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A characterization of $\mathbf{\Sigma}^1_2$ sets

by

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Abstract. We show that a subset X of a given Polish space $\mathcal{X}$ is $\mathbf{\Sigma}^1_2$ iff there is an open set $O \subseteq \mathcal{X} \times [\omega]^\omega$ such that

$$X = \{x \in \mathcal{X} \colon \exists r \in [\omega]^\omega \ \{x\} \times [r]^\omega \subseteq O\}.$$

This implies that if a set $U \subseteq \omega^\omega \times (\mathcal{X} \times [\omega]^\omega)$ is universal for G_δ subsets of $\mathcal{X} \times [\omega]^\omega$, then the set of all $(v, x) \in \omega^\omega \times \mathcal{X}$ such that the section U_{vx} has nonempty interior in the Ellentuck topology is universal for $\mathbf{\Sigma}^1_2$ subsets of $\mathcal{X}$. It follows that the σ-ideal of meager sets in the Ellentuck topology is not $\mathbf{\Sigma}^1_2$ on G_δ, a fact established recently by Sabok (2012) with the help of Kleene's Recursion Theorem.

In the *Ramsey space* $\mathcal{R} = [\omega]^\omega$ of all infinite sets of integers, the *Ellentuck topology*, given by the basis

$$[n, r] = \{s \in \mathcal{R} : r \cap n \subseteq s \subseteq r\}, \qquad n \in \omega,\ r \in \mathcal{R},$$

is the least common extension of the *Ramsey topology*, given by the basis

$$[r]^\omega = \{s \in \mathcal{R} : s \subseteq r\}, \qquad r \in \mathcal{R},$$

and the *Baire topology*, induced from the Baire space $\mathcal{N} = \omega^\omega$ by identifying any $r \in \mathcal{R}$ with its increasing enumeration $(r_i)_{i\in\omega}$.

See [2] for more about the Ellentuck topology. Below, the default topology on $\mathcal{R}$ is the Baire topology. Whenever we want to use the Ellentuck or Ramsey topology, we will explicitly indicate this, e.g., we write $\mathrm{int}_{\mathrm{Ell}}$ to denote the interior operator in the Ellentuck topology.

For any sets Q, a, and b, let $Q_a = \{c \colon (a, c) \in Q\}$ and $Q_{ab} = (Q_a)_b$.

THEOREM. *A subset X of a given Polish space $\mathcal{X}$ is $\mathbf{\Sigma}^1_2$ iff there exists an open set $O \subseteq \mathcal{X} \times \mathcal{R}$ such that all sections O_x, $x \in \mathcal{X}$, are closed in the*

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Ramsey topology and

$$X = \{x \in \mathcal{X}\colon \operatorname{int}_{\mathrm{Ram}} O_x \neq \emptyset\}.$$

Proof. $\Leftarrow$: Note that $\operatorname{int}_{\mathrm{Ram}} O_x \neq \emptyset$ iff

$$\exists r \in \mathcal{R}\ \forall s \in \mathcal{R} \quad s \subseteq r \ \Rightarrow\ s \in O_x,$$

which is a $\mathbf{\Sigma}^1_2$ statement. (The closedness of the sections is irrelevant for this implication.)

$\Rightarrow$: In the Cantor space $\mathcal{C} = 2^\omega$, choose a dense set $\{c_n\}_{n<\omega}$, and consider the set

$$\mathcal{S} = \{s \in \mathcal{R}\colon \forall i \in \omega\ \ c_{s_{i+1}}|i = c_{s_i}|i\},$$

and the functions

$$\ell(s) = \bigcup_i c_{s_i}|i, \qquad s \in \mathcal{S},$$
$$e(r) = \{t \in \mathcal{N}\colon \forall i < \omega\ \ t(i) \le r_i\}, \qquad r \in \mathcal{R}.$$

Note that:

(1) $\mathcal{S}$ is closed (in the default Baire topology), and $\mathcal{S}$ is dense open in the Ramsey topology.
(2) ℓ is a continuous function from $\mathcal{S}$ onto $\mathcal{C}$.
(3) The fibers of ℓ are open in the Ramsey topology, i.e.,

$$s \in \mathcal{S} \wedge r \in [s] \ \Rightarrow\ r \in \mathcal{S} \wedge \ell(r) = \ell(s).$$

All this follows because $\mathcal{C}$ is compact and $s \in \mathcal{S}$ just means that the sequence (c_{s_i}) is a Cauchy sequence converging to $\ell(s)$ in a controlled way.

