COLLOQUIUM MATHEMATICUM

VOL. 145 2016 NO. 1

SEMILINEAR ELLIPTIC EQUATIONS WITH MEASURE DATA AND QUASI-REGULAR DIRICHLET FORMS

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Abstract. We are mainly concerned with equations of the form $-Lu = f(x,u) + \mu$, where L is an operator associated with a quasi-regular possibly nonsymmetric Dirichlet form, f satisfies the monotonicity condition and mild integrability conditions, and μ is a bounded smooth measure. We prove general results on existence, uniqueness and regularity of probabilistic solutions, which are expressed in terms of solutions to backward stochastic differential equations. Applications include equations with nonsymmetric divergence form operators, with gradient perturbations of some pseudodifferential operators and equations with Ornstein–Uhlenbeck type operators in Hilbert spaces. We also briefly discuss the existence and uniqueness of probabilistic solutions in the case where L corresponds to a lower bounded semi-Dirichlet form.

1. Introduction. Let E be a metrizable Lusin space, m be a positive σ-finite measure on $\mathcal{B}(E)$ and let $(\mathcal{E}, D(\mathcal{E}))$ be a quasi-regular possibly nonsymmetric Dirichlet form on $L^2(E;m)$. In the present paper we study existence, uniqueness and regularity of solutions of semilinear equations of the form

$$-Lu = f(x,u) + \mu. \tag{1.1}$$

Here $f : E \times \mathbb{R} \to \mathbb{R}$ is a measurable function, μ is a smooth signed measure on $\mathcal{B}(E)$ with respect to the capacity determined by $\mathcal{E}$, and L is the operator associated with the form $\mathcal{E}$, i.e.

$$(-Lu, v) = \mathcal{E}(u,v), \qquad u \in D(L),\ v \in D(\mathcal{E}), \tag{1.2}$$

where $D(L) = \{u \in D(\mathcal{E}) : v \mapsto \mathcal{E}(u,v)$ is continuous with respect to $(\cdot,\cdot)^{1/2}$ on $D(\mathcal{E})\}$. We assume that f satisfies the monotonicity condition and mild integrability conditions (even weaker than the integrability conditions considered earlier in [1]). As for μ we assume that it belongs to the class

$$\mathcal{R} = \{\mu : |\mu| \text{ is smooth and } \hat{G}\phi \cdot \mu \in \mathcal{M}_{0,b} \quad \text{for some } \phi \in L^1(E;m) \text{ such that } \phi > 0,\ m\text{-a.e.}\}, \tag{1.3}$$

2010 *Mathematics Subject Classification*: 35J61, 35R06; Secondary 60H30.
Key words and phrases: semilinear elliptic equation, measure data, Dirichlet form, backward stochastic differential equation.
Received 8 December 2014; revised 2 October 2015.
Published online 14 April 2016.

DOI: 10.4064/cm6466-10-2015

where $|\mu|$ denotes the variation of μ, $\mathcal{M}_{0,b}$ is the space of all finite smooth signed measures and $\hat{G}$ is the co-potential operator associated with $\mathcal{E}$. In the important case where $\mathcal{E}$ is transient the class $\mathcal{R}$ includes $\mathcal{M}_{0,b}$, but it may happen that $\mathcal{R}$ also includes some Radon measures of infinite total variation.

The paper continues research begun in our work [14] in which equations of the form (1.1) with L associated with a symmetric regular Dirichlet form are studied. The main motivation is to extend results of [14] to encompass equations with nonsymmetric operators and equations in infinite dimensions.

As in [14], by a *solution* of (1.1) we mean a quasi-continuous function $u : E \to \mathbb{R}$ satisfying for quasi-every $x \in E$ the nonlinear Feynman–Kac formula

$$u(x) = E_x\Big(\int_0^\zeta f(X_t, u(X_t))\,dt + \int_0^\zeta dA_t^\mu\Big), \tag{1.4}$$

where $\mathbb{X} = (X, P_x)$ is a Markov process with life-time ζ associated with the form $\mathcal{E}$, E_x is the expectation with respect to P_x, and A^μ is the additive functional of $\mathbb{X}$ corresponding to μ in the Revuz sense. We show that in the case where $\mathcal{E}$ is transient the solution may be defined in purely analytic terms resembling Stampacchia's definition of solutions by duality. Namely, a solution of (1.1) can be defined equivalently as a quasi-continuous function u such that $|\langle \nu, u\rangle| = |\int_E u\,d\nu| < \infty$ for every ν in the set $\hat{S}_{00}^{(0)}$ of smooth measures of 0-order energy integral such that $\|\hat{U}\nu\|_\infty < \infty$ and

$$\langle \nu, u\rangle = (f(\cdot, u), \hat{U}\nu) + \langle \mu, \widetilde{\hat{U}\nu}\rangle, \qquad \nu \in \hat{S}_{00}^{(0)},$$

where $(\cdot,\cdot)$ is the usual scalar product in $L^2(E;m)$, $\hat{U}\nu$ is the 0-order co-potential of ν and $\widetilde{\hat{U}\nu}$ denotes its quasi-continuous version. We work exclusively with the probabilistic definition (1.4) because in our opinion it is simpler and more natural than the definition by duality, and what is even more important, it allows us to use directly powerful methods of the theory of Dirichlet forms and Markov processes.

The paper is organized as follows. In Section 2 we provide basic definitions and prove some auxiliary but important results on smooth measures and their associated additive functionals.

In Section 3 we prove the existence and uniqueness of probabilistic solutions of (1.1), and then in Section 4 we study additional regularity properties of the solutions. Our main result says that under mild assumptions on f, we have $f(\cdot, u) \in L^1(E;m)$, and for every $k > 0$ the truncation $T_k u := (-k)\vee u \wedge k$ belongs to the extended Dirichlet space $\mathcal{F}_e$ of $\mathcal{E}$. Moreover,

$$\mathcal{E}(T_k u, T_k u) \le k(\|f(\cdot, 0)\|_{L^1(E;m)} + 2\|\mu\|_{\mathrm{TV}}), \tag{1.5}$$

where $\|\mu\|_{\mathrm{TV}}$ stands for the total variation norm of μ.

We are mainly concerned with equations (1.1) with L corresponding to a Dirichlet form. It appears, however, that a slight modification of the proof of the main existence result from Section 3 yields the existence of a probabilistic solution of (1.1) in the case where L corresponds to a lower-bounded semi-Dirichlet form. Although for such forms general regularity results similar to those of Section 4 seem to be impossible, we find it interesting that one can still define probabilistic solutions and investigate them by probabilistic methods. Our results for semi-Dirichlet forms are given in Section 5. For corresponding results for parabolic equations we defer the reader to the recent paper [13].

In Section 6 some applications of general results of Sections 2–5 are indicated. In the case of Dirichlet forms we decided to describe in some detail four quite different examples. In the first one we consider equation (1.1) with L being a nonsymmetric divergence form operator, that is, an operator associated with a local nonsymmetric regular form. In the second example L is a "divergent free" gradient perturbation of a symmetric nonlocal operator on $\mathbb{R}^d$ whose model example is the α-laplacian. In that case L corresponds to a nonsymmetric nonlocal regular form. Then we consider a symmetric nonlocal operator on some finely open subset $D \subset \mathbb{R}^d$, which is associated with a symmetric but in general nonregular form. In the last example we consider the Ornstein–Uhlenbeck operator in Hilbert space, that is, an operator associated with a local nonregular form. In each case we formulate a specific theorem on existence, uniqueness and regularity of solutions. To our knowledge all these results are new. We also briefly discuss the possibility of other applications of our general results of Sections 2–4. Finally, to illustrate the results of Section 5, we consider two examples of equations with operators corresponding to semi-Dirichlet forms. In the first example L is a diffusion operator with drift term, while in the second it is the fractional laplacian with variable exponent.

2. Preliminaries. In Sections 2–4 we assume that E is a metrizable *Lusin space*, i.e. a metrizable space which is the image of a Polish space under a continuous bijective mapping. We adjoin an extra point ∂ to E as an isolated point. We define the Borel σ-algebra on $E_\partial := E \cup \{\partial\}$ by $\mathcal{B}(E_\partial) = \mathcal{B}(E) \cup \{B \cup \{\partial\} : B \in \mathcal{B}(E)\}$. We make the convention that any function $f : E \to \bar{\mathbb{R}}$ is extended to E_∂ by setting $f(\partial) = 0$. Throughout the paper, m is a σ-finite positive measure on $\mathcal{B}(E)$. We extend it to $\mathcal{B}(E_\partial)$ by setting $m(\{\partial\}) = 0$.

2.1. Quasi-regular Dirichlet forms and Markov processes. We assume throughout that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form on $L^2(E; m)$ (see [18, 19] for the definitions).

We also assume that $(\mathcal{E},D(\mathcal{E}))$ is a *semi-Dirichlet form* on $L^2(E;m)$, i.e. $(\tilde{\mathcal{E}},D(\mathcal{E}))$, where $\tilde{\mathcal{E}}(u,v)=\frac{1}{2}(\mathcal{E}(u,v)+\mathcal{E}(v,u))$ for $u,v\in D(\mathcal{E})$, is a symmetric closed form, $(\mathcal{E},D(\mathcal{E}))$ satisfies the weak sector condition and has the following contraction property: for every $u\in D(\mathcal{E})$, $u^+\wedge 1\in D(\mathcal{E})$ and $\mathcal{E}(u+u^+\wedge 1,u-u^+\wedge 1))\geq 0$. If, in addition, $\mathcal{E}(u-u^+\wedge 1,u+u^+\wedge 1))\geq 0$, then $(\mathcal{E},D(\mathcal{E}))$ is called a *Dirichlet form*. Recall that $(\mathcal{E},D(\mathcal{E}))$ satisfies the *weak sector condition* if there is $K>0$ such that

$$|\mathcal{E}_1(u,v)|\leq K\mathcal{E}_1(u,u)^{1/2}\mathcal{E}_1(v,v)^{1/2},\qquad u,v\in D(\mathcal{E}),$$

where as usual, for $\alpha\geq 0$ we set $\mathcal{E}_\alpha(u,v)=\mathcal{E}(u,v)+\alpha(u,v)$ for $u,v\in D(\mathcal{E})$ ($(\cdot,\cdot)$ stands for the usual inner product in $L^2(E;m)$). Occasionally we will assume that $(\mathcal{E},D(\mathcal{E}))$ satisfies the *strong sector condition*, i.e. there is $K>0$ such that

$$|\mathcal{E}(u,v)|\leq K\mathcal{E}(u,u)^{1/2}\mathcal{E}(v,v)^{1/2},\qquad u,v\in D(\mathcal{E}). \tag{2.1}$$

We will denote by $(G_\alpha)_{\alpha>0}$ (resp. $(\hat{G}_\alpha)_{\alpha>0}$) the strongly continuous contraction resolvent (resp. coresolvent) on $L^2(E;m)$ determined by $(\mathcal{E},D(\mathcal{E}))$ (see [19, Theorem I.2.8]), and by $(T_t)_{t>0}$ (resp. $(\hat{T}_t)_{t>0}$) the strongly continuous contraction semigroup on $L^2(E;m)$ corresponding to $(G_\alpha)_{\alpha>0}$ (resp. $(\hat{G}_\alpha)_{\alpha>0}$). Note that T_t,G_α and $\hat{T}_t,\hat{G}_\alpha$ can be extended to a semigroup and resolvent on $L^1(E;m)$ (see [20, Section 1.1]).

We denote by $(L,D(L))$ the generator of $(G_\alpha)_{\alpha>0}$ (and $(T_t)_{t>0}$). By [19, Proposition I.2.16] it can be characterized as the unique operator on $L^2(E;m)$ such that (1.2) is satisfied.

For a closed subset $F\subset E$ we set $D(\mathcal{E})_F=\{u\in D(\mathcal{E}):u=0\ m\text{-a.e. on } E\setminus F\}$. Let us recall that an increasing sequence $\{F_k\}_{k\geq 1}$ of closed subsets of E is called an $\mathcal{E}$-*nest* if $\bigcup_{k=1}^\infty D(\mathcal{E})_{F_k}$ is $\tilde{\mathcal{E}}^{1/2}$-dense in $D(\mathcal{E})$. A subset $N\subset E$ is called $\mathcal{E}$-*exceptional* if $N\subset\bigcap_{k=1}^\infty F_k^c$ for some $\mathcal{E}$-nest $\{F_k\}_{k\in\mathbb{N}}$. We say that a property of points in E holds $\mathcal{E}$-*quasi-everywhere* ($\mathcal{E}$-q.e. for short) if it holds outside some $\mathcal{E}$-exceptional set. An $\mathcal{E}$-q.e. defined function u is called $\mathcal{E}$-*quasi-continuous* if there exists a nest $\{F_k\}_{k\in\mathbb{N}}$ such that $f\in C(\{F_k\})$, where

$$C(\{F_k\})=\Big\{f:A\to\mathbb{R}:\bigcup_{k=1}^\infty F_k\subset A\subset E,\ f_{|F_k}\text{ is continuous for }k\in\mathbb{N}\Big\}.$$

The notions of $\mathcal{E}$-nest and $\mathcal{E}$-exceptional set can be characterized by certain capacities relative to $(\mathcal{E},D(\mathcal{E}))$. To formulate this characterization, fix $\varphi\in L^2(E;m)$ such that $0<\varphi\leq 1$ m-a.e., and for open $U\subset E$ set

$$\mathrm{Cap}_\varphi(U)=\inf\{\mathcal{E}_1(u,u):u\in D(\mathcal{E}),\ u\geq\tilde{G}_1\varphi\ m\text{-a.e. on }U\},$$

where $\{\tilde{G}_\alpha\}$ is the resolvent associated with $(\tilde{\mathcal{E}},D(\mathcal{E}))$. For arbitrary $A\subset E$

set

$$\text{(2.2)} \qquad \mathrm{Cap}_\varphi(A) = \inf\{\mathrm{Cap}_\varphi(U) : A \subset U \subset E,\ U \text{ open}\}.$$

Then by [19, Theorem III.2.11] an increasing sequence $\{F_k\}_{k\geq 1}$ of closed subsets of E is an $\mathcal{E}$-nest iff $\lim_{k\to\infty} \mathrm{Cap}_\varphi(E \setminus F_k) = 0$, and secondly, $N \subset E$ is $\mathcal{E}$-exceptional iff $\mathrm{Cap}_\varphi(N) = 0$. Notice that from the above it follows in particular that the capacities Cap_φ defined for different $\varphi \in L^2(E;m)$ such that $0 < \varphi \leq 1$ m-a.e. are equivalent to each other.

A Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is called *transient* if there is an m-a.e. strictly positive and bounded $g \in L^1(E;m)$ such that

$$\text{(2.3)} \qquad \int_E |u| g\, dm \leq \mathcal{E}(u,u)^{1/2}, \qquad u \in D(\mathcal{E}).$$

Notice that transience of a Dirichlet form depends only on its symmetric part. It is known (see [11, Corollary 3.5.34]) that $(\mathcal{E}, D(\mathcal{E}))$ is transient iff the corresponding sub-Markovian semigroup $(T_t)_{t\geq 0}$ is transient, i.e. for all $u \in L^1(E;m)$ such that $u \geq 0$ m-a.e.,

$$\lim_{N\to\infty} \int_0^N T_t u\, dt < \infty \qquad m\text{-a.e.}$$

Let $(\mathcal{E}, D(\mathcal{E}))$ be a Dirichlet form. The *extended Dirichlet space* $\mathcal{F}_e$ associated with the symmetric Dirichlet form $(\tilde{\mathcal{E}}, D(\mathcal{E}))$ is the family of measurable functions $u : E \to \mathbb{R}$ such that $|u| < \infty$ m-a.e. and there exists an $\tilde{\mathcal{E}}$-Cauchy sequence $\{u_n\} \subset D(\mathcal{E})$ such that $u_n \to u$ m-a.e. The sequence $\{u_n\}$ is called an *approximating sequence* for $u \in \mathcal{F}_e$.

For a given Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ and $u \in \mathcal{F}_e$ we set $\mathcal{E}(u,u) = \lim_{n\to\infty} \mathcal{E}(u_n,u_n)$, where $\{u_n\}$ is an approximating sequence for u (see [7, Theorem 1.5.2]). If moreover $\mathcal{E}$ satisfies the strong sector condition then we may extend $\mathcal{E}$ to $\mathcal{F}_e$ by setting $\mathcal{E}(u,v) = \lim_{n\to\infty} \mathcal{E}(u_n,v_n)$ with approximating sequences $\{u_n\}$ and $\{v_n\}$ for $u \in \mathcal{F}_e$ and $v \in \mathcal{F}_e$, respectively (it is easily seen that $\mathcal{E}(u,v)$ is independent of the choice of the approximating sequences). Observe that this extension satisfies the strong sector condition, i.e. (2.1) holds true for all $u, v \in \mathcal{F}_e$.

If $(\mathcal{E}, D(\mathcal{E}))$ is transient then by [7, Lemma 1.5.5], $(\mathcal{F}_e, \tilde{\mathcal{E}})$ is a Hilbert space. Also note that if $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form (see [18, 19] for the definition) then by [19, Proposition IV.3.3] each $u \in D(\mathcal{E})$ admits a quasi-continuous m-version denoted by $\tilde{u}$, and that $\tilde{u}$ is $\mathcal{E}$-q.e. unique for every $u \in D(\mathcal{E})$. If moreover $(\mathcal{E}, D(\mathcal{E}))$ is transient then the last statement holds true for $D(\mathcal{E})$ replaced by $\mathcal{F}_e$ (see [15, Remark 2.2]).

In the remainder of this section we assume that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form on $L^2(E;m)$.

By [19, Theorem IV.3.5] there exists an m-tight special standard Markov process $\mathbb{X}=(\Omega,(\mathcal{F}_t)_{t\ge0},(X_t)_{t\ge0},\zeta,(P_x)_{x\in E\cup\{\partial\}})$ with state space E, lifetime ζ and cemetery state ∂ (see, e.g., [18] or [19, Section IV.1] for precise definitions) which is properly associated with $(\mathcal{E},D(\mathcal{E}))$. Let $(p_t)_{t\ge0}$ be the transition semigroup of $\mathbb{X}$ defined by

$$p_tf(x)=E_xf(X_t),\qquad x\in E,\ t\ge0,\ f\in\mathcal{B}^+(E).$$

The statement that $\mathbb{X}$ is *properly associated with* $(\mathcal{E},D(\mathcal{E}))$ means that p_tf is a quasi-continuous m-version of T_tf for every $t>0$ and $f\in\mathcal{B}_b\cap L^2(E;m)$ (and hence for every $t>0$ and $f\in L^2(E;m)$ by [19, Exercise IV.2.9]). Equivalently, by [19, Proposition IV.2.8], the proper association means that $R_\alpha f$ is an $\mathcal{E}$-quasi-continuous m-version of $G_\alpha f$ for every $\alpha>0$ and f in $\mathcal{B}_b\cap L^2(E;m)$, where $(R_\alpha)_{\alpha>0}$ is the *resolvent* of $\mathbb{X}$, i.e.

$$R_\alpha f(x)=E_x\int_0^\infty e^{-\alpha t}f(X_t)\,dt,\qquad x\in E,\ \alpha>0,\ f\in\mathcal{B}^+(E).$$

By [19, Theorem IV.6.4] the process $\mathbb{X}$ is uniquely determined by $(\mathcal{E},D(\mathcal{E}))$ in the sense that if $\mathbb{X}'$ is another process with state space E properly associated with $(\mathcal{E},D(\mathcal{E}))$ then $\mathbb{X}$ and $\mathbb{X}'$ are *m-equivalent*, i.e. there is $S\in\mathcal{B}(E)$ such that $m(E\setminus S)=0$, S is both $\mathbb{X}$-invariant and $\mathbb{X}'$-invariant, and $p_tf(x)=p'_tf(x)$ for all $x\in S$, $f\in\mathcal{B}_b(E)$ and $t>0$, where $(p'_t)_{t>0}$ is the transition semigroup of $\mathbb{X}'$.

2.2. Smooth measures. Recall that a positive measure μ on $\mathcal{B}(E)$ is said to be *$\mathcal{E}$-smooth* (we write $\mu\in S$) if $\mu(B)=0$ for all $\mathcal{E}$-exceptional sets $B\in\mathcal{B}(E)$ and there exists an $\mathcal{E}$-nest $\{F_k\}_{k\in\mathbb{N}}$ of compact sets such that $\mu(F_k)<\infty$ for $k\in\mathbb{N}$. A measure $\mu\in S$ is said to be *of finite energy integral* (written $\mu\in S_0$) if there is $c>0$ such that

$$\int_E|\tilde v(x)|\,\mu(dx)\le c\mathcal{E}_1(v,v)^{1/2},\qquad v\in D(\mathcal{E}).\tag{2.4}$$

If additionally $(\mathcal{E},D(\mathcal{E}))$ is transient then $\mu\in S$ is said to be of *finite 0-order* energy integral (written $\mu\in S_0^{(0)}$) if there is $c>0$ such that

$$\int_E|\tilde v(x)|\,\mu(dx)\le c\mathcal{E}(v,v)^{1/2},\qquad v\in\mathcal{F}_e.$$

If $(\mathcal{E},D(\mathcal{E}))$ is regular and E is a locally compact separable metric space then the notion of smooth measures defined above coincides with that in [7]. Moreover, if μ is a positive Radon measure on E such that (2.4) is satisfied for all $v\in C_0(E)\cap D(\mathcal{E})$ then μ charges no $\mathcal{E}$-exceptional set (see [17, Remark A.2]) and hence $\mu\in S_0$.

By [18, Proposition 2.18(ii)] (or [19, Proposition III.3.6]) the reference measure m is $\mathcal{E}$-smooth. Therefore if $f \in L^1(E;m)$ then $\mu = f\cdot m$ is bounded and smooth. A general result on the structure of bounded smooth measures is found in [15].

Let $\mu \in S_0$ and $\alpha > 0$. Then from the Lax–Milgram theorem (see, e.g., [10, Theorem 2.7.41]) it follows that there exist unique $U_\alpha\mu, \hat{U}_\alpha\mu \in D(\mathcal{E})$ such that

$$\mathcal{E}_\alpha(U_\alpha\mu, v) = \int_E \tilde{v}(x)\,\mu(dx) = \mathcal{E}_\alpha(v, \hat{U}_\alpha\mu), \qquad v \in D(\mathcal{E}).$$

Similarly, if $(\mathcal{E}, D(\mathcal{E}))$ satisfies the strong sector condition and $\mu \in S_0^{(0)}$ then from the Lax–Milgram theorem applied to the Hilbert space $(\mathcal{F}_e, \tilde{\mathcal{E}})$, the form $\mathcal{E}$ and the operator $J : \mathcal{F}_e \to \mathbb{R}$ defined by $J(v) = \int_E \tilde{v}(x)\,\mu(dx)$ it follows that there exist unique $U\mu, \hat{U}\mu \in \mathcal{F}_e$ such that

$$\mathcal{E}(U\mu, v) = \int_E \tilde{v}(x)\,\mu(dx) = \mathcal{E}(v, \hat{U}\mu), \quad v \in \mathcal{F}_e\,.$$

Let $\mathcal{M}_{0,b}$ denote the subset of S consisting of all measures μ such that $\|\mu\|_{\mathrm{TV}} < \infty$, where $\|\mu\|_{\mathrm{TV}}$ denotes the total variation of μ, and let $\mathcal{M}_{0,b}^+$ denote the subset of $\mathcal{M}_{0,b}$ consisting of all positive measures.

The lemma below follows from the 0-order version of [7, Theorem 2.2.4] by the so-called transfer method.

Lemma 2.1. *Assume that $(\mathcal{E}, D(\mathcal{E}))$ is transient. If $\mu \in S$ then there exists a nest $\{F_n\}$ such that $\mathbf{1}_{F_n}\cdot\mu \in S_0^{(0)}$ for each $n \in \mathbb{N}$.*

Proof. See [15, Lemma 2.1]. ∎

Lemma 2.2. *Assume that $(\mathcal{E}, D(\mathcal{E}))$ is transient and satisfies the strong sector condition. If $\mu \in S_0^{(0)}$ then $\{U_\alpha\mu\}$ is weakly $\mathcal{E}$-convergent to $U\mu$ as $\alpha \downarrow 0$.*

Proof. Let $v \in \mathcal{F}_e$ and let $\{v_k\} \subset D(\mathcal{E})$ be an approximating sequence for v. We have

$$\mathcal{E}(U\mu - U_\alpha\mu, v_k) = \alpha(U_\alpha\mu, v_k), \qquad \mathcal{E}(G_0U_\alpha\mu, v_k) = (U_\alpha\mu, v_k).$$

Hence $\mathcal{E}(U\mu - U_\alpha\mu, v_k) = \mathcal{E}(\alpha G_0U_\alpha\mu, v_k)$. Letting $k \to \infty$ we deduce $\mathcal{E}(U\mu - U_\alpha\mu, v) = \mathcal{E}(\alpha G_0U_\alpha\mu, v)$. Consequently, $U\mu - U_\alpha\mu = \alpha G_0U_\alpha\mu$. In the same manner we can see that $\hat{U}\mu - \hat{U}_\alpha\mu = \alpha\hat{G}_0\hat{U}_\alpha\mu$. Hence,

$$\begin{aligned}\mathcal{E}(U\mu - U_\alpha\mu, \hat{U}\mu - \hat{U}_\alpha\mu) &= \alpha^2\mathcal{E}(G_0U_\alpha\mu, \hat{G}_0\hat{U}_\alpha\mu)\\ &= \alpha^2(G_0U_\alpha\mu, \hat{U}_\alpha\mu) \geq 0.\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathcal{E}(U\mu-U_\alpha\mu,\hat U\mu-\hat U_\alpha\mu)&=\mathcal{E}(U\mu,\hat U\mu-\hat U_\alpha\mu)-\mathcal{E}(U_\alpha\mu,\hat U\mu)\\&\quad+\mathcal{E}_\alpha(U_\alpha\mu,\hat U_\alpha\mu)-\alpha(U_\alpha\mu,\hat U_\alpha\mu)\\&=\mathcal{E}(U\mu,\hat U\mu-\hat U_\alpha\mu)-\alpha(U_\alpha\mu,\hat U_\alpha\mu)\\&\le\mathcal{E}(U\mu,\hat U\mu-\hat U_\alpha\mu)=\langle\mu,\widetilde{U\mu}\rangle-\langle\mu,\widetilde{\hat U_\alpha\mu}\rangle.\end{aligned}$$

Since $\langle\mu,\widetilde{\hat U_\alpha\mu}\rangle=\mathcal{E}_\alpha(U_\alpha\mu,\hat U_\alpha\mu)=\langle\mu,\widetilde{U_\alpha\mu}\rangle$, it follows from the above that

$$\mathcal{E}(U_\alpha\mu,U_\alpha\mu)+\alpha(U_\alpha\mu,U_\alpha\mu)=\mathcal{E}_\alpha(U_\alpha\mu,U_\alpha\mu)=\langle\mu,\widetilde{U_\alpha\mu}\rangle\le\langle\mu,\widetilde{U\mu}\rangle$$

for $\alpha>0$. Hence $\{U_\alpha\mu\}_{\alpha>0}$ is $\tilde{\mathcal{E}}$-bounded and for each $k\in\mathbb{N}$, $\alpha(U_\alpha\mu,v_k)\to0$ as $\alpha\downarrow0$. Suppose that $\{U_\alpha\mu\}$ converges $\tilde{\mathcal{E}}$-weakly to some $f\in\mathcal{F}_e$ as $\alpha\downarrow0$. Since

$$\mathcal{E}(U_\alpha\mu,v_k)=\langle\mu,\tilde v_k\rangle-\alpha(U_\alpha\mu,v_k),$$

letting $\alpha\downarrow0$ shows that $\mathcal{E}(f,v_k)=\langle\mu,\tilde v\rangle=\mathcal{E}(U\mu,v_k)$. Letting $k\to\infty$ we get $\mathcal{E}(f,v)=\mathcal{E}(U\mu,v)$ for $v\in\mathcal{F}_e$. Thus $f=U\mu$. ∎

2.3. Smooth measures and additive functionals. Let $\mathbb{X}$ be the Markov process properly associated with $(\mathcal{E},D(\mathcal{E}))$. In what follows for a Borel measure ν on E we set $P_\nu(\cdot)=\int_E P_x(\cdot)\,\nu(dx)$, and we denote by E_ν the expectation with respect to P_ν.

By [19, Theorem VI.2.4] there is a one-to-one correspondence between $\mathcal{E}$-smooth measures μ on $\mathcal{B}(E)$ and positive continuous additive functionals (PCAFs) A of $\mathbb{X}$. It is given by the relation

$$\lim_{t\downarrow0}\frac1t E_m\int_0^t f(X_s)\,dA_s=\int_E f\,d\mu,\qquad f\in\mathcal{B}^+(E).\tag{2.5}$$

The additive functional corresponding to μ in the sense of (2.5) will be denoted by A^μ. In the important case where $\mu=f\cdot m$ for some $f\in L^1(E;m)$ the additive functional A^μ is given by

$$A^\mu_t=\int_0^t f(X_s)\,ds,\qquad t\ge0.$$

The following lemma generalizes [14, Lemma 4.3].

LEMMA 2.3. *If A is a PCAF of $\mathbb{X}$ such that $E_xA_\zeta<\infty$ for m-a.e. $x\in E$ then $u:E\to\bar{\mathbb{R}}$ defined as*

$$u(x)=E_xA_\zeta,\qquad x\in E,$$

is $\mathcal{E}$-quasi-continuous. In particular, u is $\mathcal{E}$-q.e. finite.

