ANNALES
POLONICI MATHEMATICI
116.3 (2016)

Certain contact metrics satisfying the Miao–Tam critical condition

DHRITI SUNDAR PATRA (Kolkata) and
AMALENDU GHOSH (Chandannagar)

Abstract. We study certain contact metrics satisfying the Miao–Tam critical condition. First, we prove that a complete K-contact metric satisfying the Miao–Tam critical condition is isometric to the unit sphere S^{2n+1}. Next, we study (κ, μ)-contact metrics satisfying the Miao–Tam critical condition.

1. Introduction. In [MT1], Miao–Tam studied the critical points of the volume functional restricted to the space of constant scalar curvature metrics on a given compact manifold with boundary. They also derived a necessary and sufficient condition for a metric to be critical. This leads to the following definition:

DEFINITION 1.1. Let (M^n, g), $n > 2$, be a compact Riemannian manifold with a smooth boundary metric ∂M. Then g is said to be a *critical metric* if there exists a smooth function $\lambda : M^n \to \mathbb{R}$ such that

$$-(\Delta_g \lambda) g + \nabla_g^2 \lambda - \lambda S = g \tag{1.1}$$

on M and $\lambda = 0$ on ∂M, where Δ_g and $\nabla_g^2 \lambda$ are the Laplacian and the Hessian operator with respect to the metric g, and S is the $(0, 2)$ Ricci curvature of g. The function λ is known as the *potential function*.

For simplicity, these metrics will be called *Miao–Tam critical metrics* and equation (1.1) the *Miao–Tam critical condition*. Any Riemannian metric g satisfying (1.1) must have constant scalar curvature [MT1]. The existence of such metrics was confirmed on certain classes of warped product spaces which include the usual spatial Schwarzschild metrics and Ads–

2010 *Mathematics Subject Classification*: Primary 53C25, 53C21; Secondary 53D15.
Key words and phrases: contact metric manifolds, Miao–Tam critical condition, K-contact metric, (κ, μ)-contact metric.
Received 27 April 2015; revised 23 November 2015.
Published online 18 April 2016.

DOI: 10.4064/ap3704-11-2015

Schwarzschild metrics restricted to certain domains containing their horizon and bounded by two spherically symmetric spheres (cf. [MT2, Corollaries 3.1 and 3.2]).

Recently, Miao–Tam critical metrics have been studied under Einstein and conformally flat assumptions (see [MT2]). In particular, it was proved that any connected, compact, Einstein manifold with smooth boundary satisfying the Miao–Tam critical condition is isometric to a geodesic ball in a simply connected space form $\mathbb{R}^n$, $\mathbb{H}^n$ or $\mathbb{S}^n$. The same conclusion is true if one replaces the Einstein condition by the conformally flat assumption. The last result has been considered further under the Bach-flat assumption in dimension 4, which is weaker than being conformally flat (see e.g. [BDR]).

Let M be an almost contact metric manifold of dimension $2n+1$ with almost contact metric structure (φ, ξ, η). We note that a Riemannian metric g on an almost contact manifold M^{2n+1} is said to be an *almost contact metric* if

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

The metric g is known as the *associated metric* of the almost contact metric structure. Moreover, if $d\eta = g(\cdot, \varphi\, \cdot)$, then M is said to be a *contact metric manifold* (see Preliminaries). Since the metric involved in a contact metric structure is not just a Riemannian metric, we are interested in studying such metrics which satisfy the Miao–Tam critical condition. In Section 3, we study K-contact metrics satisfying the Miao–Tam critical condition and prove that these are isometric to the unit sphere S^{2n+1}. Next, we study (κ, μ)-metrics satisfying the Miao–Tam critical condition and prove that these are flat in dimension 3, and isometric to the trivial sphere bundle $E^{n+1} \times S^n(4)$ in higher dimensions.

2. Preliminaries. In this section, we recall some basic definitions and fundamental formulas for contact metric manifolds. A Riemannian manifold of dimension $2n+1$ is said to be a *contact manifold* if it admits a global 1-form η such that $\eta \wedge (d\eta)^n$ is non-vanishing everywhere on M. The 1-form η is known as the *contact form*. Corresponding to this η one can find a unit vector field ξ, called the *Reeb vector field*, such that $\eta(\xi) = 1$ and $d\eta(\xi, \cdot) = 0$. It is well known that every contact manifold admits an underlying almost contact structure (φ, ξ, η), where φ is a global tensor field of type $(1,1)$, such that

$$\eta(X) = g(X, \xi), \qquad \varphi\xi = 0, \qquad \eta \circ \varphi = 0, \qquad \varphi^2 = -I + \eta \otimes \xi.$$

Further, an almost contact structure is said to be a *contact metric* if it satisfies

$$d\eta(X, Y) = g(X, \varphi Y), \qquad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.1}$$

A Riemannian manifold M^{2n+1} together with the structure (φ,ξ,η,g) is said to be a *contact metric manifold*.

