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The terms of the form $7kx^2$ in the generalized Lucas sequence with parameters P and Q

by

Olcay Karaatli (Sakarya)

1. Introduction. Let P and Q be nonzero integers, let $D = P^2 + 4Q$ be called the *discriminant*, and assume that $D > 0$ (to exclude degenerate cases). Consider the polynomial $X^2 - PX - Q$, called the *characteristic polynomial*, which has the roots

$$\alpha = \frac{P + \sqrt{D}}{2} \quad \text{and} \quad \beta = \frac{P - \sqrt{D}}{2}.$$

For each $n \geq 0$, define $U_n = U_n(P, Q)$ and $V_n = V_n(P, Q)$ as follows:

$$U_0 = 0, \quad U_1 = 1, \quad U_{n+1} = PU_n + QU_{n-1} \quad (\text{for } n \geq 1),$$
$$V_0 = 2, \quad V_1 = P, \quad V_{n+1} = PV_n + QV_{n-1} \quad (\text{for } n \geq 1).$$

We shall consider special cases of the generalized Fibonacci and Lucas sequences. For $(P, Q) = (1, 1)$, (U_n) is the sequence of Fibonacci numbers and (V_n) is the sequence of Lucas numbers. For $(P, Q) = (2, 1)$, (U_n) and (V_n) are the sequences of Pell numbers, respectively Pell–Lucas numbers.

It is convenient to extend these sequences also to negative indices:

$$U_{-n} = -\frac{U_n}{(-Q)^n}, \quad V_{-n} = \frac{V_n}{(-Q)^n}$$

for $n \geq 1$. Then the two relations above hold for all integers n.

Binet's formulas express the numbers U_n and V_n in terms of α and β:

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = \alpha^n + \beta^n.$$

Note that by Binet's formulas we also have

$$U_n(-P, Q) = (-1)^{n-1} U_n(P, Q), \quad V_n(-P, Q) = (-1)^n V_n(P, Q).$$

So, it will be assumed that $P \geq 1$.

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Investigations of the properties of second-order linear recurring sequences have given rise to questions concerning whether, for certain pairs (P,Q), U_n or V_n is square ($=\square$) or k times a square ($=k\square$). From a result of Ljunggren [20], Robbins [29] deduced that if $(P,Q)=(2,1)$, and $n\geq 2$, then $U_n=\square$ precisely for $n=7$, and Pethő [24] (independently Cohn [13]) showed that these are the only perfect powers in the Pell sequence. In 1963, Moser and Carlitz [22] and Rollett [31] proposed the problem of finding all square Fibonacci numbers. This problem was solved by Cohn [7]. Cohn proved that if $(P,Q)=(1,1)$, then the only perfect square greater than 1 in the sequence (U_n) is $U_{12}=144$ (see also Alfred [1], Wyler [34], and Burr [6]). Cohn [9, 8] also solved the equations $U_n(1,1)=2\square$ and $V_n(1,1)=\square, 2\square$.

In 2006, Bugeaud, Mignotte and Siksek [5] showed that the perfect powers in Fibonacci and Lucas sequences are exactly $F_0=0$, $F_1=F_2=1$, $F_6=8$, $F_{12}=144$ and $L_1=1$, $L_3=4$, respectively. Robbins [28], under the conditions that $P=1$, $Q=1$, found all solutions of the equation $U_n=px^2$ such that p is prime and either $p\equiv 3 \pmod 4$ or $p<10000$, and then in 1991 the same author [30], using elementary techniques, found all solutions of the equation $V_n=px^2$, where p is prime and $p<1000$. Furthermore, Cohn [10, 11] determined the squares and double squares in (U_n) and (V_n) when P is odd and $Q=\pm 1$.

The determination of squares in generalized Fibonacci and Lucas sequences (with odd relatively prime parameters and nonzero discriminant) was obtained by various authors. Ribenboim and McDaniel [25] determined all indices n such that $U_n=\square$, $2U_n=\square$, $V_n=\square$ or $2V_n=\square$ for all odd relatively prime integers P and Q. Bremner and Tzanakis [2] extended the result of the equation $U_n=\square$ by determining all generalized Fibonacci sequences (U_n) with $U_{12}=\square$, subject only to the restriction that $(P,Q)=1$. In a later paper, the same authors [3] showed that for $2,\dots,7$, U_n is a square for infinitely many coprime P, Q and determined all sequences (U_n) with $U_n=\square, n=8,10,11$. And also in [4], they discussed the more general problem of finding all integers n, P, Q for which $U_n=k\square$ for a given integer k.

Although the question for even values of P seems to be harder, in 1998, Kagawa and Terai [14] considered a similar problem, such as that addressed by Ribenboim and McDaniel [25], for the case when P is even and $Q=1$. Using elementary properties of elliptic curves, they showed that if $P=2t$ with t even, then each of $U_n(P,1)=\square$, $2U_n(P,1)=\square$, $V_n(P,1)=\square$, or $2V_n(P,1)=\square$ implies $n\leq 3$ under some assumptions. Moreover, Mignotte and Pethő [21] proved that if $n>4$, then $U_n(P,-1)=wx^2$ is impossible when $w\in\{1,2,3,6\}$; moreover these equations have solutions for $n=4$ only if $P=338$. Extending the method of Mignotte and Pethő, Nakamula and Pethő [23] gave the solutions of the equations $U_n(P,-1) = wx^2$ where $w \in \{1, 2, 3, 6\}$.