Proof. Let $(\mathcal{E}^\#,D(\mathcal{E}^\#))$ denote the regular extension of $(\mathcal{E},D(\mathcal{E}))$ specified by [19, Theorem VI.1.2]. By [19, Theorem VI.1.6], $\mathbb{X}$ can be trivially extended to a Hunt process $\mathbb{X}^\#$ defined on $\Omega\cup(E^\#\setminus E)$ with state space

$E^{\#}$ properly associated with the form $(\mathcal{E}^{\#}, D(\mathcal{E}^{\#}))$. Let us extend A to a PCAF of $\mathbb{X}^{\#}$ by setting

$$(2.6) \quad A_t^{\#}(\omega) = A_t(\omega), \quad t \ge 0, \omega \in \Omega, \qquad A_t^{\#}(\omega) = 0, \quad t \ge 0, \omega \in E^{\#} \setminus E.$$

By the assumption and since $m^{\#}(E^{\#} \setminus E) = 0$, we have $E_x^{\#} A_{\zeta^{\#}}^{\#} < \infty$ $m^{\#}$-a.e. Therefore, by [14, Lemma 4.3], the function $u^{\#}(x) = E_x^{\#} A_{\zeta^{\#}}^{\#}$ is $\mathcal{E}^{\#}$-quasi-continuous on $E^{\#}$. By [19, Corollary VI.1.4], $u_{|E}^{\#}$ is $\mathcal{E}$-quasi-continuous on E, which proves the first part of the lemma since $u_{|E}^{\#}(x) = E_x A_\zeta$, $x \in E$. The second part is immediate from the definition of quasi-continuity. ∎

Lemma 2.4. *Assume that $(\mathcal{E}, D(\mathcal{E}))$ is transient and satisfies the strong sector condition. If $\mu \in S_0^{(0)}$ then u defined by*

$$u(x) = E_x A_\zeta^\mu, \qquad x \in E,$$

is a quasi-continuous version of $U\mu$.

Proof. By [17, Proposition A.7], for every $\alpha > 0$ the function $R_\alpha \mu$ defined by $R_\alpha\mu(x) = E_x \int_0^\infty e^{-\alpha t}\, dA_t^\mu$, $x \in E$, is a quasi-continuous version of $U_\alpha \mu$. Therefore, by Lemma 2.2 and the Banach–Saks theorem, there exist sequences $\alpha_n \downarrow 0$ and $\{n_k\}$ such that the Cesàro mean sequence $\{w_n = (1/n)\sum_{k=1}^n u_{n_k}\}$, where $u_n = R_{\alpha_n}\mu$, is $\tilde{\mathcal{E}}$-convergent to $U\mu$. On the other hand, by the monotone convergence theorem, $u_n(x) \to u(x)$ for $x \in E$, and hence $w_n(x) \to u(x)$ for $x \in E$. Consequently, $\{w_n\}$ is an approximating sequence for u. Therefore $u \in \mathcal{F}_e$ and

$$\begin{aligned} &\tilde{\mathcal{E}}(u - U\mu, u - U\mu)^{1/2} \\ &\quad \le \lim_{n\to\infty} \big(\tilde{\mathcal{E}}(u - w_n, u - w_n)^{1/2} + \tilde{\mathcal{E}}(U\mu - w_n, U\mu - w_n)^{1/2}\big) = 0. \end{aligned}$$

Since $(\tilde{\mathcal{E}}, \mathcal{F}_e)$ is a Hilbert space, it follows that u is an m-version of $U\mu$. To show that u is quasi-continuous, first note that by [19, Proposition III.3.3] there is an $\mathcal{E}$-nest $\{F_k\}$ such that $\{u_n\} \subset C(\{F_k\})$. Since $\mathcal{E}$ is quasi-regular, there exists an $\mathcal{E}$-nest $\{E_k\}$ consisting of compact sets. Write $F_k' = F_k \cap E_k$. Then $\{F_k'\}$ is an $\mathcal{E}$-nest consisting of compact sets and $\{u_n\} \subset C(\{F_k'\})$. Since $u_{n|F_k'} \nearrow u_{|F_k'}$ as $n \to \infty$ for each $k \in \mathbb{N}$, Dini's theorem shows that u is in $C(\{F_k'\})$, which is our claim. ∎

Let $S_{00}^{(0)}$ (resp. $\hat{S}_{00}^{(0)}$) denote the subset of $S_0^{(0)}$ consisting of all measures ν such that $\nu(E) < \infty$ and $\|U\nu\|_\infty < \infty$ (resp. $\|\hat{U}\nu\|_\infty < \infty$).

Lemma 2.5. *Assume that $(\mathcal{E}, D(\mathcal{E}))$ is transient and satisfies the strong sector condition. If $\mu \in S$ and $\nu \in \hat{S}_{00}^{(0)}$ then for any nonnegative Borel function f,*

$$(2.7) \qquad E_\nu \int_0^\zeta f(X_t)\, dA_t^\mu = \langle f \cdot \mu, \widetilde{\hat{U}\nu} \rangle.$$

Proof. By Lemma 2.1 there exists a nest $\{F_n\}$ with $\mathbf{1}_{F_n}|f|\cdot|\mu|\in S_0^{(0)}$. By [17, Theorem A.8], for $\alpha>0$ the function $x\mapsto E_x\int_0^\zeta e^{-\alpha t}\mathbf{1}_{F_n}f(X_t)\,dA_t^\mu$ is a quasi-continuous version of $U_\alpha(\mathbf{1}_{F_n}f\cdot\mu)$. Hence,

$$E_\nu\int_0^\zeta e^{-\alpha t}\mathbf{1}_{F_n}f(X_t)\,dA_t^\mu=\langle \widetilde{U_\alpha(\mathbf{1}_{F_n}f}\cdot\mu),\nu\rangle=\langle\mathbf{1}_{F_n}f\cdot\mu,\widetilde{\hat{U}_\alpha\nu}\rangle. \tag{2.8}$$

Letting $\alpha\downarrow 0$ and applying the monotone convergence theorem to the left-hand side of (2.8) and Lemma 2.2 to the right-hand side of (2.8), we obtain

$$E_\nu\int_0^\zeta \mathbf{1}_{F_n}f(X_t)\,dA_t^\mu=\langle\mathbf{1}_{F_n}f\cdot\mu,\widetilde{\hat{U}\nu}\rangle. \tag{2.9}$$

Letting $n\to\infty$ in (2.9) yields (2.9) with F_n replaced by $\bigcup_{n=1}^\infty F_n$, which implies (2.7) because $(\bigcup_{n=1}^\infty F_n)^c$ is an exceptional set. ∎

LEMMA 2.6. *Assume that $(\mathcal{E},D(\mathcal{E}))$ is transient, $\mu_1\in S$, $\mu_2\in\mathcal{M}_{0,b}^+$. If*

$$E_x\int_0^\zeta dA_t^{\mu_1}\le E_x\int_0^\zeta dA_t^{\mu_2}$$

for m-a.e. $x\in E$ then $\|\mu_1\|_{\mathrm{TV}}\le\|\mu_2\|_{\mathrm{TV}}$.

Proof. Let $(\mathcal{E}^\#,D(\mathcal{E}^\#))$, $\mu^\#$ be defined as in the proof of Lemma 2.3, and let $(A^\mu)^\#$ be defined by (2.6) with A replaced by A^μ. It is an elementary check that $(A^\mu)^\#=A^{\mu^\#}$. By the assumptions and since $m^\#(E^\#\setminus E)=0$,

$$E_x^\#\int_0^{\zeta^\#} dA_t^{\mu_1^\#}\le E_x^\#\int_0^{\zeta^\#} dA_t^{\mu_2^\#}$$

for m-a.e. $x\in E^\#$. Clearly $\mu_2^\#\in\mathcal{M}_{0,b}(E^\#)$. Therefore $\|\mu_1^\#\|_{\mathrm{TV}}\le\|\mu_2^\#\|_{\mathrm{TV}}$ by [14, Lemma 5.4], and hence $\|\mu_1\|_{\mathrm{TV}}\le\|\mu_2\|_{\mathrm{TV}}$. ∎

The following lemma is probably known, but we do not have a reference.

LEMMA 2.7. *Assume that $(\mathcal{E},D(\mathcal{E}))$ is transient and satisfies the strong sector condition. Let $B\in\mathcal{B}(E)$. If $\nu(B)=0$ for every $\nu\in S_{00}^{(0)}$ then B is $\mathcal{E}$-exceptional.*

Proof. Let $(\mathcal{E}^\#,D(\mathcal{E}^\#))$ be the regular extension of $(\tilde{\mathcal{E}},D(\mathcal{E}))$ specified in [19, Theorem VI.1.2]. Let $\nu^\#$ be a smooth measure on $\mathcal{B}(\mathcal{E}^\#)$ and let $\nu=\nu^\#{}_{|\mathcal{B}(E)}$. If $A\in\mathcal{B}(E)$ is $\mathcal{E}$-exceptional then by [19, Corollary VI.1.4], A is $\mathcal{E}^\#$-exceptional, and hence $\nu(A)=\nu^\#(A)=0$. Moreover, if $\{F_k\}$ is a nest in $E^\#$ such that $\nu^\#(F_k)<\infty$ for $k\in\mathbb{N}$ and $\{E_k\}$ is a nest in E as in [19, Theorem VI.1.2], then $\{F_k\cap E_k\}$ is an $\mathcal{E}$-nest of compact sets in E such that $\nu(F_k\cap E_k)<\infty$, $k\in\mathbb{N}$. Thus ν is a smooth measure on $\mathcal{B}(E)$. If moreover $\nu^{\#} \in S_{00}^{(0)}(E^{\#})$ then $\nu(E) < \infty$ and, for $\eta \in D(\mathcal{E})$,

$$\langle \nu, \tilde{\eta} \rangle = \langle \nu^{\#}, \tilde{\eta} \rangle \le c\mathcal{E}^{\#}(\eta, \eta)^{1/2} = c\mathcal{E}(\eta, \eta)^{1/2}.$$

From this in the same manner as in the proof of Lemma 2.1 one can deduce that $\langle \nu, \tilde{\eta} \rangle \le c\mathcal{E}(\eta, \eta)^{1/2}$ for $\eta \in \mathcal{F}_e$, i.e. that $\nu \in S_0^{(0)}$. From Lemma 2.4 and the fact that $A^{\nu^{\#}} = (A^{\nu})^{\#}$ it follows now that $U\nu^{\#}{}_{|E}$ is an m-version of $U\nu$. Therefore $\|U\nu\|_{\infty} < \infty$, which proves that $\nu \in S_{00}^{(0)}$. Hence $\nu(B) = 0$, and consequently $\nu^{\#}(B) = \nu(B) = 0$ for every $\nu^{\#}$ in $S_{00}^{(0)}(E^{\#})$. Therefore from the 0-order version of [7, Theorem 2.2.3] (see remark following [7, Corollary 2.2.2]) we conclude that $\mathrm{Cap}_{1,1}^{\#}(B) = 0$, where $\mathrm{Cap}_{1,1}^{\#}$ denotes the capacity relative to $(\mathcal{E}^{\#}, D(\mathcal{E}^{\#}))$ defined in [19, Definition III.2.4] (see also [19, Exercise III.2.10]). Hence $\mathrm{Cap}_{\varphi}^{\#}(B) = 0$ by [19, Proposition VI.1.5], and consequently $\mathrm{Cap}_{\varphi}(B) = 0$ by [19, Corollary VI.1.4] ($\mathrm{Cap}_{\varphi}$is defined by (2.2)). By a remark following (2.2) this implies that B is $\mathcal{E}$-exceptional. ∎

3. Probabilistic solutions. In this section we assume that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form on $L^2(E; m)$. We will need the following assumptions on f from the right-hand side of (1.1):

(A1) $f : E \times \mathbb{R} \to \mathbb{R}$ is measurable and $y \mapsto f(x, y)$ is continuous for every $x \in E$,
(A2) $(f(x, y_1) - f(x, y_2))(y_1 - y_2) \le 0$ for all $y_1, y_2 \in \mathbb{R}$ and $x \in E$,
(A3) $f(\cdot, y) \in L^1(E; m)$ for every $y \in \mathbb{R}$,
(A4) $\mu \in \mathcal{M}_{0,b}$,

and

(A3*) for every $y \in \mathbb{R}$ the function $f(\cdot, y)$ is quasi-L^1 with respect to the form $(\mathcal{E}, D(\mathcal{E}))$, i.e. $t \mapsto f(X_t, y)$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ P_x-a.s. for q.e. $x \in E$,
(A4*) $E_x \int_0^{\zeta} |f(X_t, 0)|\, dt < \infty$ and $E_x \int_0^{\zeta} d|A^{\mu}|_t < \infty$ for q.e. $x \in E$.

Note that in our previous paper [14] devoted to equations of the form (1.1) we followed [1] in assuming that f satisfies (A1), (A2), (A4) and the following condition: for every $r > 0$, $F_r \in L^1(E; m)$, where $F_r(x) = \sup_{|y| \le r} |f(x, y)|$. Obviously (A3) is weaker than the last condition. Likewise, (A3*) is weaker than the corresponding condition (A3′) in [14] saying that for every $r > 0$ the function $t \mapsto F_r(X_t, y)$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ P_x-a.s. for q.e. $x \in E$. Observe, however, that (A3) together with (A1), (A2) imply that $F_r \in L^1(E; m)$. Likewise, (A3*) together with (A1), (A2) imply condition (A3′) from [14].