Note also that:

(4) e is a continuous function from $\mathcal{R}$ into the hyperspace of $\mathcal{N}$ (the space of all compact subsets with the Vietoris topology) such that

$$\forall s \in \mathcal{R}\ \ \mathcal{N} = \bigcup_{r\in[s]} e(r).$$

Fix now $X \in \mathbf{\Sigma}^1_2(\mathcal{X})$. Since X is the projection of a $\mathbf{\Pi}^1_1$ subset of $\mathcal{X} \times \mathcal{C}$, there is a closed set $E \subseteq \mathcal{X} \times (\mathcal{C} \times \mathcal{N})$ such that

$$X = \{x \in \mathcal{X}\colon \exists c \in \mathcal{C}\ \ E_{xc} = \emptyset\}.$$

We have

$$\begin{aligned}
&\text{by (2)} \quad \exists c \in \mathcal{C}\ \ E_{xc} = \emptyset && \Leftrightarrow \exists r \in \mathcal{S}\ \ E_{x\ell(r)} = \emptyset\\
&\text{by (4)} && \Leftrightarrow \exists r \in \mathcal{S}\ \ \forall s \in [r]^\omega\ \ e(s) \cap E_{x\ell(r)} = \emptyset\\
&\text{by (3)} && \Leftrightarrow \exists r \in \mathcal{S}\ \ \forall s \in [r]^\omega\ \ e(s) \cap E_{x\ell(s)} = \emptyset\\
&\text{by (1)} && \Leftrightarrow \exists r \in \mathcal{R}\ \ [r]^\omega \subseteq \{s \in \mathcal{R}\colon s \in \mathcal{S} \Rightarrow e(s) \cap E_{x\ell(s)} = \emptyset\}.
\end{aligned}$$

Let

$$O = \{(x,s) \in \mathcal{X} \times \mathcal{R} \colon s \in \mathcal{S} \Rightarrow e(s) \cap E_{x\ell(s)} = \emptyset\}.$$

We check that O is open. Since $\mathcal{S}$ is closed in $\mathcal{R}$, it is enough to see that $O \cap (\mathcal{X} \times \mathcal{S})$ is relatively open in $\mathcal{X} \times \mathcal{S}$. By the basic properties of the Vietoris topology, if $\mathcal{Y}$ and $\mathcal{Z}$ are any topological spaces, $\mathcal{K}$ is the hyperspace of $\mathcal{Z}$, $F \subseteq \mathcal{Y} \times \mathcal{Z}$ is closed, and $f \colon \mathcal{Y} \to \mathcal{Y}$ and $g \colon \mathcal{Y} \to \mathcal{K}$ are continuous, then the set of $y \in \mathcal{Y}$ with $g(y) \cap F_{f(y)} = \emptyset$ is open. Use this for $\mathcal{Y} = \mathcal{X} \times \mathcal{S}$, $\mathcal{Z} = \mathcal{N}$, $F = E$, and f and g given by $(x,r) \mapsto (x, \ell(r))$ and $(x,r) \mapsto e(r)$.

