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On embeddings of $C_0(K)$ spaces into $C_0(L,X)$ spaces

by

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Abstract. For a locally compact Hausdorff space K and a Banach space X let $C_0(K,X)$ denote the space of all continuous functions $f : K \to X$ which vanish at infinity, equipped with the supremum norm. If X is the scalar field, we denote $C_0(K,X)$ simply by $C_0(K)$. We prove that for locally compact Hausdorff spaces K and L and for a Banach space X containing no copy of c_0, if there is an isomorphic embedding of $C_0(K)$ into $C_0(L,X)$, then either K is finite or $|K| \leq |L|$. As a consequence, if there is an isomorphic embedding of $C_0(K)$ into $C_0(L,X)$ where X contains no copy of c_0 and L is scattered, then K must be scattered.

1. Introduction. For a locally compact Hausdorff space K and a Banach space X, $C_0(K,X)$ denotes the Banach space of all continuous functions $f : K \to X$ which vanish at infinity, provided with the norm $\|f\| = \sup_{x\in K} \|f(x)\|$. If X is the field $\mathbb{R}$ of real numbers, we denote $C_0(K,X)$ simply by $C_0(K)$. If K is compact these spaces will be denoted by $C(K,X)$ and $C(K)$ respectively. As usual, we also denote $C_0(\mathbb{N})$ by c_0.

For Banach spaces X and Y, a linear operator $T : X \to Y$ is called an *isomorphic embedding* if there are $A,B > 0$ such that $A\|u\| \leq \|Tu\| \leq B\|u\|$ for all $u \in X$. If such an embedding exists, we say that Y *contains a copy of* X and write $X \hookrightarrow Y$. On the other hand, we write $X \not\hookrightarrow Y$ if Y contains no copy of X. An isomorphic embedding of X onto Y is called an *isomorphism*. Whenever an isomorphism exists, we say that the spaces are isomorphic and write $X \sim Y$.

We also adopt other standard notational conventions. For a Banach space X, B_X stands for its unit ball and X^* for its (topological) dual space. We denote by $\mathrm{span}(E)$ the linear span of a set $E \subset X$ and by $\overline{\mathrm{span}}(E)$ the closure of $\mathrm{span}(E)$ in X. The cardinality of any set Γ will be denoted by $|\Gamma|$.

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For compact Hausdorff spaces K and L, a natural question is what properties are transferred from L to K if there is an isomorphism of $C(K)$ onto $C(L)$. A more general question can be posed in the following way:

PROBLEM 1.1. *Suppose that there is an isomorphic embedding of $C(K)$ into $C(L)$ and L has some property $\mathcal{P}$. Does K have property $\mathcal{P}$?*

Since the classical paper of Banach [1], there have been several fascinating developments related to the questions above (for some very recent ones, see [5] and [6]). In the field of vector-valued continuous functions, the following question seems to be a natural extension of the previous one:

PROBLEM 1.2. *Suppose that there is an isomorphic embedding of $C(K)$ into $C(L, X)$, where X is a Banach space containing no copy of c_0 and L has some property $\mathcal{P}$. Does K have property $\mathcal{P}$?*

We observe that Problem 1.2 makes no sense without the condition 'X contains no copy of c_0' since $C(K)$ embeds in $C(L, C(K))$ for any compacta K and L.

In the present paper, for locally compact Hausdorff spaces K and L, we study isomorphic embeddings of $C_0(K)$ into $C_0(L, X)$ in the spirit of Problem 1.2. The results are the following:

THEOREM 1.3. *Let K and L be locally compact Hausdorff spaces and let X be a Banach space containing no copy of c_0. If $C_0(K) \hookrightarrow C_0(L, X)$, then either K is finite or $|K| \leq |L|$.*

REMARK 1.4. During the time the original manuscript was being processed, the above result was obtained independently by E. M Galego and M. A. Rincón-Villamizar [3]. In our original manuscript, X had the additional hypotheses: X is separable or X^* has the Radon–Nikodým property. These restrictions were later removed by the referee.

