$$e(I)=\dim \mathcal{O}_n/I.$$

Moreover, in this case $e(I)$ has a well known geometric description: $e(I)=i_0(f)$ where $i_0(f)$ is the number of points in the generic fiber of f. Using

2010 *Mathematics Subject Classification*: 32B10, 14Q99, 12Y99.
Key words and phrases: multiplicity, effective formula, Łojasiewicz exponent.
Received 6 November 2015; revised 28 February 2016.
Published online 18 March 2016.

DOI: 10.4064/ba8041-3-2016

results of R. Achilles, P. Tworzewski and T. Winiarski [1], it is possible to extend the geometric definition of $i_0(f)$ to the case $m > n$. Namely, let $i_0(f)$ be the improper intersection multiplicity of the graph of f and $\mathbb{C}^n \times \{0\} \subset \mathbb{C}^n \times \mathbb{C}^m$ at the point $(0,0) \in \mathbb{C}^n \times \mathbb{C}^m$. In the case $m = n$ this notion was defined by R. Draper [2] (see also [12], [15]). In fact, with this generalization the multiplicity $i_0(f)$ is still equal to $e(I)$. Indeed, let $L\colon \mathbb{C}^m \to \mathbb{C}^n$ be a generic linear map. By [11] we have $i_0(f) = i_0(L \circ f)$ (see Theorem 1 below). On the other hand, the ideal generated by $L \circ f$ in $\mathcal{O}_n$ is a reduction of I, hence has the same Hilbert–Samuel multiplicity [5, Theorems 14.13, 14.14]. In what follows, we will denote the multiplicity of f by $i_0(f)$.

Let us now proceed to the second invariant. Since f is analytic, there exist $C > 0$ and $\nu \geq 1$ such that

$$|f(z)| \geq C|z|^\nu$$

in some neighbourhood of the origin in $\mathbb{C}^n$. By definition, the *Łojasiewicz exponent* of f, denoted by $\mathcal{L}_0(f)$, is the infimum of the exponents ν in the above inequality. In [3] it was proved that $\mathcal{L}_0(f)$ is a rational number and the infimum is in fact a minimum. Moreover, in [3] an algebraic formula for the Łojasiewicz exponent was given:

$$\mathcal{L}_0(f) = \inf\left\{\frac{p}{q} : \mathfrak{m}_n^p \subset \overline{I^q}\right\},$$

where for any ideal J in $\mathcal{O}_n$, $\overline{J}$ denotes the integral closure of J in $\mathcal{O}_n$.

Now, let $h\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function defining an isolated singularity at $0 \in \mathbb{C}^n$ (i.e. the gradient ∇h of h has an isolated zero). Then $\mu := i_0(\nabla h)$ is the Milnor number of h. In [13], B. Teissier proved that if $s \mapsto h_s$ is an analytic family of functions with isolated singularities with constant Milnor number, then the function $s \mapsto \mathcal{L}_0(\nabla h_s)$ is lower semicontinuous. Moreover, he showed that if we do not assume that this family is μ-constant then $\mathcal{L}_0(\nabla h)$ is neither upper nor lower semicontinuous [14]. The above result was generalized by A. Płoski [7] in the following way: If $s \mapsto f_s$ is an analytic family of holomorphic maps $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ with an isolated zero and of constant multiplicity, then the function $s \mapsto \mathcal{L}_0(f_s)$ is lower semicontinuous.

One may consider a further generalization of this result. Since the multiplicity i_0 is well defined for ideals which are not generated by a system of parameters, it is reasonable to ask if this assumption in the above result of Płoski is necessary. It was proved in [8] that it is enough to assume that the f_s are maps $(\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ with m possibly greater than n, with isolated zero of constant multiplicity. Under these assumptions the function $s \mapsto \mathcal{L}_0(f_s)$ is lower semicontinuous.