In 1998, Ribenboim and McDaniel [26] showed that if P is even, $Q \equiv 3 \pmod 4$, and $U_n = \square$, then n is a square or twice an odd square and all prime factors of n divide $P^2 + 4Q$. In a later paper, for all odd values of P and Q, the same authors [27] determined all indices n such that $U_n = kx^2$ under the assumption that for all integers $u \geq 1$, k is such that, for each odd divisor h of k, the Jacobi symbol $\left(\frac{-V_{2u}}{h}\right)$ is defined and equal to 1. Afterwards, they solved the equation $V_n = 3\square$ for $P \equiv 1, 3 \pmod 8$, $Q \equiv 3 \pmod 4$, $(P, Q) = 1$, and solved $U_n = 3\square$ for all odd relatively prime integers P and Q.

Also, in [33], Şiar and Keskin determined all indices n such that $V_n = kx^2$ when $k \mid P$ and P is odd and $Q = 1$. In [16], Karaatlı and Keskin dealt with Lucas numbers of the form $V_n(P, Q)$ with the special restriction that $P \geq 3$ is odd and $Q = -1$. Under these assumptions, they solved the equations $V_n = wkx^2$, $w \in \{5, 7\}$, when $k \mid P$ with $k > 1$. Afterwards, Karaatlı [15] added to the above list the values of n for which $V_n(P, 1)$ is of the form $5kx^2$ and $7kx^2$ when $k \mid P$ with $k > 1$. Furthermore, as an application of some of these results, he gave all positive integer solutions to the equations $V_n = wx^2$, $w \in \{15, 21, 35\}$. Actually, for $k = 1$, Keskin and Karaatlı even solved the equations $V_n(P,-1) = 5x^2$ in [17] and $V_n(P, 1) = 5x^2$ and $V_n(P, 1) = 7x^2$ in [19] and [18], respectively.

In this study, we determine all indices n such that $V_n = 7kx^2$ if P and Q are odd and relatively prime and $k \mid P$ with $k > 1$. Moreover, as an application, we determine the indices n such that the equation $V_n = 21x^2$ has solutions.

We organize the paper as follows. Section 2 consists of preliminaries where all the required facts are gathered for the convenience of the reader. The last section will be devoted to the main theorem. Throughout this study, $\left(\frac{*}{*}\right)$ will denote the Jacobi symbol. Our method of proof is similar to that presented by Cohn, McDaniel and Ribenboim [10, 11, 12, 25].

2. Preliminaries. Among the numerous identities and divisibility properties satisfied by the generalized Fibonacci and Lucas numbers, we list below those to be used in this paper:

(2.1) $\qquad V_{-n} = V_n/(-Q)^n,$

(2.2) $\qquad V_{2n} = V_n^2 - 2(-Q)^n,$

(2.3) $\qquad V_n^2 - DU_n^2 = 4(-Q)^n,$

(2.4) $\qquad$ if $V_m \neq 1$ and $m \neq 0$, then $V_m \mid V_n \Leftrightarrow m \mid n$ and n/m is odd,

$$\text{if } n \text{ is odd, then } V_n \equiv (-Q)^{(n-1)/2}P \pmod{P^2+4Q}, \tag{2.5}$$

$$\text{if } P \text{ is odd, then } \left(\frac{-1}{V_{2^r}}\right) = -1 \text{ for } r \geq 1, \tag{2.6}$$

$$\text{if } 7 \mid P, \text{ then } V_{2^r} \equiv 2Q^{2^r/2} \pmod 7 \text{ for } r \geq 1. \tag{2.7}$$

From (2.6) and (2.7), we have

$$\left(\frac{7}{V_{2^r}}\right) = \begin{cases} (-1)\left(\frac{Q}{7}\right) & \text{if } r = 1, \\ -1 & \text{if } r \geq 2. \end{cases} \tag{2.8}$$

Theorem 2.1 (Şiar and Keskin [32, Corollary 3.3]). *Let P and Q be integers with $Q \neq \pm 1$. Then for all $n, m \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{Z}$ such that $mn + r \geq 0$ we get*

$$V_{2mn+r} \equiv (-(-Q)^m)^n V_r \pmod{V_m}. \tag{2.9}$$

Theorem 2.2 (Şiar and Keskin [32, Corollary 3.5]). *Let P and Q be integers with $Q \neq \pm 1$. Then for all $n \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$, and $r \in \mathbb{Z}$ such that $mn + r \geq 0$, we get*

$$V_{2mn+r} \equiv (-Q)^{mn} V_r \pmod{U_m}. \tag{2.10}$$

By using (2.10), since $8 \mid U_6$ we get

$$V_{12q+r} \equiv V_r \pmod 8, \tag{2.11}$$

for any nonnegative integer q.

Lemma 2.3 (Ribenboim and McDaniel [25, Lemma 3]). *Let r be a positive integer. Then*

$$\left(\frac{Q}{V_{2^r}}\right) = \left(\frac{-1}{Q}\right), \tag{2.12}$$

$$\left(\frac{P^2+3Q}{V_{2^r}}\right) = \begin{cases} \left(\frac{-1}{Q}\right) & \textit{if } r = 1, \\ 1 & \textit{if } r \geq 2, \end{cases} \tag{2.13}$$

$$\left(\frac{P}{V_{2^r}}\right) = \begin{cases} \left(\frac{-2Q}{P}\right) & \textit{if } r = 1, \\ \left(\frac{-2}{P}\right) & \textit{if } r \geq 2. \end{cases} \tag{2.14}$$