Theorem 1.3 provides an extension of the main result of [2]. Another application of Theorem 1.3 gives the following:

THEOREM 1.5. *Let K and L be locally compact Hausdorff spaces and let X be a Banach space containing no copy of c_0. If $C_0(K) \hookrightarrow C_0(L, X)$ and L is scattered, then K is scattered.*

REMARK 1.6. Since the unit interval $[0, 1]$ is an uncountable perfect set and clearly

$$C([0, 1]) \hookrightarrow C_0(\mathbb{N}, C([0, 1])),$$

we conclude that in general the hypothesis $c_0 \not\hookrightarrow X$ can be removed neither in Theorem 1.3 nor in Theorem 1.5.

2. Auxiliary results. In order to prove our theorems, we need first to establish some auxiliary results. Proposition 2.1 and Lemma 2.2 below will be applied in the proof of Theorem 1.3, and Lemma 2.3 will be used in the proof of Theorem 1.5.

We identify via the *Riesz representation theorem* the space $C_0(K)^*$ with the Banach space $M(K)$ of signed Radon measures on K of finite variation. For any measure $\mu \in M(K)$, we denote by $|\mu|$ its variation.

Given $y \in L$ and $\varphi \in X^*$, we denote by $\Delta_{y,\varphi}$ the functional in $C_0(L,X)^*$ defined by $\Delta_{y,\varphi}(f) = \varphi(f(y))$ for $f \in C_0(L,X)$.

PROPOSITION 2.1. *Let K and L be locally compact Hausdorff spaces, X be a Banach space containing no copy of c_0, and T be an isomorphic embedding of $C_0(K)$ into $C_0(L,X)$. For every $y \in L$ and $\epsilon > 0$ the set*

$$K_y(\epsilon) = \bigcup_{\varphi \in B_{X^*}} \{x \in K : |T^*(\Delta_{y,\varphi})|(\{x\}) > \epsilon\}$$

is finite.

Proof. Towards a contradiction, assume that $K_y(\epsilon)$ is infinite for some $y \in L$. Then we can find distinct points $x_1, x_2, \ldots$ in K_y and $\varphi_1, \varphi_2, \ldots$ in B_{X^*} such that

$$|T^*(\Delta_{y,\varphi_n})|(\{x_n\}) = |T^*(\Delta_{y,\varphi_n})(\{x_n\})| > \epsilon, \qquad \forall n \in \mathbb{N}.$$

For each n, by regularity of the measure $T^*(\Delta_{y,\varphi_n})$, we can find an open neighborhood V_n of x_n such that

$$|T^*(\Delta_{y,\varphi_n})|(V_n \setminus \{x_n\}) \leq \epsilon/2. \tag{2.1}$$

Since $x_1, x_2, \ldots$ are all distinct and K is a locally compact Hausdorff space, by passing to a subsequence if necessary, there are pairwise disjoint open sets $U_1, U_2, \ldots$ such that $x_n \in U_n \subseteq V_n$ for every $n \in \mathbb{N}$.

By using the Urysohn Lemma, we take functions $f_n \in C_0(K)$ such that $0 \leq f_n \leq 1$, $f_n(x_n) = 1$ and $f_n = 0$ outside U_n. Then, for any $\nu \in M(K)$, if $|\nu(\{x_n\})| > \epsilon$ and $|\nu(U_n)| \leq \epsilon/2$ then $|\int f_n \, d\nu| \geq \epsilon/2$. By applying this argument for $\nu_n = T(\Delta_{y,\varphi_n})$ we infer that

$$\|Tf_n(y)\| \geq |\varphi_n(Tf_n(y))| = |T^*(\Delta_{y,\varphi_n})(f_n)| = \left|\int f_n \, dT^*(\Delta_{y,\varphi_n})\right| \geq \frac{\epsilon}{2}.$$

Let $S : c_0 \to X$ be defined by $S((a_n)_n) = T(\sum_n a_n f_n)(y)$. Clearly, S is a bounded linear operator and if $\{e_n : n \in \mathbb{N}\}$ are the unit vectors in c_0, we have $\|S(e_n)\| = \|Tf_n(y)\| \geq \epsilon/2$ for every $n \in \mathbb{N}$. We deduce that $\inf_{n\in\mathbb{N}} \|S(e_n)\| \geq \epsilon/2$ and according to a result due to Rosenthal [7, Remark following Theorem 3.4], there exists an infinite $N \subseteq \mathbb{N}$ such that S restricted to $C_0(N)$ is an isomorphism onto its image. In other words, $c_0 \hookrightarrow X$, a contradiction. ∎

The next lemma is a consequence of a recent result of G. Plebanek [5, Theorem 3.3]. Although [5, Theorem 3.3] was proved for compact spaces, it also remains true for locally compact spaces. We are indebted to the referee for the proof given below. The proof of the lemma can also be found in [3, proof of Theorem 1.4]. In our original manuscript it was additionally assumed that X is separable or X^* has the Radon–Nikodým property.