Finally, we check that all sections O_x are closed in the Ramsey topology. Suppose $r \notin O_x$. Then $r \in \mathcal{S}$ and $e(r) \cap E_{x\ell(r)} \neq \emptyset$. Note that the whole open neighborhood $[r]$ of r is disjoint from O_x: if $s \in [r]$, then $\ell(s) = \ell(r)$ and $e(s) \supseteq e(r)$, so $e(s) \cap E_{x\ell(s)} \neq \emptyset$, and thus $s \notin O_x$. ∎

COROLLARY. *A subset X of a given Polish space $\mathcal{X}$ is $\mathbf{\Sigma}^1_2$ iff there exists a G_δ set $G \subseteq \mathcal{X} \times \mathcal{R}$ such that all sections G_x, $x \in \mathcal{X}$, are closed in the Ramsey topology and*

$$X = \{x \in \mathcal{X} \colon \operatorname{int}_{\mathrm{Ell}} G_x \neq \emptyset\}.$$

Proof. $\Leftarrow$: Note that $\operatorname{int}_{\mathrm{Ell}} G_x \neq \emptyset$ iff

$$\exists n \in \omega\ \exists r \in \mathcal{R}\ \forall s \in \mathcal{R} \quad r \cap n \subseteq s \subseteq r \ \Rightarrow\ s \in G_x,$$

which is a $\mathbf{\Sigma}^1_2$ statement. (The closedness of the sections is irrelevant for this implication.)

$\Rightarrow$: The Ramsey interior of $Q \subseteq \mathcal{R}$ is nonempty iff the Ellentuck interior of

$$Q^* = \{r \in \mathcal{R} \colon \forall n \in \omega \ \ r \setminus n \in Q\}$$

is nonempty. (We have: $[r] \subseteq Q \ \Rightarrow\ [0,r] \subseteq Q^*$ and $[r \setminus m] \subseteq Q \ \Leftarrow\ [m,r] \subseteq Q^*$.)

Use the O of the Theorem and define G by $G_x = (O_x)^*$ for $x \in \mathcal{X}$. ∎

REMARK 1. The Corollary is a slightly rephrased result of Sabok [3] that $X \in \mathbf{\Sigma}^1_2(\mathcal{X})$ iff there exists a G_δ set $G \subseteq \mathcal{X} \times \mathcal{R}$ such that

$$X = \{x \in \mathcal{X} \colon G_x \text{ is Ellentuck nonmeager}\}.$$

Just recall that: Borel sets (in the Baire topology) have the Baire property in the Ellentuck topology; and in the Ellentuck topology, meager sets are nowhere dense. It follows that in the Ellentuck topology, G_x has nonempty interior iff it is nonmeager.

REMARK 2. Suppose that $\mathcal{V}$ and $\mathcal{X}$ are Polish spaces. Let a set $U \subseteq \mathcal{V} \times (\mathcal{X} \times \mathcal{R})$ be universal for open subsets of $\mathcal{X} \times \mathcal{R}$, i.e., for any open set $H \subseteq \mathcal{X} \times \mathcal{R}$ there is $v \in \mathcal{V}$ with $H = U_v$. Then the set

$$P = \{(v,x) \in \mathcal{V} \times \mathcal{X} \colon \operatorname{int}_{\mathrm{Ram}} U_{vx} \neq \emptyset\}$$

is universal for $\mathbf{\Sigma}^1_2$ subsets of $\mathcal{X}$.

Indeed, given a $\mathbf{\Sigma}^1_2$ set $X \subseteq \mathcal{X}$, take O given by the Corollary, find $v \in \mathcal{V}$ with $U_v = O$, and note that

$$x \in X \;\Leftrightarrow\; \operatorname{int}_{\mathrm{Ram}} O_x \neq \emptyset \;\Leftrightarrow\; \operatorname{int}_{\mathrm{Ram}} U_{vx} \neq \emptyset \;\Leftrightarrow\; x \in P_v.$$

Remark 3. By the $\Leftarrow$ argument of the Corollary, given a Polish space $\mathcal{X}$ and a $\mathbf{\Pi}^1_1$ set $Q \subseteq \mathcal{X} \times \mathcal{R}$, the set

$$\{x \in \mathcal{X} \colon \operatorname{int}_{\mathrm{Ell}} Q_x \neq \emptyset\}$$

is $\mathbf{\Sigma}^1_2$. This is also true if Q is $\mathbf{\Sigma}^1_1$ (folklore, see [3, Lemma 5]). Similar remarks apply to the Ramsey topology.