Define the co-potential operator as

$$\hat{G}\phi = \lim_{n\to\infty} \hat{G}_{1/n}\phi, \qquad \phi \in L^1(E;m),\ \phi \ge 0,$$

and for $\mu \in S$ set

$$R\mu(x) = E_x \int_0^{\zeta} dA_t^{\mu}, \qquad x \in E.$$

LEMMA 3.1. *If $(\mathcal{E}, D(\mathcal{E}))$ is transient then for any $\mu \in S$ and $\phi \in L^1(E;m)$ such that $\phi \ge 0$ we have*

$$(R\mu, \phi) = \langle \mu, \widetilde{\hat{G}\phi} \rangle. \tag{3.1}$$

Proof. By Lemma 2.1 there is a nest $\{F_n\}$ such that $\mathbf{1}_{F_n} \cdot \mu \in S_0^{(0)}$ for each $n \in \mathbb{N}$. Let

$$R_\alpha(\mathbf{1}_{F_n} \cdot \mu)(x) = E_x \int_0^{\zeta} e^{-\alpha t} \mathbf{1}_{F_n}(X_t)\, dA_t^{\mu}, \qquad \alpha > 0,\ x \in E.$$

Since $R_\alpha(\mathbf{1}_{F_n} \cdot \mu)$ is an m-version of $U_\alpha(\mathbf{1}_{F_n} \cdot \mu)$, for any nonnegative ϕ in $L^1(E;m) \cap L^2(E;m)$ we have

$$\langle \mathbf{1}_{F_n} \cdot \mu, \widetilde{\hat{G}_\alpha \phi} \rangle = \mathcal{E}_\alpha(U_\alpha(\mathbf{1}_{F_n} \cdot \mu), \hat{U}_\alpha \phi) = \mathcal{E}_\alpha(R_\alpha(\mathbf{1}_{F_n} \cdot \mu), \hat{U}_\alpha \phi).$$

Hence,

$$\langle \mathbf{1}_{F_n} \cdot \mu, \widetilde{\hat{G}_\alpha \phi} \rangle = (R_\alpha(\mathbf{1}_{F_n} \cdot \mu), \phi). \tag{3.2}$$

In fact, approximating $\phi \in L^1(E;m)$ by a sequence $\{\phi_k\} \subset L^1(E;m) \cap L^2(E;m)$ such that $0 \le \phi_k \nearrow \phi$ yields (3.2) for any $\phi \in L^1(E;m)$ such that $\phi \ge 0$. Finally, letting $\alpha \downarrow 0$ and then $n \to \infty$ in (3.2) gives (3.1). ∎

Let $\mathcal{R}$ be defined by (1.3). If μ is smooth and $R|\mu| < \infty$ m-a.e. then from (3.1) and the fact that m is σ-finite it follows that $\mu \in \mathcal{R}$. Furthermore, if $\mu \in \mathcal{R}$ then by (3.1), $R|\mu| < \infty$ m-a.e. Thus $\mathcal{R}$ can be equivalently defined as

$$\mathcal{R} = \{\mu : \mu \text{ is smooth},\ R|\mu| < \infty,\ m\text{-a.e.}\}.$$

It follows in particular that (A4*) is satisfied iff $f(\cdot, 0) \cdot m \in \mathcal{R}$ and $\mu \in \mathcal{R}$.

PROPOSITION 3.2. *If $(\mathcal{E}, D(\mathcal{E}))$ is transient then $\mathcal{M}_{0,b} \subset \mathcal{R}$.*

Proof. Apply [20, Corollary 1.3.6] to the dual form $(\hat{\mathcal{E}}, D(\mathcal{E}))$. ∎

In general the inclusion in Proposition 3.2 is strict. To see this let us consider the classical form

$$\mathbb{D}(u, v) = \frac{1}{2} \int_D \langle \nabla u, \nabla v \rangle_{\mathbb{R}^d}\, dx, \qquad u, v \in H_0^1(D), \tag{3.3}$$

on $L^2(D; dx)$, where D is a bounded open subset of $\mathbb{R}^d$. If $d \ge 3$ and D has smooth boundary then $R1$ is a continuous strictly positive function such that $R1(x) \approx \delta(x)$ for $x \in D$, where $\delta(x) = \mathrm{dist}(x, \partial D)$ (for the last property see [16, Proposition 4.9]). Since $R1$ is an m-version of $G1 = \hat{G}1$, it follows that $L^1(D; \delta(x)dx) \in \mathcal{R}$, so $\mathcal{R}$ contains positive Radon measures of infinite total variation. Elliptic and parabolic equations with right-hand side in $L^q(D; \delta(x)dx)$ ($q \ge 1$) are studied for instance in [5].

REMARK 3.3. Assume that $(\mathcal{E}, D(\mathcal{E}))$ is transient. Then by Lemma 2.3 and Proposition 3.2, (A3) implies (A3*) and (A4) implies (A4*).

3.1. BSDEs. Let $(\Omega, (\mathcal{F}_t)_{t\ge0}, P)$ be a filtered probability space. We will need the following classes of processes defined on Ω.

$\mathcal{D}$ is the space of all $(\mathcal{F}_t)$-progressively measurable càdlàg processes, and $\mathcal{D}^q$, $q > 0$, is the subspace of $\mathcal{D}$ consisting of all processes Y such that $E\sup_{t\ge0}|Y_t|^q < \infty$.

$\mathcal{M}$ (resp. $\mathcal{M}_{\mathrm{loc}}$) is the space of all càdlàg $((\mathcal{F}_t), P)$-martingales (resp. local martingales) M such that $M_0 = 0$, and $\mathcal{M}^2$ is the subspace of $\mathcal{M}$ consisting of martingales such that $E[M]_\infty < \infty$.

We say that a càdlàg $(\mathcal{F}_t)$-adapted process Y is *of Doob's class* (D) if the collection $\{Y_\tau : \tau \in \mathcal{T}\}$, where $\mathcal{T}$ is the set of finite-valued $(\mathcal{F}_t)$-stopping times, is uniformly integrable. For a process Y of class (D) we set $\|Y\|_1 = \sup\{E|Y_\tau| : \tau \in \mathcal{T}\}$.

In the present subsection ξ is an $\mathcal{F}_T$-measurable random variable, ζ is an $(\mathcal{F}_t)$-stoping time, V is a continuous $(\mathcal{F}_t)$-adapted finite variation process such that $V_0 = 0$ and $f : [0,\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that $f(\cdot, y)$ is a $(\mathcal{F}_t)$-progressively measurable process for every $y \in \mathbb{R}$ (for brevity in notation we omit the dependence of f on ω).

DEFINITION. We say that a pair (Y, M) of processes is a solution of the *backward stochastic differential equation on* $[0, T]$ *with terminal condition* ξ *and coefficient* $f + dV$ (BSDE$(\xi, f + dV)$ for short) if

(a) $Y \in \mathcal{D}$, Y is of class (D) and $M \in \mathcal{M}_{\mathrm{loc}}$,
(b) the mapping $[0, T] \ni t \mapsto f(t, Y_t)$ belongs to $L^1(0, T)$ P-a.s. and

$$Y_t = \xi + \int_t^T f(r, Y_r)\,dr + \int_t^T dV_r - \int_t^T dM_r, \quad t \in [0, T],\ P\text{-a.s.}$$

DEFINITION. We say that a pair (Y, M) is a solution of the *backward stochastic differential equation with terminal condition* 0 *at terminal time* ζ *and coefficient* $f + dV$ (BSDE$^\zeta(f + dV)$ for short) if

(a) $Y \in \mathcal{D}$, $Y_{t\wedge\zeta} \to 0$ P-a.s. as $t \to \infty$, Y is of class (D) and $M \in \mathcal{M}_{\mathrm{loc}}$,
(b) for every $T > 0$, $[0, T] \ni t \mapsto f(t, Y_t)$ belongs to $L^1(0, T)$ P-a.s. and

$$Y_t = Y_{T\wedge\zeta} + \int_{t\wedge\zeta}^{T\wedge\zeta} f(r, Y_r)\,dr + \int_{t\wedge\zeta}^{T\wedge\zeta} dV_r - \int_{t\wedge\zeta}^{T\wedge\zeta} dM_r, \qquad t \in [0, T],\ P\text{-a.s.}$$

Let us consider the following hypotheses:

(H1) For every $t \in [0,T]$ the function $\mathbb{R} \ni y \mapsto f(t,y)$ is continuous P-a.s.

(H2) For every $t \in [0,T]$ the function $\mathbb{R} \ni y \mapsto f(t,y)$ is P-a.s. nondecreasing.

(H3) For every $y \in \mathbb{R}$ the function $[0,T] \ni t \mapsto f(t,y)$ belongs to $L^1(0,T)$ P-a.s.

REMARK 3.4. The following Theorems 3.6 and 3.7 were stated in [14] (see Theorems 2.7 and 3.4 there). Unfortunately, there are some gaps in the proofs of these results in [14]. Namely, in the proof of [14, Theorem 2.7] we applied [14, Lemma 2.6], which is true, but its proof is correct under the additional assumption that the coefficient f is bounded from below by some linear function of y (otherwise the function f_n appearing in the proof is not well defined). Secondly, in the proof of [14, Theorem 3.4] we applied [14, Lemma 2.5], which is correct only for $p \geq 2$ or under the additional assumption that the solution (Y,M) is continuous (the reason is that in the proof of [14, Lemma 2.5] we used the Burkholder–Davis–Gundy inequality with exponent $p/2$). Here we give the proofs of [14, Theorems 2.7 and 3.4] in full generality.

In what follows we denote by T_c, $c \geq 0$, the *truncation operator*, i.e.

$$T_c(x) = (-c) \vee x \wedge c, \qquad x \in E. \tag{3.4}$$

LEMMA 3.5. *Assume that* (H1)–(H3) *are satisfied and there exists* $c > 0$ *such that*

$$T \cdot \sup_{0 \leq t \leq T} |f(t,0)| + |V|_T + |\xi| \leq c.$$

Then there exists a unique solution $(Y,M) \in \mathcal{D}^2 \otimes \mathcal{M}^2$ *of* $\mathrm{BSDE}(\xi, f + dV)$.

Proof. Let $f_c(t,y) = f(t, T_c(y))$. Then $|\inf_{y\in\mathbb{R}} f_c(t,y)| < \infty$ and the proof of [14, Lemma 2.6] shows (see Remark 3.4) that there exists a unique solution $(Y,M) \in \mathcal{D}^2 \otimes \mathcal{M}^2$ of $\mathrm{BSDE}(\xi, f_c + dV)$. But by the Tanaka–Meyer formula and the assumptions,

$$\begin{aligned} |Y_t| &\leq E\Big(|\xi| + \int_t^T \operatorname{sgn}(Y_r) f_c(r, Y_r)\,dr + \int_t^T \operatorname{sgn}(Y_r)\,dV_r \,\Big|\, \mathcal{F}_t\Big) \\ &\leq E\Big(|\xi| + \int_0^T |f(r,0)|\,dr + \int_0^T d|V|_r \,\Big|\, \mathcal{F}_t\Big) \leq c, \end{aligned}$$

so in fact (Y,M) is a solution of $\mathrm{BSDE}(\xi, f + dV)$. ∎

THEOREM 3.6. *Assume that* (H1)–(H3) *are satisfied and*

$$E\Big(|\xi| + \int_0^T |f(t,0)|\,dt + \int_0^T d|V|_t\Big) < \infty.$$

Then there exists a solution (Y,M) *of* $\mathrm{BSDE}(\xi, f+dV)$ *such that* $Y\in\mathcal{D}^q$ *for every* $q\in(0,1)$ *and* M *is a uniformly integrable martingale.*

Proof. Set $\xi^n = T_n(\xi)$ and

$$f_n(t,y)=f(t,y)-f(t,0)+T_n(f(t,0)),\qquad V^n_t=\int_0^t \mathbf{1}_{\{|V|_r\le n\}}\,dV_r.$$

By Lemma 3.5, for $n\ge1$ there is a solution (Y^n,M^n) of $\mathrm{BSDE}(\xi^n,f^n+dV^n)$. As in the proof of [14, Theorem 2.7] we show that there exists a process Y of class (D) such that $Y\in\mathcal{D}^q$ for $q\in(0,1)$ and

$$E\sup_{0\le t\le T}|Y^n_t-Y_t|^q\to0 \tag{3.5}$$

for every $q\in(0,1)$. By the Tanaka–Meyer formula and (H2),

$$|Y^n_t|\le E\Big(|\xi|+\int_0^T|f(r,0)|\,dr+\int_0^T d|V|_r\ \Big|\ \mathcal{F}_t\Big),\qquad t\in[0,T].$$

Let R denote a càdlàg process such that for every $t\in[0,T]$, R_t is equal to the right-hand side of the above inequality. Then

$$|Y^n_t|\le R_t,\qquad t\in[0,T],\ n\ge1.$$

For $k,N\in\mathbb{N}$ set

$$\tau_{k,N}=\inf\Big\{t\ge0: R_t\ge k \text{ or } \int_0^t(|f(r,-k)|+|f(r,k)|)\,dr\ge N\Big\}\wedge T. \tag{3.6}$$

By the definition of a solution of $\mathrm{BSDE}(\xi^n,f_n+dV^n)$,

$$Y^n_{t\wedge\tau_{k,N}}=E\Big(Y^n_{\tau_{k,N}}+\int_{t\wedge\tau_{k,N}}^{\tau_{k,N}}f_n(r,Y^n_r)\,dr+\int_{t\wedge\tau_{k,N}}^{\tau_{k,N}}dV^n_r\ \Big|\ \mathcal{F}_t\Big). \tag{3.7}$$

From the definition of $\tau_{k,N}$ it follows that

$$\Big|\int_{t\wedge\tau_{k,N}}^{\tau_{k,N}}f_n(r,Y^n_r)\,dr\Big|\le\int_0^{\tau_{k,N}}|f(r,Y^n_r)|\,dr\le N.$$

From this, (H1) and (3.5) we conclude that

$$E\int_0^{\tau_{k,N}}|f_n(t,Y^n_t)-f(t,Y_t)|\,dt\to0$$

as $n\to\infty$. Therefore letting $n\to\infty$ in (3.7) and using (3.5) and Doob's maximal inequality (for details see the argument following (3.15)) we obtain

$$Y_{t\wedge\tau_{k,N}}=E\Big(Y_{\tau_{k,N}}+\int_{t\wedge\tau_{k,N}}^{\tau_{k,N}}f(r,Y_r)\,dr+\int_{t\wedge\tau_{k,N}}^{\tau_{k,N}}dV_r\ \Big|\ \mathcal{F}_t\Big). \tag{3.8}$$

By [14, Lemma 2.3],

$$E\int_0^T |f_n(t,Y_t^n)|\,dt \le E\Big(|\xi| + \int_0^T |f(t,0)|\,dt + \int_0^T d|V|_t\Big), \qquad n\ge 1,$$

so applying Fatou's lemma and (3.5) gives

$$E\int_0^T |f(t,Y_t)|\,dt < \infty. \tag{3.9}$$

By (H3), $\tau_{k,N}\to\tau_k$ P-a.s. as $N\to\infty$, where

$$\tau_k = \inf\{t\ge 0 : R_t \ge k\}\wedge T. \tag{3.10}$$

Therefore letting $N\to\infty$ in (3.8) and using (3.9) and the fact that Y is of class (D) we get

$$Y_{t\wedge\tau_k} = E\Big(Y_{\tau_k} + \int_{t\wedge\tau_k}^{\tau_k} f(r,Y_r)\,dr + \int_{t\wedge\tau_k}^{\tau_k} dV_r \,\Big|\, \mathcal{F}_t\Big). \tag{3.11}$$

Since R is a càdlàg process, $\tau_k\to T$ P-a.s. as $k\to\infty$. Therefore letting $k\to\infty$ in (3.11) and using once again (3.9) and the fact that Y is of class (D) gives

$$Y_t = E\Big(\xi + \int_t^T f(r,Y_r)\,dr + \int_t^T dV_r \,\Big|\, \mathcal{F}_t\Big).$$