In the paper we prove that for a given finite complex stratification $\{\Gamma_\nu^i\}$ of the space of parameters such that f_s is multiplicity-constant on each stratum Γ_ν^i, the function $s \mapsto \mathcal{L}_0(f_s)$ is lower semicontinous on this stratum and there exists a refinement $\{\Sigma_\mu^j\}$ of $\{\Gamma_\nu^i\}$ such that the function $s \mapsto \mathcal{L}_0(f_s)$ is constant on each stratum Σ_μ^j (Theorem 7). The proof is based on an algorithm which allows us to effectively compute the multiplicity $i_0(f)$ (Theorem 4, cf. [10]). As a corollary we get the above-mentioned semicontinuity theorem (Corollary 11).

1. A formula for multiplicity. Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ be a holomorphic mapping with an isolated zero. Denote by $\mathbb{L}(m,n)$ the set of all linear mappings $\mathbb{C}^m \to \mathbb{C}^n$.

The basis for our further considerations is

THEOREM 1 ([11, Theorem 1.1]). *For any $L \in \mathbb{L}(m,n)$ such that the mapping $L \circ f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ has an isolated zero we have*

$$i_0(f) \le i_0(L \circ f). \tag{1}$$

Moreover, for generic $L \in \mathbb{L}(m,n)$, the mapping $L \circ f$ has an isolated zero and

$$i_0(f) = i_0(L \circ f). \tag{2}$$

The next proposition will be used to pass from holomorphic to polynomial germs of mappings.

PROPOSITION 2 ([6, 11]). *We have*

$$\mathcal{L}_0(f) \le i_0(f).$$

Moreover, if $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ is a holomorphic mapping such that $\operatorname{ord}(f-g) > \mathcal{L}_0(f)$ *then g has an isolated zero and*

$$\mathcal{L}_0(g) = \mathcal{L}_0(f) \quad \textit{and} \quad i_0(g) = i_0(f).$$

From now on we will assume that $f = (f_1, \ldots, f_m)\colon \mathbb{C}^n \to \mathbb{C}^m$ is a polynomial mapping such that $0 \in \mathbb{C}^n$ is an isolated point of $f^{-1}(0)$.

PROPOSITION 3 ([6, 11]). *Let $d_j = \deg f_j$, $j = 1, \ldots, m$. Assume that $d_1 \ge \cdots \ge d_m$. Then*

$$\mathcal{L}_0(f) \le d_1 \cdots d_n.$$

The algorithm which computes $i_0(f)$ is given in the following construction.

Let $d = \max\{\deg f_1, \ldots, \deg f_m\}$. Define a mapping $H_L\colon \mathbb{C}^n \to \mathbb{C}^n$ by

$$H_L(z) = L(f(z)) + (z_1^{d^n+1}, \ldots, z_n^{d^n+1}), \tag{3}$$

where $L \in \mathbb{L}(m,n)$. Set

$$\mathbb{M}(m,n) = \mathbb{L}(m,n) \times \mathbb{L}(n,1) \times \mathbb{C}^n$$

and let

$$\Phi\colon \mathbb{M}(m,n) \to \mathbb{M}(m,n) \times \mathbb{C}$$

be given by

$$\Phi(L,N,z) = (L,N,H_L(z),N(z)).$$

The mapping Φ is proper and consequently $\Phi(\mathbb{M}(m,n))$ is an algebraic set of pure dimension $mn+2n$. So, there exists an irreducible polynomial $P \in \mathbb{C}[L,N,y,t]$, where $y=(y_1,\dots,y_n)$ and $y_1,\dots,y_n,t$ are independent variables, of the form

$$P(L,N,y,t) = \sum_{j=0}^{p} P_j(L,N,y)t^j \tag{4}$$

such that $P_p \neq 0$ and $\Phi(\mathbb{M}(m,n)) = P^{-1}(0)$. Since P vanishes exactly on the image of the polynomial map Φ, it could be computed by means of Gröbner bases.

Theorem 4. *We have*

$$i_0(f) = \min\{j \in \mathbb{Z} : \operatorname{ord}_y P_j = 0\}.$$

The right hand side above is well defined in view of the following proposition, which is a special case of [9, Theorem 7].