LEMMA 2.2. *Let K and L be infinite locally compact Hausdorff spaces, X be a Banach space and T be an isomorphic embedding of $C_0(K)$ into $C_0(L,X)$. If X has no subspace isomorphic to c_0, then for each $x \in K$ there are $y \in L$ and $\varphi \in B_{X^*}$ such that $|T^*(\Delta_{y,\varphi})|(\{x\}) > 0$.*

Proof. Let $T : C_0(K) \to C_0(L,X)$ be an isomorphic embedding. Let B_{X^*} be the dual unit ball equipped with the weak*-topology. Then we can define an isomorphic embedding $S : C_0(K) \to C_0(L \times B_{X^*})$ by

$$S(f)(y,\varphi) = \varphi(Tf(y)), \quad y \in L,\ \varphi \in B_{X^*},\ f \in C_0(K).$$

For $y \in L$ and $\varphi \in B_{X^*}$, we denote by $\delta_{(y,\varphi)} \in C_0(L \times B_{X^*})^*$ the Dirac measure at (y,φ) and observe that for any $f \in C_0(L)$,

$$S^*(\delta_{(y,\varphi)})(f) = S(f)(y,\varphi) = \varphi(Tf(y)) = T^*(\Delta_{y,\varphi})(f).$$

Thus, $S^*(\delta_{(y,\varphi)}) = T^*(\Delta_{y,\varphi})$ for every $y \in L$ and $\varphi \in B_{X^*}$. According to the locally compact version of [5, Theorem 3.3], for each $x \in K$ there is some $(y,\varphi) \in L \times B_{X^*}$ such that $|S^*(\delta_{(y,\varphi)})|(\{x\}) = |T^*(\Delta_{y,\varphi})|(\{x\}) > 0$. ■

LEMMA 2.3. *Let K be a scattered compact Hausdorff space and let X and Y be Banach spaces, where Y is separable. If $Y \hookrightarrow C(K,X)$, then there is a countable metrizable compact space K_0 and a separable Banach space $X_0 \subset X$ such that*

$$Y \hookrightarrow C(K_0,X_0) \hookrightarrow C(K,X).$$

Proof. Since Y is a separable space isomorphically embedded in $C(K,X)$ and $\operatorname{span}(\{f \cdot u : f \in C(K),\ 0 \le f \le 1 \text{ and } u \in X\})$ is dense in $C(K,X)$, one can find sets $\{f_n : n \in \mathbb{N}\} \subset C(K)$, where $0 \le f_n \le 1$ for each $n \in \mathbb{N}$, and $\{u_n : n \in \mathbb{N}\} \subset X$ such that $Y \hookrightarrow \overline{\operatorname{span}}(\{f_n u_m : n,m \in \mathbb{N}\})$.

Consider the following equivalence relation: for $x,y \in K$, $x \sim y$ if $f_n(x) = f_n(y)$ for every n. Let $K_0 = K/{\sim}$ be the quotient space, $q : K \to K_0$ be the quotient map and $X_0 = \overline{\operatorname{span}}(\{u_n : n \in \mathbb{N}\})$. It follows that K_0 is a compact scattered Hausdorff space and X_0 is a separable Banach space. Clearly, $C(K_0,X_0)$ can be isometrically embedded into $C(K,X)$ via the composition map $f \mapsto f \circ q$.

For each $n \in \mathbb{N}$, let $g_n : K_0 \to \mathbb{R}$ be given by $g_n(q(x)) = f_n(x)$, $x \in K$. It is clearly well defined and continuous and we can deduce that $\overline{\operatorname{span}}(\{f_n u_m : n,m \in \mathbb{N}\}) \sim \overline{\operatorname{span}}(\{g_n u_m : n,m \in \mathbb{N}\}) \subseteq C(K_0,X_0)$, and furthermore $Y \hookrightarrow C(K_0,X_0) \hookrightarrow C(K,X)$.