It follows that the pair (Y,M), where M is a càdlàg process such that

$$M_t = E\Big(\xi + \int_0^T f(r,Y_r)\,dr + \int_0^T dV_r \,\Big|\, \mathcal{F}_t\Big) - Y_0, \qquad t\in[0,T],$$

is a solution of BSDE$(\xi, f+dV)$. ∎

THEOREM 3.7. *Assume that* (H1)–(H3) *are satisfied for every* $T>0$, *and that*

$$E\Big(\int_0^\zeta |f(t,0)|\,dt + \int_0^\zeta d|V|_t\Big) < \infty.$$

Then there exists a unique solution (Y,M) *of* BSDE$^\zeta(f+dV)$. *Moreover,* $Y\in\mathcal{D}^q$ *for* $q\in(0,1)$, M *is a uniformly integrable* $(\mathcal{F}_t)$*-martingale and*

$$E\int_0^\zeta |f(t,Y_t)|\,dt \le E\Big(\int_0^\zeta |f(t,0)|\,dt + \int_0^\zeta d|V|_t\Big). \tag{3.12}$$

Proof. The uniqueness part is a direct consequence of [14, Corollary 3.2]. To prove the existence we slightly modify the proof of [14, Theorem 3.4]. By Theorem 3.6, for each $n\in\mathbb{N}$ there exists a unique solution (Y^n,M^n) of

BSDE$(0,\mathbf{1}_{[0,\zeta]}f+dV_{\cdot\wedge\zeta})$ on $[0,n]$ such that $Y^n\in\mathcal{D}^q$ for $q\in(0,1)$ and M^n is a uniformly integrable $(\mathcal{F}_t)$-martingale. By the definition of a solution,

$$Y^n_t=\int_t^n \mathbf{1}_{[0,\zeta]}(r)f(r,Y^n_r)\,dr+\int_t^n dV_{r\wedge\zeta}-\int_t^n dM^n_r,\qquad t\in[0,n]. \tag{3.13}$$

Set $(Y^n_t,M^n_t)=(0,M^n_n)$ for $t\ge n$. Then as in the proof of [14, Theorem 3.4] we show (see the proof of [14, (3.11)] and the inequality following it) that for every $m>n$ and $q\in(0,1)$,

$$E\sup_{t\ge0}|Y^m_t-Y^n_t|^q\le\frac{1}{1-q}\Big(E\int_{n\wedge\zeta}^{\zeta}|f(r,0)|\,dr+\int_{n\wedge\zeta}^{\zeta}d|V|_r\Big)^q,$$

$$\|Y^m-Y^n\|_1\le\frac{1}{1-q}\Big(E\int_{n\wedge\zeta}^{\zeta}|f(r,0)|\,dr+\int_{n\wedge\zeta}^{\zeta}d|V|_r\Big)^q.$$

Therefore there exists Y such that $Y\in\mathcal{D}^q$ for $q\in(0,1)$, Y is of class (D) and $Y^n\to Y$ in the norm $\|\cdot\|_1$ and in $\mathcal{D}^q$ for $q\in(0,1)$. Since $Y^n_\zeta=0$ P-a.s. for $n\in\mathbb{N}$, from the latter convergence it follows in particular that $Y_{t\wedge\zeta}\to0$ as $t\to\infty$. In much the same way as in the proof of [14, (3.5)] we show that

$$|Y^n_t|\le E\Big(\int_{t\wedge\zeta}^{n\wedge\zeta}\operatorname{sgn}(Y^n_{r-})(f(r,Y^n_r)\,dr+dV_r)\;\Big|\;\mathcal{F}_t\Big).$$

From this and (H2) we get

$$\begin{aligned}|Y^n_t|&\le E\Big(\int_{t\wedge\zeta}^{\zeta}(|f(r,0)|\,dr+d|V|_r)\;\Big|\;\mathcal{F}_t\Big)\\&\le E\Big(\int_0^{\zeta}(|f(r,0)|\,dr+d|V|_r)\;\Big|\;\mathcal{F}_t\Big)=:R_t,\qquad t\ge0.\end{aligned} \tag{3.14}$$

Let $\tau_{k,N}$ be defined by (3.6) but with R_t from (3.14). By (3.13), for $T<n$ we have

(3.15)

$$Y^n_{t\wedge\tau_{k,N}}=E\Big(Y^n_{\zeta\wedge\tau_{k,N}}+\int_{t\wedge\zeta\wedge\tau_{k,N}}^{\zeta\wedge\tau_{k,N}}(f(r,Y^n_r)\,dr+dV_r)\;\Big|\;\mathcal{F}_t\Big),\quad t\in[0,T],\ P\text{-a.s.}$$

By Doob's maximal inequality, for every $\varepsilon>0$,

$$\lim_{n\to\infty}P\Big(\sup_{t\le T}|E(Y^n_{\zeta\wedge\tau_{k,N}}-Y_{\zeta\wedge\tau_{k,N}}|\mathcal{F}_t)|>\varepsilon\Big)\le\varepsilon^{-1}\lim_{n\to\infty}E|Y^n_{\zeta\wedge\tau_{k,N}}-Y_{\zeta\wedge\tau_{k,N}}|=0.$$

Since $\sup_{t\geq 0}|Y^n_t - Y_t| \to 0$ in probability P, it follows from the definition of $\tau_{k,N}$ that

$$\lim_{n\to\infty} E \int_0^{\zeta\wedge\tau_{k,N}} |f(r,Y^n_r) - f(r,Y_r)|\,dr = \lim_{n\to\infty} E \int_0^{\zeta\wedge\tau_{k,N}-} |f(r,Y^n_r) - f(r,Y_r)|\,dr = 0.$$

Applying Doob's maximal inequality we conclude from the above that

$$\lim_{n\to\infty} P\Big(\sup_{t\leq T} E\Big(\int_t^{\zeta\wedge\tau_{k,N}} |f(r,Y^n_r) - f(r,Y_r)|\,dr \,\Big|\, \mathcal{F}_t\Big) > \varepsilon\Big) = 0$$

for $\varepsilon > 0$. Therefore letting $n\to\infty$ in (3.15) we can assert that P-a.s. we have

$$Y_{t\wedge\tau_{k,N}} = E\Big(Y_{\zeta\wedge\tau_{k,N}} + \int_{t\wedge\zeta\wedge\tau_{k,N}}^{\zeta\wedge\tau_{k,N}} (f(r,Y_r)\,dr + dV_r) \,\Big|\, \mathcal{F}_t\Big), \quad t\in[0,T]. \tag{3.16}$$

By (H3), $\tau_{k,N}\to\tau_k$ P-a.s. as $N\to\infty$, where τ_k is defined by (3.10) but with R_t defined by (3.14). Hence $Y_{\zeta\wedge\tau_{k,N}} \to Y_{\zeta\wedge\tau_k}$ P-a.s. as $N\to\infty$, and consequently $E|Y_{\zeta\wedge\tau_{k,N}} - Y_{\zeta\wedge\tau_k}| \to 0$ since Y is of class (D). Therefore letting $N\to\infty$ in (3.16) we obtain

$$Y_{t\wedge\tau_k} = E\Big(Y_{\zeta\wedge\tau_k} + \int_{t\wedge\zeta\wedge\tau_k}^{\zeta\wedge\tau_k} (f(r,Y_r)\,dr + dV_r) \,\Big|\, \mathcal{F}_t\Big), \quad t\in[0,T]. \tag{3.17}$$

Since we may assume that R is a càdlàg process, $\tau_k\to T$, P-a.s. as $k\to\infty$. Hence $Y_{\zeta\wedge\tau_k}\to Y_{T\wedge\zeta}$ P-a.s. as $k\to\infty$, and consequently $E|Y_{\zeta\wedge\tau_k} - Y_{T\wedge\zeta}| \to 0$ since Y is of class (D). Also, $E|Y_{T\wedge\zeta}|\to 0$ as $T\to\infty$ since we know that $Y_{T\wedge\zeta}\to 0$ P-a.s. By [14, Lemma 2.3], for every $n\geq 1$,

$$E\int_0^{n\wedge\zeta} |f(r,Y^n_r)|\,dr \leq E\Big(|Y^n_{n\wedge\zeta}| + \int_0^{n\wedge\zeta} |f(r,0)|\,dr + \int_0^{n\wedge\zeta} d|V|_r\Big).$$

Letting $n\to\infty$ in the above inequality and applying Fatou's lemma and the first inequality in (3.14) we get (3.12). Therefore letting $k\to\infty$ in (3.17) and then letting $T\to\infty$ and using Doob's maximal inequality we obtain

$$Y_t = E\Big(\int_{t\wedge\zeta}^{\zeta} (f(r,Y_r)\,dr + dV_r) \,\Big|\, \mathcal{F}_t\Big), \quad t\geq 0,\ P\text{-a.s.}$$

From this, one can easily deduce that the pair (Y,M), where

$$M_t = E\Big(\int_0^{\zeta} f(r,Y_r)\,dr + \int_0^{\zeta} dV_r \,\Big|\, \mathcal{F}_t\Big) - Y_0, \quad t\geq 0,$$

is a solution of $\mathrm{BSDE}^{\zeta}(f+dV)$. Finally, since the martingale M is closed, it is uniformly integrable. ■

3.2. Existence and uniqueness of probabilistic solutions. Let $(L, D(L))$ be the operator defined by (1.2).

DEFINITION. Let $\mu \in \mathcal{R}$. We say that an $\mathcal{E}$-quasi-continuous function $u : E \to \mathbb{R}$ is a *probabilistic solution* of the equation

$$-Lu = f_u + \mu, \tag{3.18}$$

where $f_u(x) = f(x, u(x))$ for $x \in E$, if $E_x \int_0^{\zeta} |f_u(X_t)|\, dt < \infty$ and

$$u(x) = E_x\Big(\int_0^{\zeta} f_u(X_t)\, dt + \int_0^{\zeta} dA_t^{\mu}\Big) \tag{3.19}$$

for q.e. $x \in E$.

In what follows we say that a function $u : E \to \mathbb{R}$ is *of class* (FD) if the process $t \mapsto u(X_t)$ is of class (D) under the measure P_x for q.e. $x \in E$. Similarly, we write $u \in \mathcal{FD}^q$ if the process $t \mapsto u(X_t)$ belongs to $\mathcal{D}^q$ under P_x for q.e. $x \in E$. The notation BSDE_x means that the backward stochastic differential equation under consideration is defined on the filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, P_x)$.

THEOREM 3.8. *Assume that* (A1), (A2), (A3*), (A4*) *are satisfied. Then there exists a unique probabilistic solution u of* (3.18). *Actually, u is of class* (FD) *and $u \in \mathcal{FD}^q$ for $q \in (0,1)$. Moreover, for q.e. $x \in E$ there exists a unique solution (Y^x, M^x) of* $\mathrm{BSDE}_x^{\zeta}(f + dA^{\mu})$. *In fact,*

$$u(X_t) = Y_t^x, \quad t \geq 0,\ P_x\text{-}a.s.$$

Proof. From Lemma 2.3 it follows that under (A4*) the assumptions of Theorem 3.7 are satisfied under the measure P_x with coefficient $f(\cdot, X_\cdot) + dA^{\mu}$ and terminal time ζ for q.e. $x \in E$. To prove the theorem it now suffices to use Theorem 3.7 and repeat step by step arguments from the proof of [14, Theorem 4.7]. ■

Let us note that from Theorem 3.7 it follows that under the assumptions of Theorem 3.8,

$$E_x \int_0^{\zeta} |f_u(X_t)|\, dt \leq E_x\Big(\int_0^{\zeta} |f(X_t, 0)|\, dt + \int_0^{\zeta} d|A^{\mu}|_t\Big) \tag{3.20}$$

for m-a.e. $x \in E$, where u is a probabilistic solution of (3.18).

3.3. Probabilistic solutions vs. solutions in the sense of duality. Assume that $(\mathcal{E}, D(\mathcal{E}))$ is transient and satisfies the strong sector condition. Let $\mathcal{A}$ denote the space of all $\mathcal{E}$-quasi-continuous functions $u : E \to \mathbb{R}$ such that $u\in L^1(E;\nu)$ for every $\nu\in\hat{S}^{(0)}_{00}$. Following [14] we adopt the following definition.

Definition. Let $\mu\in\mathcal{M}_{0,b}$. We say $u:E\to\mathbb{R}$ is a *solution* of (3.18) *in the sense of duality* if $u\in\mathcal{A}$, $f_u\in L^1(E;m)$ and

$$\langle\nu,u\rangle=(f_u,\hat{U}\nu)+\langle\mu,\widetilde{\hat{U}\nu}\rangle,\qquad \nu\in\hat{S}^{(0)}_{00}. \tag{3.21}$$

Note that by the very definition of $S^{(0)}_0$, if $\nu\in S^{(0)}_0$ and $u\in\mathcal{F}_e$ then $\tilde{u}\in L^1(E;\nu)$. As a consequence, if $u\in\mathcal{F}_e$ then $\tilde{u}\in\mathcal{A}$.