If M is any positive divisor of P, then (2.14) implies that

$$\left(\frac{M}{V_{2^r}}\right) = \begin{cases} (-1)^{(M-1)/2}(-1)^{(M^2-1)/8}\left(\frac{Q}{M}\right) & \text{if } r = 1, \\ (-1)^{(M-1)/2}(-1)^{(M^2-1)/8} & \text{if } r \geq 2. \end{cases} \tag{2.15}$$

If $7 \mid P$, then by (2.8) and (2.14), we have

$$\left(\frac{7}{V_{2^r}}\right)\left(\frac{P}{V_{2^r}}\right) = \begin{cases} (-1)(-1)^{(P-1)/2}(-1)^{(P^2-1)/8}\left(\frac{Q}{7}\right)\left(\frac{Q}{P}\right) & \text{if } r = 1, \\ (-1)(-1)^{(P-1)/2}(-1)^{(P^2-1)/8} & \text{if } r \geq 2. \end{cases} \tag{2.16}$$

And by (2.16), we have

$$(2.17)\qquad \left(\frac{7}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right) = \begin{cases} (-1)(-1)^{(M-1)/2}(-1)^{(M^2-1)/8}\left(\frac{Q}{7}\right)\left(\frac{Q}{M}\right) & \text{if } r=1, \\ (-1)(-1)^{(M-1)/2}(-1)^{(M^2-1)/8} & \text{if } r\ge 2. \end{cases}$$

From now on, we assume that P and Q are odd and relatively prime. We omit the proof of the following lemma, as it is a straightforward induction.

LEMMA 2.4. *If n is a positive even integer, then $V_n \equiv 2Q^{n/2} \pmod{P^2}$, and if n is a positive odd integer, then $V_n \equiv nPQ^{(n-1)/2} \pmod{P^2}$.*

LEMMA 2.5. *Let n be a positive integer. If $3 \mid P$, then $3 \mid V_n$ if and only if n is odd. If $3 \nmid P$, then $3 \mid V_n$ if and only if $n \equiv 2 \pmod 4$ and $Q \equiv 1 \pmod 3$.*

Proof. Let $3 \mid P$. If n is odd, then by Lemma 2.4 we have $V_n \equiv nPQ^{(n-1)/2} \pmod{P^2}$, implying that $3 \mid V_n$. Conversely, assume that $3 \mid V_n$ and n is even. Then by Lemma 2.4, it follows that $V_n \equiv 2Q^{n/2} \pmod{P^2}$. Since $3 \mid P$ and $3 \mid V_n$, it follows that $3 \mid 2Q^{n/2}$. But this is impossible since $3 \nmid Q$.

Now assume that $3 \nmid P$. Also, assume that $3 \mid V_n$. Then by (2.3), we readily obtain $3 \nmid U_n$. So, $3 \mid 4(-Q)^n + 1 + 4Q$. If n is odd, then a simple calculation shows that $3 \nmid V_n$ in all the cases $Q \equiv 0, 1, 2 \pmod 3$. Hence, n is even. Therefore, when $Q \equiv 0, 2 \pmod 3$, it follows that $3 \nmid V_n$. So, $Q \equiv 1 \pmod 3$. The fact that $P^2 \equiv 1 \pmod 3$ and $Q \equiv 1 \pmod 3$ gives $3 \mid V_2$. On the other hand, since n is even, we can write $n = 2t$ for some positive integer t. Assume now that t is even, $t = 2r$, say. Then $n = 4r$ and so $V_n = V_{4r} \equiv (-Q)^r V_0 \pmod{V_2}$ by (2.9). Using the fact that $3 \mid V_2$, we see that $V_n \equiv 2(-Q)^r \pmod 3$. But this is impossible since $3 \nmid 2(-Q)^r$. Hence, $n \equiv 2 \pmod 4$. As a consequence, if $3 \mid V_n$, then $n \equiv 2 \pmod 4$ and $Q \equiv 1 \pmod 3$. Conversely, assume that $n \equiv 2 \pmod 4$ and $Q \equiv 1 \pmod 3$. Then by (2.4), $V_2 \mid V_n$. On the other hand, since $P^2 \equiv 1 \pmod 3$ and $Q \equiv 1 \pmod 3$, it follows that $3 \mid V_2$. And so, $3 \mid V_n$. ■

The idea behind the proof of the following lemma is similar to that of the lemma above and we omit the proof.

LEMMA 2.6. *Let n be a positive integer. If $7 \mid P$, then $7 \mid V_n$ if and only if n is odd. If $P^2 \equiv 1 \pmod 7$, then $7 \mid V_n$ if and only if $n \equiv 2 \pmod 4$, $Q \equiv 3 \pmod 7$ or $n = 4t$, $2 \nmid t$, $Q \equiv 1 \pmod 7$ or $n = 3t$, $2 \nmid t$, $Q \equiv 2 \pmod 7$. If $P^2 \equiv 2 \pmod 7$, then $7 \mid V_n$ if and only if $n \equiv 2 \pmod 4$, $Q \equiv 6 \pmod 7$ or $n = 4t$, $2 \nmid t$, $Q \equiv 1 \pmod 7$ or $n = 3t$, $2 \nmid t$, $Q \equiv 4 \pmod 7$. If $P^2 \equiv 4 \pmod 7$, then $7 \mid V_n$ if and only if $n \equiv 2 \pmod 4$, $Q \equiv 5 \pmod 7$ or $n = 4t$, $2 \nmid t$, $Q \equiv 2 \pmod 7$ or $n = 3t$, $2 \nmid t$, $Q \equiv 1 \pmod 7$.*

Under the conditions that $P^2 \equiv 1 \pmod 7$, $Q \equiv 2 \pmod 7$ or $P^2 \equiv 2 \pmod 7$, $Q \equiv 4 \pmod 7$ or $P^2 \equiv 4 \pmod 7$, $Q \equiv 1 \pmod 7$ in Lemma 2.6, we have

$$\left(\frac{7}{V_{2^r}}\right) = 1 \tag{2.18}$$

for all $r \geq 1$. Therefore, by (2.18) and (2.15), we obtain

$$\left(\frac{7}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right) = \begin{cases} (-1)^{(M-1)/2}(-1)^{(M^2-1)/8}\left(\frac{Q}{M}\right) & \text{if } r = 1, \\ (-1)^{(M-1)/2}(-1)^{(M^2-1)/8} & \text{if } r \geq 2. \end{cases} \tag{2.19}$$

The following lemma can be proven by induction.