Proposition 5. *There exists $r \in \mathbb{Z}$ with $0 \le r < p$ such that*

$$\operatorname{ord}_y P_j > 0 \quad \text{for } j=0,\dots,r \quad \text{and} \quad \operatorname{ord}_y P_{r+1} = 0. \tag{5}$$

Set

$$\Delta(P) = \min_{j=0}^{r} \frac{\operatorname{ord}_y P_j}{r+1-j}.$$

Then

$$\mathcal{L}_0(f) = \frac{1}{\Delta(P)} < d^n + 1. \tag{6}$$

We will also use this proposition in the proof of the main result in the next section.

Proof of Theorem 4. Let r be the integer given in Proposition 5. We must prove that $i_0(f) = r+1$. Observe that there exists a Zariski open, nonempty set $\mathcal{U} \subset \mathbb{L}(m,n) \times \mathbb{L}(n,1)$ such that if $(L,N) \in \mathcal{U}$ then:

- $L \circ f$ has an isolated zero at the origin,
- condition (5) is satisfied,
- $N|_{H_L^{-1}(y)}$ is injective for generic $y \in \mathbb{C}^n$,
- $H_L^{-1}(0) \cap \ker N = \{0\}$.

Fix $(L,N) \in \mathcal{U}$. Then

$$i_0(H_L) = r+1. \tag{7}$$

Indeed, by (5) the polynomial $P_{L,N}(y,t)$ is a t-regular function of order $r+1$.

Using the Weierstrass preparation theorem we may write

$$P_{L,N}(y,t) = Q_{L,N}(y,t)\widetilde{P}_{L,N}(y,t),$$

where $Q_{L,N}$ is an invertible power series in (y,t). By the properties of $\mathcal{U}$ the image of the local map $(H_L, N)\colon (\mathbb{C}^n,0) \to (\mathbb{C}^{n+1},0)$ is equal to the germ of the zero set of $\widetilde{P}_{L,N}$. Since $P_{L,N}$ is irreducible, so is $\widetilde{P}_{L,N}$. On the other hand, with any y in a sufficiently small neighbourhood of the origin in $\mathbb{C}^n$ we may associate two sets: all roots $\{(y,t_1),\ldots,(y,t_{r+1})\}$ of $\widetilde{P}_{L,N}$ (since $\widetilde{P}_{L,N}$ is a polynomial of degree $r+1$ in t) and the fiber $H_L^{-1}(y) = \{z_1,\ldots,z_u\}$ (where H_L is treated as a local map $(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$). If additionally y is generic then:

- $\#\{(y,t_1),\ldots,(y,t_{r+1})\} = r+1$,
- $u = i_0(H_L)$,
- (H_L,N) restricted to $\{z_1,\ldots,z_u\}$ is a bijection onto $\{(y,t_1),\ldots,(y,t_{r+1})\}$.

As a result we get (7).

By Proposition 3 we have $\operatorname{ord}(L\circ f - H_L) > \mathcal{L}_0(L\circ f)$. Thus $i_0(H_L) = i_0(L\circ f)$ by Proposition 2. By (7) and Theorem 1, this ends the proof. ∎

Corollary 6. *Let $Q \in \mathbb{C}\{L,N,y,t\}$ be a series of the form*

$$Q(L,N,y,t) = \sum_{j=0}^{\infty} Q_j(L,N,y)t^j. \tag{8}$$

If Q is irreducible in $\mathcal{O}_{mn+2n+1}$ and $Q\circ\Phi = 0$ at the level of germs, then

$$i_0(f) = \min\{j\in\mathbb{Z} : \operatorname{ord}_y Q_j = 0\}. \tag{9}$$

Proof. Since P and Q are irreducible in $\mathcal{O}_{mn+2n+1}$ and the germs of the sets $P^{-1}(0)$ and $Q^{-1}(0)$ are equal, P and Q differ by an invertible factor in $\mathcal{O}_{mn+2n+1}$. Hence Theorem 4 yields the assertion. ∎

2. Semicontinuity of the Łojasiewicz exponent. Let $f\colon(\mathbb{C}^n,0)\to(\mathbb{C}^m,0)$ be a holomorphic mapping. We say that $F = F_s(z) = F(z,s)\colon (\mathbb{C}^n\times\mathbb{C}^k,0)\to(\mathbb{C}^m,0)$ is a *deformation* of f if F is holomorphic, $F_0 = f$ and $F_s(0) = 0$ for s in some neighbourhood of the origin in $\mathbb{C}^k$.