Proposition 3.9. *Assume that $(\mathcal{E},D(\mathcal{E}))$ is transient, satisfies the strong sector condition and that $\mu\in\mathcal{M}_{0,b}$. If u is $\mathcal{E}$-quasi-continuous and f_u is in $L^1(E;m)$, then u is a probabilistic solution of* (3.18) *iff it is a solution of* (3.18) *in the sense of duality.*

Proof. Let u be a solution of (3.18) in the sense of duality. Denote by $w(x)$ the right-hand side of (3.19) if it is finite, and set $w(x)=0$ otherwise. By Proposition 3.2, w is finite m-a.e., and hence, by Lemma 2.3, w is quasi-continuous. By Lemma 2.5, $w\in\mathcal{A}$ and $\langle\nu,w\rangle$ is equal to the right-hand side of (3.21). Thus $\langle\nu,u\rangle=\langle\nu,w\rangle$ for $\nu\in\hat{S}^{(0)}_{00}$. Lemma 2.7 applied to the form $\hat{\mathcal{E}}$ now shows that $u=w$ $\mathcal{E}$-q.e. since u,v are $\mathcal{E}$-quasi-continuous. Conversely, assume that u is a probabilistic solution of (3.18). Then, again by Lemma 2.5, $u\in\mathcal{A}$ and u satisfies (3.21). ∎

Proposition 3.10. *Assume that $(\mathcal{E},D(\mathcal{E}))$ is transient and* (A4) *is satisfied.*

(i) *If u is a probabilistic solution of* (3.18) *then $f_u\in L^1(E;m)$ and*

$$\|f_u\|_{L^1(E;m)}\le\|f(\cdot,0)\|_{L^1(E;m)}+\|\mu\|_{\mathrm{TV}}. \tag{3.22}$$

(ii) *If moreover $(\mathcal{E},D(\mathcal{E}))$ satisfies the strong sector condition then u is a probabilistic solution of* (3.18) *iff it is a solution of* (3.18) *in the sense of duality.*

Proof. Assertion (i) follows from (3.20) and Lemma 2.6, whereas (ii) follows from (i) and Proposition 3.9. ∎

4. Regularity of probabilistic solutions. Below, T_k denotes the truncation operator defined by (3.4).

Lemma 4.1. *Assume that $(\mathcal{E},D(\mathcal{E}))$ is a Dirichlet form. Then for every $k>0$,*

$$\mathcal{E}(T_ku,T_ku)\le\mathcal{E}(u,T_ku) \tag{4.1}$$

for all $u\in D(\mathcal{E})$. If moreover $(\mathcal{E},D(\mathcal{E}))$ satisfies the strong sector condition then (4.1) *holds for all $u\in\mathcal{F}_e$.*

Proof. Let $u \in D(\mathcal{E})$. Since G_α is Markov,

$$\alpha(T_k(u) - \alpha G_\alpha T_k(u), u - T_k(u)) \geq 0$$

for all $k, \alpha > 0$. Therefore the first assertion of the lemma follows from [19, Theorem I.2.13]. Now assume that $\mathcal{E}$ satisfies (2.1) and $u \in \mathcal{F}_e$. Let $\{u_n\} \subset D(\mathcal{E})$ be an approximating sequence for u. By [7, Theorem 1.5.3], $T_k u_n \in \mathcal{F}_e$ and $\tilde{\mathcal{E}}(T_k u_n, T_k u_n) \leq \tilde{\mathcal{E}}(u_n, u_n)$ for each $n \in \mathbb{N}$. Since $\{u_n\}$ is $\tilde{\mathcal{E}}$-convergent, $\sup_{n\geq 1} \tilde{\mathcal{E}}(T_k u_n, T_k u_n) < \infty$. Since $(\mathcal{F}_e, \tilde{\mathcal{E}})$ is a Hilbert space, applying the Banach–Saks theorem we can find a subsequence $\{n_l\}$ such that the Cesàro mean sequence $\{w_N = (1/N)\sum_{l=1}^N T_k(u_{n_l})\}$ is $\tilde{\mathcal{E}}$-convergent to some $w \in \mathcal{F}_e$. Since $\tilde{\mathcal{E}}$ is transient, there is an m-a.e. strictly positive and bounded $g \in L^1(E; m)$ such that

$$\int_E |w_N - v| g \, dm \leq \mathcal{E}(w_N - w, w_N - w)^{1/2} \to 0.$$

On the other hand, since $u_n \to u$ m-a.e., applying the Lebesgue dominated convergence theorems shows that $\int_E |w_N - T_k u| g \, dm \to 0$. Consequently, $w = T_k u$ and $\{T_k u_n\}$ converges $\tilde{\mathcal{E}}$-weakly to $T_k u$. From this and the first part of the proof it follows that

$$\mathcal{E}(T_k u, T_k u) \leq \liminf_{n\to\infty} \mathcal{E}(T_k u_n, T_k u_n) \leq \liminf_{n\to\infty} \mathcal{E}(u_n, T_k u_n). \tag{4.2}$$

Moreover, using (2.1) and the facts that $\{u_n\}$ is $\tilde{\mathcal{E}}$-convergent to u and $\{T_k u_n\}$ is $\tilde{\mathcal{E}}$-weakly convergent to $T_k u$ we conclude that the last limit in (4.2) equals $\mathcal{E}(u, T_k u)$, which completes the proof of the second assertion of the lemma. ■

THEOREM 4.2. *Assume that* $(\mathcal{E}, D(\mathcal{E}))$ *is a quasi-regular transient Dirichlet form and* $\mu \in \mathcal{M}_{0,b}$. *If* u *is a solution of* (3.18) *and* $f_u \in L^1(E; m)$ *then* $T_k u \in \mathcal{F}_e$ *for every* $k > 0$. *Moreover, for every* $k > 0$,

$$\mathcal{E}(T_k u, T_k u) \leq k(\|f_u\|_{L^1(E;m)} + \|\mu\|_{\mathrm{TV}}). \tag{4.3}$$

Proof. By Lemma 2.1 there exists a nest $\{F_n\}$ with $\mathbf{1}_{F_n}|f_u| \cdot m + \mathbf{1}_{F_n} \cdot |\mu| \in S_0^{(0)}$. For $\alpha > 0$ set

$$u_n^\alpha(x) = E_x \int_0^\zeta e^{-\alpha t} \mathbf{1}_{F_n} f_u(X_t) \, dt + E_x \int_0^\zeta e^{-\alpha t} \mathbf{1}_{F_n}(X_t) \, dA_t^\mu, \qquad x \in E,$$

and $\mu_n = \mathbf{1}_{F_n} f_u \cdot m + \mathbf{1}_{F_n} \cdot \mu$. By [17, Theorem A.8],

$$u_n^\alpha(x) = \widetilde{U_\alpha \mu_n}(x)$$

for q.e. $x \in E$. Hence $u_n^\alpha \in D(\mathcal{E})$ and $T_k u_n^\alpha \in D(\mathcal{E})$ since every normal contraction operates on $(\mathcal{E}, D(\mathcal{E}))$. Therefore,

$$\mathcal{E}_\alpha(u_n^\alpha, T_k u_n^\alpha) = \mathcal{E}_\alpha(U_\alpha \mu_n, T_k u_n^\alpha) = \int_E \widetilde{T_k u_n^\alpha} \, d\mu_n \leq k(\|f_u\|_{L^1(E;m)} + \|\mu\|_{\mathrm{TV}}).$$

By Lemma 4.1 applied to the form $\mathcal{E}_\alpha$,

$$\mathcal{E}_\alpha(T_k u_n^\alpha, T_k u_n^\alpha) \le \mathcal{E}_\alpha(u_n^\alpha, T_k u_n^\alpha). \tag{4.4}$$

Consequently,

$$\mathcal{E}(T_k u_n^\alpha, T_k u_n^\alpha) \le k(\|f_u\|_{L^1(E;m)} + \|\mu\|_{\mathrm{TV}}).$$

By the Banach–Saks theorem we can choose a sequence $\{\alpha_l\}$ such that $\alpha_l \downarrow 0$ as $l \to \infty$, and the sequence $\{w_N = (1/N)\sum_{l=1}^N T_k u_n^{\alpha_l}\}$ is $\tilde{\mathcal{E}}$-convergent. Moreover, from Lemma 2.4 one can deduce that $u_n^\alpha(x) \to u_n(x)$ as $\alpha \downarrow 0$ for q.e. $x \in E$. Hence $T_k u_n^\alpha \to T_k u_n$ m-a.e., and consequently $w_N \to T_k u_n$ m-a.e. Thus $\{w_N\}$ is an approximating sequence for $T_k u_n$. By (4.4), $\mathcal{E}(w_N, w_N) \le k(\|f_u\|_{L^1(E;m)} + \|\mu\|_{\mathrm{TV}})$ for every $N \in \mathbb{N}$. Hence,

$$\mathcal{E}(T_k u_n, T_k u_n) = \lim_{N\to\infty} \mathcal{E}(w_N, w_N) \le k(\|f_u\|_{L^1(E;m)} + \|\mu\|_{\mathrm{TV}}).$$

Since $u_n \to u$ q.e. we now apply the above arguments again, with $T_k u_n^\alpha$ replaced by $T_k u_n$, to obtain (4.3). ∎

Corollary 4.3. *If* $(\mathcal{E}, D(\mathcal{E}))$ *is a quasi-regular transient Dirichlet form and* f, μ *satisfy* (A1), (A2), (A3*), (A4) *then there exists a unique solution* u *of* (3.18). *Moreover,* u *is of class* (FD), $u \in \mathcal{FD}^q$ *for* $q \in (0,1)$ *and* (3.22), (4.3) *are satisfied.*

Proof. Follows immediately from Theorem 3.8, Proposition 3.10 and Theorem 4.2. ∎

Remark 4.4. Assume that $(\mathcal{E}, D(\mathcal{E}))$ is transient, satisfies the strong sector condition, and that $\mu \in S_0^{(0)}$. If u is a solution of (3.18) such that $f_u \cdot m \in S_0^{(0)}$ then u is a *weak solution* of (3.18), i.e. for every $v \in \mathcal{F}_e$,

$$\mathcal{E}(u, v) = (f_u, v) + \langle \mu, \tilde{v}\rangle. \tag{4.5}$$

Indeed, by Lemma 2.1, if $\mu, f_u \cdot m \in S_0^{(0)}$ then u satisfying (3.19) is a quasi-continuous version of $U(f_u \cdot m + \mu)$, which implies (4.5). Note that the condition $f_u \cdot m \in S_0^{(0)}$ is satisfied if $f_u \in L^2(E;m)$ and there is $c > 0$ such that $(u, u) \le c\mathcal{E}(u, u)$ for $u \in D(\mathcal{E})$. Indeed, the last inequality implies that $S_0 = S_0^{(0)}$, and from the fact that $f_u \in L^2(E;m)$ it follows that $f_u \cdot m \in S_0$.

Remark 4.5. (i) Example 5.7 in [14] shows that in general under (A1)–(A4) the solution u of (3.18) may not be locally integrable.

(ii) Assume that (A1), (A2), (A3*) and (A4*) hold, and let u be a probabilistic solution of (3.18) as in Theorem 3.8. Then from (3.19) and (3.20) it follows that $|u(x)| \le R(|f(\cdot,0)| \cdot m + 2|\mu|)$. Therefore the condition

$$(|f(\cdot,0)|, \hat{G}1) + \langle |\mu|, \widetilde{\hat{G}1}\rangle < \infty \tag{4.6}$$

is sufficient to guarantee integrability of u. One interesting situation in which (4.6) holds true is given at the end of Section 6.

5. The case of semi-Dirichlet forms. In the present section, E is a locally compact separable metric space, m is an everywhere dense positive Radon measure on $\mathcal{B}(E)$, and $(\mathcal{E}, D(\mathcal{E}))$ is a transient lower-bounded semi-Dirichlet form on $L^2(E;m)$ in the sense of [20, Section 1.1]. We also assume that $(\mathcal{E}, D(\mathcal{E}))$ is regular (see [20, Section 1.2]). By $\mathbb{X}$ we denote a Hunt process associated with $(\mathcal{E}, D(\mathcal{E}))$ (see [20, Theorem 3.3.4]). We fix $\gamma > \alpha_0$, where α_0 is the constant from conditions $(\mathcal{E}.1)$, $(\mathcal{E}.2)$ in [20, Section 1.1], and we set $\mathrm{Cap} = \mathrm{Cap}^{(\gamma)}$, where $\mathrm{Cap}^{(\gamma)}$ is the capacity defined in [20, Section 2.1]. For $B \subset E$ we define $\sigma_B = \inf\{t > 0 : X_t \in B\}$, and for $\psi \in L^1(E;m)$ we set $P_{\psi\cdot m}(\cdot) = \int_E P_x(\cdot)\psi(x)\, m(dx)$.

LEMMA 5.1. *Let $\psi \in L^1(E;m)$ be strictly positive and let $\{A_n\}$ be a decreasing sequence of subsets of E with $\mathrm{Cap}(A_1) < \infty$. If $P_{\psi\cdot m}(\sigma_{A_n} < \infty) \to 0$ as $n \to \infty$ then $\mathrm{Cap}(A_n) \to 0$.*

Proof. For a Borel subset B of E, denote by e^γ_B (resp. $\hat{e}^\gamma_B$) the γ-equilibrium potential (resp. γ-coequilibrium potential) of B (see [20, Section 2.1] for the definitions). By [20, Theorem 2.2.7],

$$\mathrm{Cap}(A_n) = \mathcal{E}_\gamma(e^\gamma_{A_n}, \hat{e}^\gamma_{A_n}) \le \mathcal{E}_\gamma(e^\gamma_{A_n}, \hat{e}^\gamma_{A_1}) = \int_E e^\gamma_{A_n}\, d\hat{\mu}^\gamma_{A_1},$$

where we have used [20, Lemma 2.1.1] and the fact that $e^\gamma_{A_n}$ is excessive. By the assumption, $H^\gamma_{A_n}(x) := E_x e^{-\gamma\sigma_{A_n}} \searrow 0$ m-a.e. Hence $e^\gamma_{A_n} \searrow 0$ m-a.e. by [20, Theorem 3.4.8], and consequently $e^\gamma_{A_n} \searrow 0$ q.e. by [20, Lemma 3.4.6]. This combined with the fact that $\hat{\mu}^\gamma_{A_1}$ is smooth gives the desired result. ∎

LEMMA 5.2. *Let A be a PCAF of $\mathbb{X}$ such that $E_x A_\zeta < \infty$ for m-a.e. $x \in E$. Then the assertion of Lemma 2.3 holds true.*

Proof. By a standard argument, u is finite q.e. Let μ be a smooth measure such that $A = A^\mu$ and let $\{F_n\}$ be a nest such that $\mathbf{1}_{F_n} \cdot \mu \in S_0$ (see [20, Lemma 4.1.14]). By [20, Lemma 4.1.11], for all $\alpha > 0$ and $n \ge 1$ the function $u^{n,\alpha}$ defined as

$$u^{n,\alpha}(x) = E_x \int_0^\zeta e^{-\alpha r}\mathbf{1}_{F_n}(X_r)\, dA^\mu_r$$

is quasi-continuous. Let $\psi \in L^1(E;m)$ be a strictly positive function such that $\|\psi\|_{L^1} = 1$. Then by [2, Lemma 6.1], for $q \in (0,1)$,

$$E_{\psi\cdot m} \sup_{t\ge 0} |u^{n,\alpha}(X_t) - u(X_t)|^q \le \frac{1}{1-q}\Big(E_{\psi\cdot m}\int_0^\zeta (1 - e^{-\alpha r}\mathbf{1}_{F_n}(X_r))\, dA^\mu_r\Big)^q.$$

Set $A^\varepsilon_{n,\alpha} = \{x \in E : |u^{n,\alpha}(x) - u(x)| > \varepsilon\}$. Then the above inequality yields

$$P_{\psi\cdot m}(\sigma_{A^\varepsilon_{n,\alpha}} < \infty) = P_{\psi\cdot m}\Big(\sup_{t\ge 0}|u^{n,\alpha}(X_t) - u(X_t)| > \varepsilon\Big) \to 0$$

as $\alpha \downarrow 0$ and $n \to \infty$. Hence, by Lemma 5.1, $\mathrm{Cap}(A^{\varepsilon}_{n,\alpha}) \to 0$. Because of arbitrariness of $\varepsilon > 0$, $u^{n,\alpha} \to u$ quasi-uniformly as $\alpha \downarrow 0$ and $n \to \infty$, which implies that u is quasi-continuous. ∎

LEMMA 5.3. *For every $\phi \in L^1(E;m) \cap \mathcal{B}^+(E)$ and $\mu \in S$,*

$$(R\mu, \phi) = \langle \mu, \widetilde{\hat{G}\phi} \rangle.$$

Proof. Follows from [20, Lemma 4.1.5]. ∎

THEOREM 5.4. *Assume that* (A1), (A2), (A3*), (A4*) *are satisfied. Then all the assertions of Theorem* 3.8 *hold true.*

Proof. It is enough to repeat the proof of Theorem 3.8 with the only exception that we now use Lemma 5.2 instead of Lemma 2.3. ∎

We close this section with some remarks on the class $\mathcal{R}$. Proposition 3.2 says that $\mathcal{M}_{0,b} \subset \mathcal{R}$ in case $\mathcal{E}$ is a transient Dirichlet form. We shall show that for semi-Dirichlet forms the same inclusion holds under the following duality condition considered in [13]:

(Δ) there exists a nest $\{F_n\}$ such that for every $n \in \mathbb{N}$ there is a non-negative $\eta_n \in L^2(E;m)$ such that $\eta_n > 0$ m-a.e. on F_n and $\hat{G}\eta_n$ is bounded.