Lemma 2.7. *Let r be a positive integer. Then*

$$V_{2^r} \equiv \begin{cases} Q^{2^{r-1}-1}V_2 \pmod A & \textit{if } r \textit{ is odd}, \\ -Q^{2^{r-1}-1}(P^2+3Q) \pmod A & \textit{if } r \textit{ is even}, \end{cases}$$

where $A = P^4 + 5P^2Q + 5Q^2$.

We see easily from this lemma that if $Q \equiv 3 \pmod 8$, then

$$\left(\frac{A}{V_{2^r}}\right) = \left(\frac{V_{2^r}}{A}\right) = -1, \tag{2.20}$$

since $A \equiv 5 \pmod 8$.

3. Main theorem

Theorem 3.1. *If $V_n = 7kx^2$ for some $k \mid P$ with $k > 1$, then $n = 1, 3, 5$.*

Proof. Assume that $V_n = 7kx^2$ for some $k \mid P$ with $k > 1$. Obviously, $k \mid V_n$ and so, by Lemma 2.4, n is odd. Moreover, since $k \mid P$, we have $P = kM$ for some odd $M > 0$. Suppose $n > 3$. Then we can write $n = 4q + 1$ or $n = 4q + 3$ for some $q > 0$. Now we distinguish two cases.

Case I: Let $7 \mid P$.

Subcase I(i): Assume that $\left(\frac{Q}{7}\right)\left(\frac{Q}{M}\right) = -1$. If $n = 4q+1$, then it follows from (2.5) that

$$7kx^2 = V_n = V_{4q+1} \equiv Q^{2q}P \pmod{P^2 + 4Q}.$$

Multiplying both sides of the congruence above by M and using the fact that $(P, P^2 + 4Q) = 1$, we immediately get

$$7x^2 \equiv Q^{2q}M \pmod{P^2 + 4Q}.$$

This shows that $1 = \left(\frac{7}{P^2+4Q}\right)\left(\frac{M}{P^2+4Q}\right)$. However, this is impossible since

$$\left(\frac{7}{P^2+4Q}\right)\left(\frac{M}{P^2+4Q}\right) = \left(\frac{P^2+4Q}{7}\right)\left(\frac{P^2+4Q}{M}\right) = \left(\frac{Q}{7}\right)\left(\frac{Q}{M}\right) = -1.$$

Now assume that $n = 4q+3$. Again using (2.5), $(P, P^2+4Q) = 1$ and $P = kM$, we have

$$7kx^2 = V_n = V_{4q+3} \equiv -Q^{2q+1}P \pmod{P^2+4Q},$$

i.e.,

$$7x^2 \equiv -Q^{2q+1}M \pmod{P^2+4Q}.$$

This means that $1 = \left(\frac{-1}{P^2+4Q}\right)\left(\frac{7}{P^2+4Q}\right)\left(\frac{M}{P^2+4Q}\right)\left(\frac{Q}{P^2+4Q}\right)$. Obviously, $\left(\frac{Q}{P^2+4Q}\right) = \left(\frac{P^2}{Q}\right) = 1$ and $\left(\frac{-1}{P^2+4Q}\right) = 1$. So,

$$\begin{aligned} 1 &= \left(\frac{7}{P^2+4Q}\right)\left(\frac{M}{P^2+4Q}\right) = \left(\frac{P^2+4Q}{7}\right)\left(\frac{P^2+4Q}{M}\right) \\ &= \left(\frac{Q}{7}\right)\left(\frac{Q}{M}\right) = -1, \end{aligned}$$

a contradiction.

Subcase I(ii): Assume that $\left(\frac{Q}{7}\right)\left(\frac{Q}{M}\right) = 1$. If $n = 4q+1$, then set $n = 4q+1 = 2\cdot 2^r a + 1$ with $2 \nmid a$ and $r \geq 1$. Therefore by Theorem 2.1,

$$7kx^2 = V_n = V_{4q+1} = V_{2\cdot 2^r a+1} \equiv -Q^{2^r a}V_1 \pmod{V_{2^r}},$$

implying that

$$7x^2 \equiv -MQ^{2^r a} \pmod{V_{2^r}}.$$

Hence,

$$1 = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{7}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right).$$

We first assume that $M \equiv 5, 7 \pmod 8$. Then $\left(\frac{7}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right) = 1$ by (2.17) and $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (2.6). So, the above is impossible. Assume now that $M \equiv 1, 3 \pmod 8$ and also $Q \equiv 1, 5 \pmod 8$. If we write $n = 4q+1 = 4(q+1) - 3 = 2\cdot 2^r a - 3$ with a odd and $r \geq 1$, then by Theorem 2.1,

$$7kx^2 = V_n = V_{2\cdot 2^r a-3} \equiv Q^{2^r a-3}P(P^2+3Q) \pmod{V_{2^r}},$$

implying that

$$7x^2 \equiv Q^{2^r a-3}M(P^2+3Q) \pmod{V_{2^r}}.$$

This shows that

$$1 = \left(\frac{7}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right)\left(\frac{Q}{V_{2^r}}\right)\left(\frac{P^2+3Q}{V_{2^r}}\right).$$

But this is impossible since $\left(\frac{7}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right) = -1$, $\left(\frac{Q}{V_{2^r}}\right) = 1$, and $\left(\frac{P^2+3Q}{V_{2^r}}\right) = 1$ by (2.17), (2.12), and (2.13).