Next, we check that K_0 is metrizable and countable. Consider the function $\Phi : K_0 \to [0,1]^{\mathbb{N}}$ defined as $\Phi(q(x)) = (f_n(x))_{n\in\mathbb{N}}$, $x \in K$. Since the map $x \mapsto (f_n(x))_{n\in\mathbb{N}}$ is continuous, because the projections onto each coordinate are continuous, and $q : K \to K_0$ is a quotient map, Φ is continuous. Because K_0 is compact and $[0,1]^{\mathbb{N}}$ is a Hausdorff space, Φ is a closed map, and moreover, if $\Phi(q(x)) = \Phi(q(y))$ then $f_n(x) = f_n(y)$ for each $n \in \mathbb{N}$, so $x \sim y$ and $q(x) = q(y)$. We deduce that K_0 is homeomorphic to a subspace of $[0,1]^{\mathbb{N}}$, so it is metrizable; since K_0 is also scattered, it must be countable (see [4] and also [9, Theorem 8.6.10, p. 155]). ■

3. Proofs of the main results

Proof of Theorem 1.3. Let T be an isomorphic embedding of $C_0(K)$ into $C_0(L,X)$. If L is finite, then there is $n \in \mathbb{N}$ such that the space $X \oplus \overset{n}{\cdots} \oplus X$ (denoted simply by X^n) is isomorphic to $C_0(L,X)$. If K were infinite, then $C_0(K)$, and consequently X^n, would contain a copy of c_0. According to a result of C. Samuel [8, Theorem 1], X would have a copy of c_0, contrary to hypothesis. We conclude that if L is finite, then K must be finite as well.

Next, assume that both L and K are infinite. For each $y \in L$ and $n \in \mathbb{N}$ consider

$$K_y(1/n) = \bigcup_{\varphi\in B_{X^*}} \{x \in K : |T^*(\Delta_{y,\varphi})|(\{x\}) > 1/n\}$$

and

$$K_y = \bigcup_{n=1}^{\infty} K_y(1/n).$$

By Proposition 2.1, $K_y(1/n)$ is finite for every $n \in \mathbb{N}$ and so K_y is countable for every $y \in L$. According to Lemma 2.2, for every $x \in K$ there exists $y \in L$ such that $x \in K_y$. Thus,

$$K \subset \bigcup_{y\in L} K_y,$$

and it follows that

$$|K| \le \Big|\bigcup_{y\in L} K_y\Big| \le |L|\,\omega_0 = |L|. \ ■$$

Proof of Theorem 1.5. Suppose that $C_0(K) \hookrightarrow C_0(L,X)$ with L scattered and K non-scattered. If $\alpha K = K \,\dot\cup\, \{\infty\}$ is the Aleksandrov one-point compactification of K, then αK is also non-scattered and there must be a continuous surjection $\psi : \alpha K \to [0,1]$ (see [9, Theorem 8.5.4, p. 148]).

Assume that $\psi(\infty) = t \in [0,1]$. The composition map $f \mapsto f \circ \psi$ induces an isometric embedding of $C_0([0,1] \setminus \{t\})$ into $C_0(K)$.

If $\alpha L = L \,\dot\cup\, \{\infty\}$ is the Aleksandrov one-point compactification of L, then the space $C_0(L,X)$ can be isometrically identified as a subspace of

$C(\alpha L, X)$, namely the subspace of all continuous functions $f : \alpha L \to X$ such that $f(\infty) = 0$. We deduce

$$C_0([0,1] \setminus \{t\}) \hookrightarrow C_0(K) \hookrightarrow C_0(L, X) \hookrightarrow C(\alpha L, X).$$

Since $C_0([0,1] \setminus \{t\})$ is separable and αL is a scattered compact space, according to Lemma 2.3, there is a countable metrizable compact space L_0 and a separable Banach space $X_0 \subset X$ such that

$$C_0([0,1] \setminus \{t\}) \hookrightarrow C(L_0, X_0) \hookrightarrow C(\alpha L, X).$$

Since X_0 contains no copy of c_0, we can apply Theorem 1.3 to obtain

$$2^{\omega_0} = |[0,1] \setminus \{t\}| \leq |L_0| = \omega_0,$$

a contradiction, which establishes the theorem. ∎

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