Remark 4. Becker–Kahane–Louveau [1] give the following characterization: $X \in \mathbf{\Sigma}^1_2(\mathcal{C})$ iff there exist continuous $f_n \colon \mathcal{C} \times \mathcal{C} \to 2$, $n \in \omega$, such that

$$X = \{x \in \mathcal{C} \colon \exists r \in \mathcal{R}\ \ \langle f_n(x,\cdot)\rangle_{n\in r} \text{ converges pointwise to } 0\}. \tag{1}$$

This can be deduced from our characterization as follows.

Fix $X \in \mathbf{\Sigma}^1_2(\mathcal{C})$. Pick an open $O \subseteq \mathcal{C} \times \mathcal{R}$ with all vertical sections closed in the Ramsey topology such that

$$X = \{x \in \mathcal{C} \colon \operatorname{int}_{\mathrm{Ram}} O_x \neq \emptyset\}.$$

Find clopen $C^0 \subseteq C^1 \subseteq C^2 \subseteq \cdots \subseteq \mathcal{C} \times \mathcal{C}$ such that

$$O = (\mathcal{C} \times \mathcal{R}) \cap \bigcup_n C^n.$$

(Subsets of ω are identified with their characteristic functions here.)

Define a continuous function $f_n \colon \mathcal{C} \times \mathcal{C} \to 2$ by

$$f_n(x,s) = 1 \;\Leftrightarrow\; n \in s \notin C^n_x.$$

For $\subseteq$ in (1), fix $x \in X$ and take $r \in \mathcal{R}$ with $[r]^\omega \subseteq O_x$. Given $s \subseteq \omega$, to see that $\langle f_n(x,s)\rangle_{n\in r}$ converges to 0 note that if $s \cap r$ is finite, then the clause $n \in s$ fails for almost all $n \in r$. If $s \cap r$ is infinite, then $s \cap r \in O_x$ by $[r]^\omega \subseteq O_x$, so $s \in O_x$ since O_x is closed in the Ramsey topology. As $O_x = \bigcup_n C^n_x$, the clause $s \notin C^n_x$ fails for almost all $n \in \omega$.

For $\supseteq$ in (1), fix $x \notin X$ and consider any $r \in \mathcal{R}$. Choose $s \in [r]^\omega \setminus O_x$. Then for infinitely many $n \in r$ (all $n \in s$), we have $n \in s \notin C^n_x$, so $f_n(x,s) = 1$. It follows that $\langle f_n(x,s)\rangle_{n\in r}$ does not converge to 0.

Remark 5. Note that we have not used much the assumption that $\mathcal{X}$ is a Polish space; it can be any topological space (with obvious adaptations of the notions of $\mathbf{\Sigma}^1_2$ and $\mathbf{\Pi}^1_1$ sets).

Remark 6. The following two examples show that some care has to be exercised when playing with the Mathias forcing.

Let $\boldsymbol{L}$ be the constructible universe. Consider the Σ^1_2 set $X = \boldsymbol{L} \cap \mathcal{C}$. Then the G_δ set G of the Corollary is coded in $\boldsymbol{L}$ and

$$\begin{aligned} \boldsymbol{L} &\models \forall x \in \mathcal{C}\ \ G_x \text{ is Ellentuck nonmeager,}\\ \boldsymbol{V} &\models \forall x \in \mathcal{C} \setminus \boldsymbol{L}\ \ G_x \text{ is Ellentuck meager (in fact empty).} \end{aligned}$$

Let $\boldsymbol{M}$ be the minimal countable transitive model of ZFC. Consider the Σ^1_2 set

$$X = \{x \in \mathcal{C} \colon x = x \wedge \exists m \in \mathcal{C}\ \ m \text{ codes } \boldsymbol{M}\}.$$

Then the G_δ set G of the Corollary is coded in $\boldsymbol{M}$ and

$$\begin{aligned} \boldsymbol{M} &\models \forall x \in \mathcal{C}\ \ G_x \text{ is Ellentuck meager (in fact empty),}\\ \boldsymbol{V} &\models \forall x \in \mathcal{C}\ \ G_x \text{ is Ellentuck nonmeager.} \end{aligned}$$

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