Now assume that $Q \equiv 3, 7 \pmod 8$. Let $n = 4q+1$ and $3 \mid q$. Then $q = 3t$ for some $t > 0$ and therefore $n = 12t+1$. Thus,

$$7kx^2 = V_n = V_{12t+1} = V_{2\cdot 6t+1} \equiv V_1 \equiv P \pmod 8,$$

implying that

$$7x^2 \equiv M \pmod 8.$$

Since $M \equiv 1, 3 \pmod 8$, the above congruence becomes $7x^2 \equiv 1, 3 \pmod 8$, which is impossible.

Let $n = 4q + 1$ and $3 \nmid q$. Then either $n = 12t + 5$ or $n = 12t + 9$ for some $t \geq 0$. Assume that $n = 12t + 5$ and $Q \equiv 7 \pmod 8$. Hence, we have $7kx^2 = V_n = V_{12t+5} \equiv V_5 \pmod 8$ by (2.11). Using the fact that $P = kM$ and $M \equiv 1, 3 \pmod 8$, we readily obtain

$$7x^2 \equiv 6 + 5Q \pmod 8 \quad \text{or} \quad 7x^2 \equiv 2 + 7Q \pmod 8.$$

However, both the congruences above are impossible since $Q \equiv 7 \pmod 8$.

Now assume that $n = 12t + 5$ and $Q \equiv 3 \pmod 8$. If we write $n = 12t + 5 = 2 \cdot 2^r \cdot a + 5$ with a odd and $r \geq 1$, then we get

$$7kx^2 = V_{12t+5} = V_{2\cdot 2^r \cdot a+5} \equiv -Q^{2^r a} V_5 \pmod{V_{2^r}},$$

which implies

$$x^2 \equiv -7Q^{2^r \cdot a} MA \pmod{V_{2^r}},$$

where $A = P^4 + 5P^2Q + 5Q^2$. This shows that

$$1 = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{7}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right)\left(\frac{A}{V_{2^r}}\right).$$

But this is impossible since $\left(\frac{-1}{V_{2^r}}\right) = -1$, $\left(\frac{7}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right) = -1$, and $\left(\frac{A}{V_{2^r}}\right) = -1$ by (2.6), (2.17), and (2.20), respectively.

Now assume that $n = 12t + 9$ and $Q \equiv 3, 7 \pmod 8$. Then by (2.11), we immediately have $V_n = V_{12t+9} \equiv V_9 \equiv 2P, 6P \pmod 8$, i.e., $7kx^2 \equiv 2P, 6P \pmod 8$, which implies $7x^2 \equiv 2M, 6M \pmod 8$. Since $M \equiv 1, 3 \pmod 8$, we get $x^2 \equiv 2, 6 \pmod 8$, which is impossible.

Now let $n = 4q + 3$. Assume that $Q \equiv 1, 5 \pmod 8$. Writing $n = 4q + 3 = 2 \cdot 2^r a + 3$ with a odd and $r \geq 1$, we get

$$7kx^2 = V_n = V_{2\cdot 2^r a+3} \equiv -Q^{2^r a} V_3 \pmod{V_{2^r}},$$

that is,

$$7x^2 \equiv -Q^{2^r a} M(P^2 + 3Q) \pmod{V_{2^r}}$$

by (2.9). This shows that

$$1 = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{7}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right)\left(\frac{P^2 + 3Q}{V_{2^r}}\right).$$

Assume that $M \equiv 5, 7 \pmod 8$. Since $Q \equiv 1, 5 \pmod 8$, it follows from (2.6), (2.17), and (2.13) that

$$1 = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{7}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right)\left(\frac{P^2 + 3Q}{V_{2^r}}\right) = (-1)(1)(1) = -1,$$

a contradiction.

Now assume that $M \equiv 1, 3 \pmod 8$. Putting $n = 4q+3 = 4(q+1)-1 = 2\cdot 2^r a-1$ with a odd and $r \geq 1$ and using (2.1) and (2.9) gives $7x^2 \equiv Q^{2^r a-1}M \pmod{V_{2^r}}$, implying that $1 = \left(\frac{7QM}{V_{2^r}}\right)$. Since $Q \equiv 1, 5 \pmod 8$, it follows from (2.12) and (2.17) that

$$1 = \left(\frac{Q}{V_{2^r}}\right)\left(\frac{7}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right) = (1)(-1) = -1,$$

a contradiction.