One can observe that under (Δ),

$$\mathcal{R} = \{\mu \in S : R\mu < \infty \ m\text{-a.e.}\}.$$

PROPOSITION 5.5. *Assume that $\mathcal{E}$ satisfies the duality condition* (Δ). *Then $\mathcal{M}_{0,b} \subset \mathcal{R}$.*

Proof. Let $\mu \in \mathcal{M}_{0,b}$ and let η_n be the functions of the definition of (Δ). Then

$$(R\mu, \eta_n) = \langle \mu, \widetilde{\hat{G}\eta_n} \rangle \le \|\mu\|_{\mathrm{TV}} \cdot \|\hat{G}\eta_n\|_\infty < \infty$$

for every $n \in \mathbb{N}$. Hence $R\mu < \infty$ m-a.e., and consequently $\mu \in \mathcal{R}$ by the remark preceding the lemma. ∎

REMARK 5.6. If $\mathcal{E}$ satisfies (Δ) then by Lemma 5.2 and Proposition 5.5, (A3) implies (A3*) and (A4) implies (A4*).

6. Applications. In this section we show by examples how our general results work in practice. Propositions 6.2–6.4 below concerning nonlocal operators and operators in Hilbert spaces are new even in the linear case, i.e. if $f \equiv 0$. To our knowledge Proposition 6.1 concerning nonsymmetric local form is also new. Note that in all the examples below concerning Dirichlet forms one can explicitly describe the structure of bounded smooth measures (see [15]). In the last subsection we consider semi-Dirichlet forms.

6.1. Classical nonsymmetric local regular forms. We start with nonsymmetric forms associated with divergence form operators. Let D be a bounded open subset of $\mathbb{R}^d$, $d \geq 3$, and let m be the Lebesgue measure on D. Assume that $a : D \to \mathbb{R}^d \otimes \mathbb{R}^d$, $b, d : D \to \mathbb{R}^d$ and $c : D \to \mathbb{R}$ are measurable functions such that

(a) $c - \sum_{i=1}^d \frac{\partial b_i}{\partial x_i} \geq 0$ and $c - \sum_{i=1}^d \frac{\partial d_i}{\partial x_i} \geq 0$ in the sense of Schwartz distributions,

(b) there exist $\lambda > 0$, $M > 0$ such that $\sum_{i,j=1}^d \tilde{a}_{ij}\xi_i\xi_j \geq \lambda|\xi|^2$ for all $\xi \in \mathbb{R}^d$ and $|\breve{a}_{ij}| \leq M$ for $i, j = 1, \dots, d$, where $\tilde{a}_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$ and $\breve{a}_{ij} = \frac{1}{2}(a_{ij} - a_{ji})$,

(c) $c \in L^{d/2}_{\mathrm{loc}}(D; dx)$, $b_i, d_i \in L^d_{\mathrm{loc}}(D; dx)$, $b_i - d_i \in L^d(D; dx) \cup L^\infty(D; dx)$ for $i = 1, \dots, d$.

Then by [19, Proposition II.2.11], the form $(\mathcal{E}, C_0^\infty(D))$, where

$$\mathcal{E}(u, v) = \int_D \langle a\nabla u, \nabla v\rangle_{\mathbb{R}^d}\, dx + \int_D (\langle b, \nabla u\rangle_{\mathbb{R}^d}\, v + \langle d, \nabla v\rangle_{\mathbb{R}^d}\, u)\, dx + \int_D cuv\, dx, \tag{6.1}$$

is closable and its closure $(\mathcal{E}, D(\mathcal{E}))$ is a regular Dirichlet form on $L^2(D; dx)$. By (a) and (b),

$$\mathcal{E}(u, u) \geq \int_D \langle a\nabla u, \nabla u\rangle_{\mathbb{R}^d}\, dx = \int_D \langle \tilde{a}\nabla u, \nabla u\rangle_{\mathbb{R}^d}\, dx \geq \lambda \int_D \langle \nabla u, \nabla u\rangle_{\mathbb{R}^d}\, dx \tag{6.2}$$

for $u \in H_0^1(D)$, and hence, by Poincaré's inequality, there is $C_1 > 0$ such that

$$\mathcal{E}(u, u) \geq C_1(u, u) \tag{6.3}$$

for $u \in H_0^1(D)$. Consequently, $(\mathcal{E}, D(\mathcal{E}))$ satisfies the strong sector condition. From the calculations in [19, pp. 50–51] it follows that there exists $C_2 > 0$ depending on λ and the coefficients a, b, c, d such that

$$\mathcal{E}(u, u) \leq C_2\mathbb{D}_1(u, u), \tag{6.4}$$

where $\mathbb{D}_1(u, u) = \mathbb{D}(u, u) + \int_D u^2\, dx$ and $\mathbb{D}$ is defined by (3.3). By (6.2)–(6.4), $D(\mathcal{E}) = H_0^1(D)$. From this, (6.2) and the fact that $(\mathbb{D}, H_0^1(D))$ is transient it follows that $(\mathcal{E}, D(\mathcal{E}))$ is transient as well.

The operator corresponding to $(\mathcal{E}, D(\mathcal{E}))$ in the sense of (1.2) has the form

$$Lu = \sum_{i,j=1}^d \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) - \sum_{i=1}^d b_i\frac{\partial u}{\partial x_i} + \sum_{i=1}^d \frac{\partial}{\partial x_i}(d_i u) - cu. \tag{6.5}$$

From the above considerations and Corollary 4.3 we obtain the following proposition.

PROPOSITION 6.1. *Let $D \subset \mathbb{R}^d$, $d \geq 3$, be a bounded domain and let a, b, c, d satisfy* (a)–(c). *If f, μ satisfy* (A1)–(A4) *then there exists a unique probabilistic solution of the problem*

$$-Lu = f_u + \mu \text{ on } D, \qquad u|_{\partial D} = 0.$$

Moreover, $f_u \in L^1(D; dx)$, $T_k u \in H^1_0(D)$ for every $k > 0$, and (3.22), (4.3) *hold true.*

6.2. Gradient perturbations of nonlocal symmetric regular forms on $\mathbb{R}^d$. The following example of a nonlocal nonsymmetric regular Dirichlet form is borrowed from [10].

Let $\psi : \mathbb{R}^d \to \mathbb{R}$ be a continuous negative definite function, i.e. $\psi(0) \geq 0$ and $\xi \mapsto e^{-t\psi(\xi)}$ is positive definite for $t \geq 0$, and for $s \in \mathbb{R}$ let $H^{\psi,s}$ denote the Hilbert space

$$H^{\psi,s} = \{u \in L^2(\mathbb{R}^d; dx) : \|u\|_{\psi,s} < \infty\},$$

where

$$\|u\|^2_{\psi,s} = \int_{\mathbb{R}^d} (1 + \psi(\xi))^s |\hat{u}(\xi)|^2 \, d\xi$$

and $\hat{u}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\xi\cdot x} u(x) \, dx$, $\xi \in \mathbb{R}^d$. If $\psi(\xi) = |\xi|^2$ then $H^{\psi,s}$ coincides with the usual fractional Sobolev space H^s. The basic properties of the spaces $H^{\psi,s}$ are found in [10, Section 3.10].

Given ψ as above and $b = (b_1, \ldots, b_d) : \mathbb{R}^d \to \mathbb{R}^d$ such that $b_i \in C^1_b(\mathbb{R}^d)$ for $i = 1, \ldots, d$ define forms Ψ and $\mathcal{B}$ by

$$\Psi(u, v) = \int_{\mathbb{R}^d} \psi(\xi) \hat{u}(\xi) \overline{\hat{v}(\xi)} \, d\xi, \qquad u, v \in H^{\psi,1},$$

$$\mathcal{B}(u, v) = \int_{\mathbb{R}^d} \langle b, \nabla u \rangle_{\mathbb{R}^d} v \, dx, \qquad u, v \in C^\infty_0(\mathbb{R}^d). \tag{6.6}$$

Consider the following assumptions on ψ, b:

(a) $1/\psi$ is locally integrable on $\mathbb{R}^d$,
(b) there exist $\alpha \in (1, 2]$ and $\kappa, R > 0$ such that $\psi(\xi) \geq \kappa |\xi|^\alpha$ if $|\xi| > R$,
(c) $b_i \in C^1_{\mathrm{b}}(\mathbb{R}^d)$ for $i = 1, \ldots, d$ and $\operatorname{div} b = 0$.

It is known (see, e.g., [7, Example 1.4.1]) that $(\Psi, H^{\psi,1})$ is a symmetric regular Dirichlet form on $L^2(\mathbb{R}^d; dx)$. By [7, Example 1.5.2] it is transient iff condition (a) is satisfied. By [10, Corollary 4.7.35] there exists $c > 0$ (depending on b) such that $|\mathcal{B}(u, v)| \leq c\|u\|_{H^{1/2}} \|v\|_{H^{1/2}}$. Hence, if (b) is satisfied then $H^{\psi,1} \subset H^{1/2}$ and

$$|\mathcal{B}(u, v)| \leq C\|u\|_{\psi,1} \|v\|_{\psi,1} \tag{6.7}$$

for some $C>0$. Since $C_0^\infty(\mathbb{R}^d)$ is dense in $H^{\psi,1}$ (see [10, Theorem 3.10.3]), it follows that under (b) we may extend (6.6) to a continuous bilinear form $(\mathcal{B},H^{\psi,1})$. If moreover (c) is satisfied, then by integration by parts,

$$\mathcal{B}(u,u)=-\frac{1}{2}\int_{\mathbb{R}^d}u^2\operatorname{div}b\,dx=0 \tag{6.8}$$

for $u\in C_0^\infty(\mathbb{R}^d)$ and hence for all $u\in H^{\psi,1}$. Using integration by parts one can also check (see [10, Example 4.7.36]) that if $\operatorname{div}b=0$ then $(\mathcal{B},H^{\psi,1})$ has the contraction properties required in the definition of a Dirichlet form and hence is a Dirichlet form. Finally, let us consider the form

$$\mathcal{E}(u,v)=\Psi(u,v)+\mathcal{B}(u,v),\qquad u,v\in H^{\psi,1}. \tag{6.9}$$

From the properties of $\Psi,\mathcal{B}$ mentioned above it follows that if (a)–(c) are satisfied then $(\mathcal{E},H^{\psi,1})$ is a regular transient Dirichlet form on $L^2(\mathbb{R}^d;dx)$, and the extended Dirichlet space for $\mathcal{E}$ coincides with the extended Dirichlet space for Ψ, which we denote here by $H_e^{\psi,1}$. The space $H_e^{\psi,1}$ can be characterized for ψ of the form $\psi(\xi)=c|\xi|^\alpha$ for some $\alpha\in(0,2]$, $c>0$ (see [7, Example 1.5.2] or [11, Example 3.5.55]). That characterization shows that if ψ satisfies (b) and $\alpha<d$ (i.e. (a) is satisfied) then $H_e^{\psi,1}\hookrightarrow L^p(\mathbb{R}^d)$ with $p=2d/(d-\alpha)$ and $\|u\|_{L^p(\mathbb{R}^d;dx)}\le C\Psi(u,u)^{1/2}$ for $u\in H_e^{\psi,1}$ (see [11, Corollary 3.5.60]).

The operator associated with Ψ is a pseudodifferential operator $\psi(\nabla)$ which for $u\in C_0^\infty(\mathbb{R}^d)$ has the form

$$\psi(\nabla)u(x)=(2\pi)^{-d/2}\int_{\mathbb{R}^d}e^{i(x,\xi)}\psi(\xi)\hat{u}(\xi)\,d\xi,\qquad x\in\mathbb{R}^d.$$

PROPOSITION 6.2. *Assume that* (A1)–(A4) *and* (a)–(c) *hold. Then there exists a unique probabilistic solution of the equation*

$$-\psi(\nabla)u-(b,\nabla u)=f_u+\mu.$$

Moreover, $f_u\in L^1(\mathbb{R}^d;dx)$*,* $T_ku\in H_e^{\psi,1}$ *for every* $k>0$*, and* (3.22), (4.3) *are satisfied.*

Proposition 6.2 holds for operators corresponding to (6.9) with Ψ replaced by an arbitrary symmetric regular Dirichlet form with domain $H^{\psi,1}$. For examples of such forms see [10, Examples 4.7.30 and 4.7.31] and [11, Remark 2.6.8 and Theorem 2.6.10].

6.3. Nonlocal symmetric forms on $D\subset\mathbb{R}^d$. Let ψ be a continuous negative definite function satisfying conditions (a) and (b) of Subsection 6.2, and let $D\subset\mathbb{R}^d$ be a nearly Borel measurable set finely open with respect to the process associated with the form Ψ. Set $L_D^2(\mathbb{R}^d;dx)=\{u\in L^2(\mathbb{R}^d;dx):$

$u = 0$ a.e. on $D^c\}$ and

$$H_D^{\psi,1} = \{u \in H^{\psi,1} : \tilde{u} = 0 \text{ q.e. on } D^c\}.$$

Then by [3, Theorem 3.3.8], $(\Psi, H_D^{\psi,1})$ is a quasi-regular Dirichlet form on $L_D^2(\mathbb{R}^d; dx)$. If $\alpha < d$ then it is transient by [7, Theorem 4.4.4]. In case Ψ is transient, we denote its extended Dirichlet space by $H_{D,e}^{\psi,1}$. The above remarks and Corollary 4.3 lead to the following proposition.

Proposition 6.3. *Let the assumptions of Proposition* 6.2 *hold and let D be a nearly Borel finely open subset of $\mathbb{R}^d$ with $d > \alpha$. Then there exists a unique probabilistic solution of the problem*

$$-\psi(\nabla)u = f_u + \mu \quad \text{in } D, \qquad u = 0 \quad \text{in } D^c. \tag{6.10}$$

Moreover, $f_u \in L^1(D; dx)$, $T_k u \in H_{D,e}^{\psi,1}$ for every $k > 0$, and (3.22), (4.3) *hold true.*

Let us remark that if D is bounded then $H_{D,e}^{\psi,1} = H_D^{\psi,1}$, because $H_D^{\psi,1} \hookrightarrow L_D^2(\mathbb{R}^d; dx)$ in that case. If D is open and has smooth boundary then as in [12] we may define the space $H_0^{\psi,1}(D)$ as follows. Given $u \in C_0^\infty(D)$ we extend it to $\mathbb{R}^d$ by setting $u = 0$ on D^c. We then obtain a function $u \in C_0^\infty(\mathbb{R}^d)$ with support in D. Consequently, we may regard $C_0^\infty(D)$ as a subspace of $H^{\psi,1}$ and therefore define $H_0^{\psi,1}(D)$ as the closure of $C_0^\infty(D)$ in $H^{\psi,1}$. By [7, Theorem 4.4.3], $C_0^\infty(D)$ is a special standard core of $(\Psi, H_D^{\psi,1})$, and hence, by [7, Lemma 2.3.4], $H_D^{\psi,1} = H_0^{\psi,1}(D)$.