In what follows we will use the notion of a complex stratification (or briefly stratification) after [4].

The main result of this section is

Theorem 7. *Let $f\colon(\mathbb{C}^n,0)\to(\mathbb{C}^m,0)$ be a germ of a holomorphic mapping with an isolated zero and let $F\colon(\mathbb{C}^n\times\mathbb{C}^k,0)\to(\mathbb{C}^m,0)$ be its deformation. Let $U = \bigcup \Gamma_\nu^i$ be a finite stratification of some sufficiently small neighbourhood $U\subset\mathbb{C}^k$ of the origin such that for each stratum Γ_ν^i the function $\Gamma_\nu^i \ni s\mapsto i_0(F_s)\in\mathbb{Z}$ is constant. Then the function $\Gamma_\nu^i\ni s\mapsto \mathcal{L}_0(F_s)\in\mathbb{Q}$ is* *lower semicontinuous and there exists a finite stratification* $\{\Sigma^j_\mu\}$, *which is a refinement of* $\{\Gamma^i_\nu\}$, *such that the function* $\Sigma^j_\mu \ni s \mapsto \mathcal{L}_0(F_s) \in \mathbb{Q}$ *is constant for any stratum* Σ^j_μ.

In the proof we will need the following

LEMMA 8 ([9, Lemma 1]). *If* $P, Q, R \in \mathbb{C}\{y,t\}$ *are series such that*

$$P(y,t) = \sum_{j=0}^{\infty} P_j(y)t^j, \quad Q(y,t) = \sum_{j=0}^{\infty} Q_j(y)t^j,$$

$\operatorname{ord} R(y,t) = 0$ *and* $Q = PR$, *and for some* $r \geq 0$ *we have* $\operatorname{ord} P_j, \operatorname{ord} Q_j > 0$, $j = 0, \ldots, r$, *then*

$$\min_{j=0}^{r} \frac{\operatorname{ord} P_j}{r+1-j} = \min_{j=0}^{r} \frac{\operatorname{ord} Q_j}{r+1-j}.$$

Proof of Theorem 7. Since the multiplicity and the Łojasiewicz exponent of a local map do not change after perturbation in monomials of orders greater than the multiplicity of the map, we may assume that F_s is a polynomial map for any s. Let $d = \max\{\deg F_s : s \in U\}$.

Define a mapping $H_{L,s} \colon \mathbb{C}^n \to \mathbb{C}^n$ by

$$H_{L,s}(z) = L(F_s(z)) + (z_1^{d^n+1}, \ldots, z_n^{d^n+1}), \tag{10}$$

where $L \in \mathbb{L}(m,n)$, $s \in U$. Set

$$\mathbb{W} = \mathbb{M}(m,n) \times U$$

and let

$$\Phi \colon \mathbb{W} \to \mathbb{W} \times \mathbb{C}$$

be given by

$$\Phi(L,N,z,s) = (L,N,H_{L,s}(z),s,N(z)). \tag{11}$$

Define $\Phi_s \colon \mathbb{M}(m,n) \to \mathbb{M}(m,n) \times \mathbb{C}$ for $s \in U$ by

$$\Phi_s(L,N,z) = (L,N,H_{L,s}(z),N(z)). \tag{12}$$

Decreasing U if necessary, we achieve that the mapping Φ is proper, and consequently, by Remmert's Proper Mapping Theorem, $\Phi(\mathbb{W})$ is an analytic set of pure dimension $mn + 2n + k = \dim \mathbb{W}$. So, for some neighbourhoods $W \subset \mathbb{W}$ and $D \subset \mathbb{W}\times\mathbb{C}$ of the origins and a holomorphic function $Q \colon D \to \mathbb{C}$ with an irreducible germ at zero we have $\Phi(W) = \{(w,t) \in D : Q(w,t) = 0\}$.