Now assume that $Q \equiv 3, 7 \pmod 8$. If $Q \equiv 7 \pmod 8$, then it can be easily seen by (2.11) that $7kx^2 = V_n \equiv 6P, P \pmod 8$, which implies that $x^2 \equiv 2M, 7M \pmod 8$. But this congruence does not hold for $M \equiv 1, 3, 5 \pmod 8$. If $Q \equiv 3 \pmod 8$ and $n \not\equiv 5 \pmod 6$, then again we have $x^2 \equiv 6M, 7M \pmod 8$, which is also impossible for $M \equiv 1, 3, 5 \pmod 8$. So, we have $M \equiv 7 \pmod 8$. Writing $n = 4q+3 = 2\cdot 2^r a+3$ with a odd and $r \geq 1$, we readily see by (2.9) that

$$7kx^2 = V_{4q+3} = V_{2\cdot 2^r a+3} \equiv -Q^{2^r a}V_3 \pmod{V_{2^r}},$$

which implies

$$7x^2 \equiv -Q^{2^r a}M(P^2+3Q) \pmod{V_{2^r}}.$$

This shows that

$$1 = \left(\frac{-7M(P^2+3Q)}{V_{2^r}}\right).$$

This is impossible for $r \geq 2$, since

$$1 = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{7}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right)\left(\frac{P^2+3Q}{V_{2^r}}\right) = (-1)(1)(1) = -1$$

by (2.6), (2.17), and (2.13).

Now assume that $r = 1$. Then $n = 4a+3$ with a odd. Writing $n = 4(a+1)-1 = 2\cdot 2^t u-1$ with $2 \nmid u$ and $t \geq 2$, by (2.9) we have

$$7kx^2 = V_n = V_{2\cdot 2^t u-1} \equiv Q^{2^t u-2}PQ \pmod{V_{2^t}},$$

that is,

$$7x^2 \equiv Q^{2^t u-2}QM \pmod{V_{2^t}}.$$

However, this is also impossible since

$$1 = \left(\frac{Q}{V_{2^t}}\right)\left(\frac{7}{V_{2^t}}\right)\left(\frac{M}{V_{2^t}}\right) = (-1)(1) = -1$$

by (2.12) and (2.17).

Lastly, let $Q \equiv 3 \pmod 8$ and $n = 6a+5$ for some $a > 0$. Then either $n = 12t+5$ for some $t > 0$ or $n = 12t+11$ for some $t \geq 0$. Hence by (2.11), we get $7kx^2 = V_n \equiv V_5, V_{11} \equiv 5P \pmod 8$, i.e., $x^2 \equiv 3M \pmod 8$, which shows that $M \equiv 3 \pmod 8$. Assume that $n = 12t+5$. Then setting $n = 12t+5 = 2\cdot 2^r b+5$ with b odd and $r \geq 1$ gives $7kx^2 = V_n \equiv -Q^{2^r b}V_5 \pmod{V_{2^r}}$ by

(2.9) and thus we immediately obtain $7x^2 \equiv -Q^{2^r b}MA \pmod{V_{2^r}}$, where $A = P^4 + 5P^2Q + 5Q^2$. This shows that

$$1 = \left(\frac{-7MA}{V_{2^r}}\right).$$

However, this is impossible since

$$1 = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{7}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right)\left(\frac{A}{V_{2^r}}\right) = (-1)(-1)(-1) = -1$$

by (2.6), (2.17), and (2.20).

Assume that $n = 12t + 11$. We can write $n = 4c + 3$ for some $c > 0$. If c is odd, then $n = 4(c+1) - 1 = 8b - 1$ for some $b > 0$. Then by (2.9),

$$7kx^2 = V_n = V_{8b-1} \equiv -Q^{4b-1}V_1 \pmod{V_2},$$

that is,

$$7x^2 \equiv -Q^{4b-2}QM \pmod{V_2}.$$

This gives

$$1 = \left(\frac{-7QM}{V_2}\right).$$

But this is impossible since $\left(\frac{-1}{V_2}\right) = -1$ by (2.6), $\left(\frac{7}{V_2}\right)\left(\frac{M}{V_2}\right) = -1$ by (2.17), and $\left(\frac{Q}{P^2+2Q}\right) = (-1)\left(\frac{P^2+2Q}{Q}\right) = -1$.

Now assume that c is even, $c = 2^r b$, $r \geq 1$ and b is odd, say. Then $n = 2 \cdot 2^{r+1}b + 3$. If $r \geq 2$, then by (2.9),

$$7kx^2 = V_n = V_{2\cdot 2^{r+1}b+3} \equiv Q^{2^{r+1}b}V_3 \pmod{V_{2^r}},$$

implying that

$$7x^2 = Q^{2^{r+1}b}M(P^2 + 3Q) \pmod{V_{2^r}}.$$

But this is also impossible since $\left(\frac{7}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right) = -1$ by (2.17) and $\left(\frac{P^2+3Q}{V_{2^r}}\right) = 1$ by (2.13).

Lastly, assume that $r = 1$. Then $n = 8b + 3$ with b odd. Writing $n = 8(b+1) - 5 = 2 \cdot 2^s u - 5$ with $2 \nmid u$ and $s \geq 3$, by (2.9), we have

$$7kx^2 = V_n = V_{2\cdot 2^s u-5} \equiv Q^{2^s u-5}V_5 \pmod{V_{2^s}},$$

that is,

$$7x^2 \equiv Q^{2^s u-5}MA \pmod{V_{2^s}},$$

where $A = P^4 + 5P^2Q + 5Q^2$. This shows that

$$1 = \left(\frac{7}{V_{2^s}}\right)\left(\frac{M}{V_{2^s}}\right)\left(\frac{Q}{V_{2^s}}\right)\left(\frac{A}{V_{2^s}}\right).$$

But this is impossible since $\left(\frac{7}{V_{2^s}}\right)\left(\frac{M}{V_{2^s}}\right) = -1$ by (2.17), $\left(\frac{Q}{V_{2^s}}\right) = -1$ by (2.12), and $\left(\frac{A}{V_{2^s}}\right) = -1$ by (2.20). Hence, we obtain $a = 0$ and therefore $n = 5$.