Assume that $d \geq 3$ and $D \subset \mathbb{R}^d$ is a bounded open set with a $C^{1,1}$ boundary. Consider the form $(\Psi, H_D^{\psi,1})$ with $\psi(\xi) = c|\xi|^\alpha$ for some $c > 0$, $\alpha \in (0, 2]$. By [16, Proposition 4.9] there exist constants $0 < c_1 < c_2$ depending only on d, α, D such that

$$c_1 \delta^{\alpha/2}(x) \leq R1(x) \leq c_2 \delta^{\alpha/2}(x), \qquad x \in D,$$

where $\delta(x) = \operatorname{dist}(x, \partial D)$. From this, Theorem 3.8 and Remark 4.5 it follows that if f satisfies (A1), (A2), (A3*) and $f(\cdot, 0) \in L^1(D; \delta^{\alpha/2}(x)\, dx)$, $\int_D \delta^{\alpha/2}(x)\, |\mu|(dx) < \infty$ then the probabilistic solution u of (3.18) belongs to $L^1(D; dx)$.

6.4. Dirichlet forms on infinite-dimensional state space. Let H be a separable real Hilbert space and let A, Q be linear operators on H. Assume that

(a) $A : D(A) \subset H \to H$ generates a strongly continuous semigroup $\{e^{tA}\}$ in H such that $\|e^{tA}\| \leq Me^{-\omega t}$, $t \geq 0$, for some $M, \omega > 0$,
(b) Q is bounded, $Q = Q^* > 0$ and $\sup_{t>0} \operatorname{Tr} Q_t < \infty$, where $Q_t = \int_0^t e^{sA} Q e^{sA^*}\, ds$,
(c) $Q_\infty(H) \subset D(A)$, where $Q_\infty = \int_0^\infty e^{tA} Q e^{tA^*}\, dt$.

A simple and important example of A, Q satisfying (a)–(c) is $Q = I$ and a self-adjoint operator A such that $\langle Ax, x\rangle_H \le -\omega|x|_H^2$, $x \in D(A)$, for some $\omega > 0$ and A^{-1} is of trace class. In this example $Q_\infty = -\frac{1}{2}A^{-1}$. Other examples are found for instance in [6].

By (a) the operators Q_t, Q_∞ are well defined, and by (b), Q_∞ is of trace class. Let γ denote the Gaussian measure on H with mean 0 and covariance operator Q_∞. We consider the form

$$\mathcal{E}(u, v) = -\int_H \langle \nabla u, AQ_\infty \nabla v\rangle_H \, d\gamma, \qquad u, v \in \mathcal{F}C_{\mathrm{b}}^\infty. \tag{6.11}$$

Here $\mathcal{F}C_{\mathrm{b}}^\infty$ is the space of finitely based smooth bounded functions, i.e.

$$\mathcal{F}C_{\mathrm{b}}^\infty = \{u : H \to \mathbb{R} : u(x) = f(\langle x, e_1\rangle, \ldots, \langle x, e_m\rangle),\ m \in \mathbb{N},\ f \in C_{\mathrm{b}}^\infty(\mathbb{R}^m)\}$$

for some orthonormal basis $\{e_k\}$ of H consisting of eigenvectors of Q_∞, and ∇ is the H-gradient defined for $u \in \mathcal{F}C_{\mathrm{b}}^\infty$ as the unique element of H such that $\langle \nabla u(x), h\rangle_H = \frac{\partial u}{\partial h}(x)$ for $x \in H$ (the last derivative is the Gateaux derivative in the direction h, i.e. $\frac{\partial u}{\partial h}(x) = \frac{d}{ds}u(x + sh)|_{s=0}$). Under (a)–(c) the form $(\mathcal{E}, \mathcal{F}C_{\mathrm{b}}^\infty)$ is closable and its closure, which will be denoted by $(\mathcal{E}, W_Q^{1,2}(H))$, is a coercive closed form on $L^2(H;\gamma)$ (see [6, Theorem 2.2, Remark 2.3 and Lemma 3.3]). Using the product rule for ∇ on $\mathcal{F}C_{\mathrm{b}}^\infty$ one can check in the same way as in [19, Section II.2(d)] (see also [19, Section II.3(e)]) that it has the Dirichlet property. Finally, by results of [19, Section IV.4], it is quasi-regular.

By [6, Theorem 3.6] the semigroup $\{P_t\}$ on $L^2(H;\gamma)$ associated with $(\mathcal{E}, W_Q^{1,2}(H))$ is the Ornstein–Uhlenbeck semigroup of the form

$$P_t f(x) = \int_H f(y)\mathcal{N}(e^{tA}x, Q_t)\,(dy), \qquad x \in H,$$

where $\mathcal{N}(e^{tA}x, Q_t)$ is the gaussian measure in H with mean $e^{tA}x$ and covariance operator Q_t. Note that $\{P_t\}$ is analytic. Actually, analyticity of $\{P_t\}$ is equivalent to the fact that it corresponds to some nonsymmetric Dirichlet form (see [8] and also [9] for related results in a more general setting). The generator of $\{P_t\}$ has the form

$$Lu(x) = \tfrac{1}{2}\operatorname{Tr}(Q\Delta u(x)) + \langle x, A^*\nabla u(x)\rangle_H, \qquad x \in H.$$

Since for every $\lambda > 0$ the form $(\mathcal{E}_\lambda, W_Q^{1,2}(D))$ is transient, from the above remarks and Corollary 4.3 we obtain the following proposition.

PROPOSITION 6.4. *Assume that* (A1)–(A4) *and* (a)–(c) *hold. Then for every* $\lambda > 0$ *there exists a unique probabilistic solution to the equation*

$$-Lu + \lambda u = f_u + \mu.$$

Moreover, $f_u \in L^1(H;\gamma)$, $T_k u \in W^{1,2}_Q(D)$ *for every* $k>0$, *and* (3.22), (4.3) *hold true.*

For generalizations of forms (6.11) to operators Q depending on x or more general measures on topological vector spaces than gaussian measures on Hilbert spaces we refer the reader to [19, Section II.3], [8], [22] and the references therein).

6.5. Additional remarks on Dirichlet forms. In this subsection we briefly outline how general results on transformation of Dirichlet forms can by applied to obtain other interesting examples of quasi-regular forms.

(i) *Perturbation of Dirichlet forms.* Let $(\mathcal{E},D(\mathcal{E}))$ be a quasi-Dirichlet form and let $\nu\in S$. Set

$$\mathcal{E}^\nu(u,v)=\mathcal{E}(u,v)+\int_E \tilde{u}\tilde{v}\,d\nu,\qquad u,v\in D(\mathcal{E}^\nu),$$

where $D(\mathcal{E}^\nu)=D(\mathcal{E})\cap L^2(E;\nu)$. By [23, Proposition 2.3], $(\mathcal{E}^\nu,D(\mathcal{E}^\nu))$ is a quasi-regular Dirichlet form on $L^2(E;m)$. In our context an important example of ν is $\nu(dx)=V(x)m(dx)$ for some nonnegative $V\in L^1(E;m)\cap L^\infty(E;m)$. In this case $D(\mathcal{E}^V)\equiv D(\mathcal{E}^\nu)=D(\mathcal{E})$. Moreover, $(\mathcal{E}^V,D(\mathcal{E}^V))$ satisfies the strong sector condition if $(\mathcal{E},D(\mathcal{E}))$ does, and from (2.3) it follows immediately that $(\mathcal{E}^V,D(\mathcal{E}^V))$ is transient if $(\mathcal{E},D(\mathcal{E}))$ is transient or V is m-a.e. strictly positive. Therefore Propositions 6.1 and 6.4 hold true for operators L replaced by $L-V$ (in Proposition 6.4 we can take $\lambda\geq 0$ if V is $m\equiv\mu$-a.e. strictly positive), and Proposition 6.3 holds for $\psi(\nabla)$ replaced by $\psi(\nabla)-V$. Note that the perturbed regular form may become nonregular. For instance, in [19, Section II.2(e)] one can find an example of V such that the classical form $(\mathbb{D},H^1(\mathbb{R}^d))$ (see (3.3)) perturbed by V is not regular.

(ii) *Superposition of closed forms.* For $k=1,2$ let $(\mathcal{E}^{(k)},D^{(k)})$ be a closable symmetric bilinear form on $L^2(E;m)$. Set

$$\mathcal{E}(u,v)=\mathcal{E}^{(1)}(u,v)+\mathcal{E}^{(2)}(u,v),\qquad u,v\in D,$$

where $D=\{u\in D^{(1)}\cap D^{(2)}:\mathcal{E}^{(1)}(u,u)+\mathcal{E}^{(2)}(u,u)<\infty\}$. By [19, Proposition I.3.7] the form $(\mathcal{E},D)$ is closable on $L^2(E;m)$. We may use this property and examples considered in Sections 6.1–6.4 to construct new quasi-regular Dirichlet forms. To illustrate how this works in practice, following [19, Remark II.3.16] we consider the form $(\mathcal{E},\mathcal{F}C^\infty_{\mathrm{b}})$ of Section 6.1 and a symmetric finite positive measure on $(H\times H,\mathcal{B}(H)\otimes\mathcal{B}(H))$ such that the form

$$J(u,v)=\int_H\int_H(u(x)-u(y))(v(x)-v(y))\,J(dx\,dy),\qquad u,v\in\mathcal{F}C^\infty_{\mathrm{b}},$$

is closable. Then the form $(\mathcal{E}+J,\mathcal{F}C^\infty_{\mathrm{b}})$ is closable and its closure is a symmetric quasi-regular Dirichlet form. Thus we have constructed an infinite-dimensional (and so nonregular) Dirichlet form which is nonlocal. For the operator corresponding to that form one can formulate an analogue of Proposition 6.4.

General results on superposition of closed forms are found in [7, Section 3.1] and [19, Proposition I.3.7].

(iii) *Parts of forms.* Let $(\mathcal{E}, D(\mathcal{E}))$ be a symmetric regular Dirichlet form on $L^2(E;m)$, and $D \subset E$ be a nearly Borel measurable finely open set with respect to the process $\mathbb{X}$ associated with $(\mathcal{E}, D(\mathcal{E}))$. Set $L^2_D(E;m) = \{u \in L^2(E;m) : u = 0 \text{ } m\text{-a.e. on } D^c\}$ and $\mathcal{F}_D = \{u \in D(\mathcal{E}) : \tilde{u} = 0 \text{ q.e. on } D^c\}$. By [3, Theorem 3.3.8] the form $(\mathcal{E}, \mathcal{F}_D)$ on $L^2_D(E;m)$, called the *part of* $(\mathcal{E}, D(\mathcal{E}))$ *on* D, is a quasi-regular Dirichlet form (if D is open then it is regular). We can use this result to get solutions of Dirichlet problems of the form (6.10) with $\psi(\nabla)$ replaced by an arbitrary operator associated with a symmetric regular Dirichlet form.

6.6. Semi-Dirichlet forms

(I) *Diffusion operator with drift.* Let $D \subset \mathbb{R}^d$, $d \geq 3$, be a bounded domain and let $a_{ij}, b_i : D \to \mathbb{R}$ be measurable functions such that b_i is bounded, $a_{ij} = a_{ji}$ and

$$\lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \leq \lambda|\xi|^2, \qquad \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d,$$

for some $\lambda \geq 1$. Consider the form $(\mathcal{E}, C_0^\infty(D))$ defined by (6.1) with $c = 0$, $d = 0$. By [20, Theorems 1.5.2 and 1.5.3] its smallest closed extension $(\mathcal{E}, H_0^1(D))$ is a regular lower-bounded semi-Dirichlet form on $L^2(D;dx)$. Therefore, if (A1), (A2), (A3*), (A4*) are satisfied, then there exists a unique probabilistic solution of (1.1) with L defined by (6.5) with $c, d = 0$.

Let G_D denote the Green function for L on D, and let $\hat{G}_D$ denote the Green function on D for the adjoint operator to L, i.e. the operator associated with the form $(\hat{\mathcal{E}}, H_0^1(D))$. It is known that $G_D(x,y) = \hat{G}_D(y,x)$ and $G_D(x,y) \leq c|x-y|^{-(d-2)}$ for $x, y \in D$ such that $x \neq y$ (see, e.g., [21, Section 4.2]). Therefore,

$$\hat{G}1(x) = \int_D \hat{G}_D(x,y)\,dy = \int_D G_D(y,x)\,dy \leq c\int_D |x-y|^{-d+2}\,dy,$$

and hence

$$\hat{G}1(x) \leq c \int_{B(x,\mathrm{diam}(D))} |x-y|^{-d+2}\,dy = c_1(\mathrm{diam}(D))^2.$$

Accordingly, $\mathcal{E}$ satisfies condition (Δ) with $\eta_n = 1$ and $F_n = D$. From this and Remark 5.6 it follows that (A3) implies (A3*) and (A4) implies (A4*).

(II) *Fractional laplacian with variable exponent.* Let $\alpha : \mathbb{R}^d \to \mathbb{R}$ be a measurable function such that $\alpha_1 \leq \alpha(x) \leq \alpha_2$, $x \in \mathbb{R}^d$, for some constants

$0 < \alpha_1 \le \alpha_2 < 2$. Let $L_t = L = \Delta^{\alpha(x)}$, i.e. L is a pseudodifferential operator such that

$$Lu(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\xi} |\xi|^{\alpha(x)} \hat{u}(\xi)\, d\xi, \qquad u \in C_c^\infty(\mathbb{R}^d). \tag{6.12}$$

If $\int_0^1 (\beta(r)|\log r|)^2 r^{-(1+\alpha_2)}\, dr < \infty$, where $\beta(r) = \sup_{|x-y|\le r} |\alpha(x) - \alpha(y)|$, then L is associated with some regular semi-Dirichlet form $\mathcal{E}$ on $L^2(\mathbb{R}^d; dx)$ (see [13, Example 5.13] for details). Therefore under the above assumptions on α and (A1), (A2), (A3*), (A4*) there exists a unique probabilistic solution of (1.1) with L defined by (6.12).

Acknowledgments. The first author was supported by the Polish National Science Centre (grant no. 2012/07/D/ST1/02107).

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