Suppose that the function Q is of the form

$$Q(L,N,y,s,t) = \sum_{j=0}^{\infty} Q_j(L,N,y,s)t^j, \tag{13}$$

and denote $Q_s(L,N,y,t) = Q(L,N,y,s,t)$, $Q_{j,s}(L,N,y) = Q_j(L,N,y,s)$. It is easy to see that Q_s is irreducible for s sufficiently close to the origin of $\mathbb{C}^k$. Set $W_s = \{u \in \mathbb{M}(m,n) : (u,s) \in W\}$, $D_s = \{(u,t) \in \mathbb{M}(m,n) \times \mathbb{C} : (u,s,t) \in D\}$. Then $\Phi_s(W_s) = \{(u,t) \in D_s : Q_s(u,t) = 0\}$. Denote by $r^i_\nu + 1$ the multiplicity of F_s on Γ^i_ν. From Corollary 6 we have $\operatorname{ord}_y Q_{s,j} > 0$, $j = 0, \ldots, r^i_\nu$, and $\operatorname{ord}_y Q_{s,r^i_\nu+1} = 0$.

Set

$$\Delta(Q_s) = \min_{j=0}^{r_\nu^i} \frac{\operatorname{ord}_y Q_{j,s}}{r_\nu^i + 1 - j}, \quad s \in \Gamma_\nu^i.$$

Observe that the mapping $\Gamma_\nu^i \ni s \mapsto \Delta(Q_s) \in \mathbb{Q}$ is upper semicontinuous and determines a stratification of Γ_ν^i. Thus, there exists a finite stratification $\{\Sigma_\mu^j\}$, which is a refinement of $\{\Gamma_\nu^i\}$, such that the function $\Sigma_\mu^j \ni s \mapsto \Delta(Q_s) \in \mathbb{Q}$ is constant for any stratum Σ_μ^j. On the other hand, by Lemma 8 and Proposition 5 we have $\mathcal{L}_0(F_s) = 1/\Delta(Q_s)$ for $s \in U$. This ends the proof. ■

Remark 9. It is well known that the stratification $\{\Gamma_\nu^i\}$ from Theorem 7 always exists. For example, from Theorem 4 we see that the polynomial Q_s used in the proof of Theorem 7 determines such a stratification.

Example 10. Let $F\colon (\mathbb{C}^2 \times \mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be given by the formula

$$F_{s_1,s_2}(x_1, x_2) := (s_1x_1 + x_2^2, s_2x_2 + x_1^2).$$

For the stratification

$$\Gamma_1^2 := \{(s_1, s_2) : s_1s_2 \neq 0\},$$
$$\Gamma_1^1 \cup \Gamma_2^1 := \{(s_1, s_2) : s_2 = 0\} \cup \{(s_1, s_2) : s_1 = 0\}$$

we have

$$i_0(F_{s_1,s_2}) = \begin{cases} 1, & (s_1, s_2) \in \Gamma_1^2, \\ 4, & (s_1, s_2) \in \Gamma_1^1 \cup \Gamma_2^1. \end{cases}$$

If we set $\Sigma_1^2 := \Gamma_1^2$, $\Sigma_1^1 := \Gamma_1^1 \setminus \{(0,0)\}$, $\Sigma_2^1 := \Gamma_2^1 \setminus \{(0,0)\}$, $\Sigma_1^0 := \{(0,0)\}$, then

$$\mathcal{L}_0(F_{s_1,s_2}) = \begin{cases} 1, & (s_1, s_2) \in \Sigma_1^2, \\ 4, & (s_1, s_2) \in \Sigma_1^1 \cup \Sigma_2^1, \\ 2, & (s_1, s_2) \in \Sigma_1^0. \end{cases}$$

Observe that in this case F_s is already a proper polynomial mapping. Using CAS the polynomial Q_s is given by

$$\begin{aligned} Q_s(N, y, t) &= t^4 + (-3s_1s_2a_1a_2 - 2y_2a_1^2 - 2y_1a_2^2)t^2 \\ &\quad + (s_1s_2^2a_1^3 + s_1^2s_2a_2^3 + 4s_2y_1a_1^2a_2 + 4s_1y_2a_1a_2^2)t \\ &\quad - s_2^2y_1a_1^4 - s_1s_2y_2a_1^3a_2 - s_1s_2y_1a_1a_2^3 - s_1^2y_2a_2^4 + y_2^2a_1^4 - 2y_1y_2a_1^2a_2^2 + y_1^2a_2^4, \end{aligned}$$

where $N(x) = a_1x_1 + a_2x_2$, $y = (y_1, y_2)$, $s = (s_1, s_2)$.