Case II: Let $7 \nmid P$. Since $7 \mid V_n$, it follows from Lemma 2.6 that $n = 12q + 3$ for some $q > 0$ or $n = 12q + 9$ for some $q \geq 0$. The remainder of the proof is split into two subcases.

Subcase II(i): Assume that $\left(\frac{Q}{M}\right) = -1$. Then by (2.5), we have

$$7kx^2 = V_n \equiv -Q^{6q+1}P \text{ or } Q^{6q+4}P \pmod{P^2 + 4Q},$$

that is,

$$7x^2 \equiv -Q^{6q+1}M \text{ or } Q^{6q+4}M \pmod{P^2 + 4Q}.$$

This shows that

$$1 = \left(\frac{-7QM}{P^2 + 4Q}\right) \quad \text{or} \quad 1 = \left(\frac{7M}{P^2 + 4Q}\right).$$

A simple calculation shows that $\left(\frac{7}{P^2+4Q}\right) = \left(\frac{P^2+4Q}{7}\right) = \left(\frac{2}{7}\right)$ or $\left(\frac{4}{7}\right)$. This leads to $\left(\frac{7}{P^2+4Q}\right) = 1$. On the other hand, we easily see that $\left(\frac{-1}{P^2+4Q}\right) = 1$, $\left(\frac{Q}{P^2+4Q}\right) = \left(\frac{P^2+4Q}{Q}\right) = \left(\frac{P^2}{Q}\right) = 1$, and $\left(\frac{M}{P^2+4Q}\right) = \left(\frac{P^2+4Q}{M}\right) = \left(\frac{Q}{M}\right) = -1$. Hence, we have a contradiction in both cases above.

Subcase II(ii): Assume that $\left(\frac{Q}{M}\right) = 1$. If $n = 12q + 3$, then by (2.11),

$$7kx^2 = V_n = V_{12q+3} \equiv V_3 \pmod 8.$$

Assume that $Q \equiv 3, 7 \pmod 8$. Then the congruence above becomes

$$7x^2 \equiv 2M, 6M \pmod 8,$$

implying that

$$x^2 \equiv 2M, 6M \pmod 8.$$

But this is impossible for any values of $M \equiv 1, 3, 5, 7 \pmod 8$. Thus $Q \equiv 1, 5 \pmod 8$. In addition, assume that $M \equiv 1, 3 \pmod 8$. Writing $n = 12q + 3 = 2 \cdot 2^r a + 3$ with a odd and $r \geq 1$, we get

$$7kx^2 = V_n = V_{2 \cdot 2^r a + 3} \equiv -Q^{2^r a}P(P^2 + 3Q) \pmod{V_{2^r}},$$

that is,

$$7x^2 \equiv -Q^{2^r a}M(P^2 + 3Q) \pmod{V_{2^r}}.$$

This shows that

$$1 = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{7}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right)\left(\frac{P^2 + 3Q}{V_{2^r}}\right).$$

But this is impossible since $\left(\frac{-1}{V_{2^r}}\right) = -1$, $\left(\frac{7}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right) = 1$, and $\left(\frac{P^2+3Q}{V_{2^r}}\right) = 1$ by (2.6), (2.19), and (2.13), respectively.

Now assume that $M \equiv 5, 7 \pmod 8$. On the other hand, suppose $n = 12q + 3$ and $2 \mid q$. Then $n = 24t + 3$ for some $t > 0$. By (2.9) this gives

$$7kx^2 = V_n = V_{24t+3} \equiv Q^{12t}V_3 \pmod{V_2},$$

that is,

$$7x^2 \equiv Q^{12t} M(P^2 + 3Q) \pmod{V_2}.$$

As a consequence,

$$1 = \left(\frac{7}{V_2}\right)\left(\frac{M}{V_2}\right)\left(\frac{P^2+3Q}{V_2}\right).$$

But this is impossible since $\left(\frac{7}{V_2}\right)\left(\frac{M}{V_2}\right) = -1$ and $\left(\frac{P^2+3Q}{V_2}\right) = \left(\frac{Q}{V_2}\right) = 1$ by (2.19) and (2.12), respectively.

Now suppose $2 \nmid q$. Then $n = 24t + 15$. We can write $n = 8c - 1$ for some $c > 0$. Setting $n = 8c - 1 = 2 \cdot 2^r a - 1$ with $2 \nmid a$ and $r \geq 2$, we get

$$7kx^2 = V_n = V_{2 \cdot 2^r a - 1} \equiv Q^{2^r a - 1} P \pmod{V_{2^r}}$$

by (2.1) and (2.9). This implies that

$$7x^2 \equiv Q^{2^r a - 2} QM \pmod{V_{2^r}}.$$

However, this is impossible since $\left(\frac{7}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right) = -1$ by (2.19) and $\left(\frac{Q}{V_{2^r}}\right) = 1$ by (2.12).

Now assume that $n = 12q + 9$. If $Q \equiv 3, 7 \pmod 8$, then $V_9 \equiv 2P, 6P \pmod 8$ and therefore $7kx^2 = V_n = V_{12q+9} \equiv V_9 \equiv 2P, 6P \pmod 8$ by (2.11), implying that $7x^2 \equiv 2M, 6M \pmod 8$. This is impossible for any $M \equiv 1, 3, 5, 7 \pmod 8$. Hence, $Q \equiv 1, 5 \pmod 8$. We also assume that $M \equiv 1, 3 \pmod 8$. Set $n = 12q + 9 = 4c + 1$ for some $c > 0$.

If c is odd, then $n = 4(c+1) - 3 = 8b - 3$ for some $b > 0$. Then by (2.1) and (2.9),

$$7kx^2 = V_n = V_{8b-3} \equiv -Q^{4b-3} P(P^2 + 3Q) \pmod{V_2},$$

that is,

$$7x^2 \equiv -Q^{4b-4} Q^2 M \pmod{V_2}.$$

This shows that

$$1 = \left(\frac{-1}{V_2}\right)\left(\frac{7}{V_2}\right)\left(\frac{M}{V_2}\right).$$

But this is impossible since $\left(\frac{-1}{V_2}\right) = -1$ and $\left(\frac{7}{V_2}\right)\left(\frac{M}{V_2}\right) = 1$ by (2.6) and (2.19), respectively.

Now assume that c is even. Since $n = 4c + 1$, it follows that $n = 8b + 1$ for some $b > 0$ and writing $n = 8b + 1 = 2 \cdot 2^r a + 1$ with $2 \nmid a$ and $r \geq 2$, we get

$$7kx^2 = V_n = V_{2 \cdot 2^r a + 1} \equiv -Q^{2^r a} V_1 \pmod{V_{2^r}}$$

by (2.9). This implies that

$$7x^2 \equiv -Q^{2^r a} M \pmod{V_{2^r}}.$$

But this is also impossible since $\left(\frac{7}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right) = 1$ by (2.19) and $\left(\frac{-1}{V_{2^r}}\right) = -1$ (2.6).

Assume now that $M \equiv 5, 7 \pmod 8$. Let q be even for the case of $n = 12q + 9$. Then using (2.9), we readily obtain

$$7kx^2 = V_n = V_{12q+9} \equiv Q^{6q} V_9 \pmod{V_2},$$

that is,

$$7x^2 \equiv Q^{6q} MB(P^2 + 3Q) \pmod{V_2},$$

where $B = P^6 + 6P^4Q + 9P^2Q^2 + 3Q^3$. This means that

$$1 = \left(\frac{7}{V_2}\right)\left(\frac{M}{V_2}\right)\left(\frac{P^2+3Q}{V_2}\right)\left(\frac{B}{V_2}\right).$$

However, this is impossible since $\left(\frac{7}{V_2}\right)\left(\frac{M}{V_2}\right) = -1$ by (2.19), $\left(\frac{P^2+3Q}{V_2}\right) = \left(\frac{Q}{V_2}\right) = 1$ and $\left(\frac{B}{V_2}\right) = \left(\frac{Q^3}{V_2}\right) = \left(\frac{Q}{V_2}\right) = 1$ by (2.12). So, q is odd, $q = 2t + 1$, say. Therefore $n = 24t + 21 = 8(3t + 3) - 3 = 8b - 3 = 2 \cdot 2^r a - 3$ with $2 \nmid a$ and $r \geq 2$. By (2.1) and (2.9), we readily obtain

$$7kx^2 = V_n = V_{2\cdot 2^r a-3} \equiv Q^{4b-3} P(P^2 + 3Q) \pmod{V_{2^r}},$$

that is,

$$7x^2 \equiv Q^{4b-4} QM(P^2 + 3Q) \pmod{V_{2^r}}.$$

However, this is also impossible since $\left(\frac{7}{V_{2^r}}\right)\left(\frac{M}{V_{2^r}}\right) = -1$, $\left(\frac{Q}{V_{2^r}}\right) = 1$, and $\left(\frac{P^2+3Q}{V_{2^r}}\right) = 1$ by (2.19), (2.12), and (2.13), respectively.

This completes the proof of Theorem 3.1. ∎

Corollary 3.2. *If $V_n = 21x^2$ for some integer x, then $n = 1, 3,$ or 5.*

Proof. Let $3 \mid P$. Then by Theorem 3.1, we have $n = 1, 3,$ or 5.

Now, let $3 \nmid P$. Assume that $7 \mid P$. Since $7 \mid V_n$, it follows from Lemma 2.6 that n is odd. On the other hand, since $3 \mid V_n$ and $3 \nmid P$, it follows from Lemma 2.5 that $n \equiv 2 \pmod 4$, a contradiction.

Assume that $7 \nmid P$. Since $3 \mid V_n$ and $7 \mid V_n$, it follows from Lemmas 2.5 and 2.6 that $n \equiv 2 \pmod 4$. Writing $n = 12q + 2$, $n = 12q + 6$, or $n = 12q + 10$ gives immediately, by (2.11),

$$21x^2 = V_n \equiv V_2, V_6, V_{10} \pmod 8, \tag{3.1}$$

respectively. A simple calculation shows that $V_2 \equiv 3 \pmod 8$, $V_6 \equiv 2 \pmod 8$, and $V_{10} \equiv 3 \pmod 8$ when $Q \equiv 1, 5 \pmod 8$ and $V_2 \equiv 7 \pmod 8$, $V_6 \equiv 2 \pmod 8$, and $V_{10} \equiv 7 \pmod 8$ when $Q \equiv 3, 7 \pmod 8$. Thus, (3.1) implies

$$21x^2 \equiv 2, 3, 7 \pmod 8,$$

that is,

$$x^2 \equiv 2, 3, 7 \pmod 8,$$

which is impossible. ∎

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Olcay Karaatlı
Department of Mathematics
Faculty of Arts and Sciences
Sakarya University
54187 Sakarya, Turkey
E-mail: okaraatli@sakarya.edu.tr