The deformation F_s is called *multiplicity-constant* if the map $s \mapsto i_0(F_s)$ has constant finite value.

Corollary 11. *If f has an isolated zero and $F\colon (\mathbb{C}^n \times \mathbb{C}^k, 0) \to (\mathbb{C}^m, 0)$ is a multiplicity-constant deformation of f, then there exists $\varepsilon > 0$ such that*

$$\mathcal{L}_0(f) \le \mathcal{L}_0(F_s) \quad \text{for } |s| \le \varepsilon.$$

Acknowledgments. This research was partially supported by the Polish National Science Centre, grant 2012/07/B/ST1/03293.

References

[1] R. Achilles, P. Tworzewski, and T. Winiarski, *On improper isolated intersection in complex analytic geometry*, Ann. Polon. Math. 51 (1990), 21–36.
[2] R. N. Draper, *Intersection theory in analytic geometry*, Math. Ann. 180 (1969), 175–204.
[3] M. Lejeune-Jalabert and B. Teissier, *Séminaire Lejeune-Teissier: Clôture intégrale des idéaux et équisingularité : Chapitre 1*, Université Scientifique et Médicale de Grenoble, Laboratoire de mathématiques pures associé au C.N.R.S., 1974.
[4] S. Łojasiewicz, *Introduction to Complex Analytic Geometry*, Birkhäuser, 1991.
[5] H. Matsumura, *Commutative Ring Theory*, Cambridge Stud. Adv. Math. 8, Cambridge Univ. Press, Cambridge, 1989.
[6] A. Płoski, *Multiplicity and the Łojasiewicz exponent*, in: Singularities (Warszawa, 1985), Banach Center Publ. 20, PWN, Warszawa, 1988, 353–364.
[7] A. Płoski, *Semicontinuity of the Łojasiewicz exponent*, Univ. Iagel. Acta Math. 48 (2010), 103–110.
[8] T. Rodak, *Reduction of a family of ideals*, Kodai Math. J. 38 (2015), 201–208.
[9] T. Rodak and S. Spodzieja, *Effective formulas for the local Łojasiewicz exponent*, Math. Z. 268 (2011), 37–44.
[10] A. Różycki, *Effective calculations of the multiplicity of polynomial mappings*, Bull. Sci. Math. 138 (2014), 343–355.
[11] S. Spodzieja, *Multiplicity and the Łojasiewicz exponent*, Ann. Polon. Math. 73 (2000), 257–267.
[12] J. Stückrad and W. Vogel, *An algebraic approach to the intersection theory*, in: The Curves Seminar at Queens, Vol. II (Kingston, Ont., 1981/1982), Queen's Papers in Pure Appl. Math. 61, exp. no. A, Queen's Univ., Kingston, ON, 1982, 32 pp.
[13] B. Teissier, *Variétés polaires. I. Invariants polaires des singularités d'hypersurfaces*, Invent. Math. 40 (1977), 267–292.
[14] B. Teissier, *Polyèdre de Newton jacobien et équisingularité*, in: Seminar on Singularities (Paris, 1976/1977), Publ. Math. Univ. Paris VII 7, Univ. Paris VII, Paris, 1980, 193–221.
[15] P. Tworzewski, *Intersection theory in complex analytic geometry*, Ann. Polon. Math. 62 (1995), 177–191.

Tomasz Rodak, Adam Różycki, Stanisław Spodzieja
Faculty of Mathematics and Computer Science
University of Łódź
S. Banacha 22
90-238 Łódź, Poland
E-mail: rodakt@math.uni.lodz.pl
rozycki@math.uni.lodz.pl
spodziej@math.uni.lodz.pl