We now define two operators h and l by

$$h=\tfrac{1}{2}\mathcal{L}_\xi\varphi \quad \text{and} \quad l=R(\cdot,\xi)\xi.$$

These tensors are self-adjoint and satisfy $\operatorname{Tr} h=0$, $\operatorname{Tr} h\varphi=0$, $l\xi=0$ and $h\varphi=-\varphi h$. On a contact metric manifold the following formulas are valid [Bl]:

$$\nabla_X\xi=-\varphi X-\varphi hX, \tag{2.2}$$

$$g(Q\xi,\xi)=\operatorname{Tr} l=2n-\operatorname{Tr} h^2, \tag{2.3}$$

$$\nabla_\xi h=\varphi-\varphi h^2-\varphi l, \tag{2.4}$$

$$(\nabla_Y\varphi)X+(\nabla_{\varphi Y}\varphi)\varphi X=2g(Y,X)\xi-\eta(X)\big(Y+hY+\eta(Y)\xi\big). \tag{2.5}$$

If the vector field ξ is Killing (equivalently, $h=0$ or $\operatorname{Tr} l=2n$), then the contact metric manifold M is said to be *K-contact*. On a K-contact manifold the following formulas are known [Bl]:

$$\nabla_X\xi=-\varphi X, \tag{2.6}$$

$$Q\xi=2n\xi, \tag{2.7}$$

$$R(X,\xi)\xi=X-\eta(X)\xi, \tag{2.8}$$

$$R(\xi,X)Y=(\nabla_X\varphi)Y, \tag{2.9}$$

where ∇ is the operator of covariant differentiation of g, Q the Ricci operator associated with the $(0,2)$ Ricci tensor S, and R the Riemann curvature tensor of g. A contact metric structure on M is said to be *normal* if the almost complex structure on $M\times\mathbb{R}$ defined by

$$J(X,fd/dt)=(\varphi X-f\xi,\eta(X)d/dt),$$

where f is a real function on $M\times\mathbb{R}$, is integrable. Equivalently, a contact metric manifold is said to *Sasakian* if

$$(\nabla_X\varphi)Y=g(X,Y)\xi-\eta(Y)X. \tag{2.10}$$

An important characterization is that a contact metric manifold is a Sasakian manifold if and only if the curvature tensor satisfies

$$R(X,Y)\xi=\eta(Y)X-\eta(X)Y. \tag{2.11}$$

In [BKP], Blair et al. introduced the notion of a (κ,μ)-*contact manifold*, which is a contact metric manifold $M^{2n+1}(\varphi,\xi,\eta,g)$ whose curvature tensor satisfies

$$R(X,Y)\xi=\kappa\{\eta(Y)X-\eta(X)Y\}+\mu\{\eta(Y)hX-\eta(X)hY\} \tag{2.12}$$

for some real numbers (κ,μ). Applying the D-homothetic deformation [T]

$$\bar\eta=a\eta,\qquad \bar\xi=\frac{1}{a}\xi,\qquad \bar\varphi=\varphi,\qquad \bar g=ag+a(a-1)\eta\otimes\eta,$$

for a positive real constant a, to a contact metric manifold satisfying $R(X,Y)\xi = 0$, one obtains this type of manifold. Note that D-homothetic deformation preserves Sasakian, K-contact and (κ,μ)-contact structures. It is interesting to point out that the class of (κ,μ)-contact structures contains Sasakian manifolds (for $\kappa = 1$) and the trivial sphere bundle $E^{n+1} \times S^n(4)$ (for $\kappa = \mu = 0$). Examples of non-Sasakian (κ,μ)-contact manifolds are the tangent sphere bundles of Riemannian manifolds of constant curvature $\neq 1$. Further, equation (2.7) determines the curvature completely for $\kappa < 1$. For (κ,μ)-contact manifolds the following formulas are known [BKP]:

$$QX = [2(n-1) - n\mu]X + [2(n-1) + \mu]hX + [2(1-n) + n(2\kappa+\mu)]\eta(X)\xi, \tag{2.13}$$

$$h^2 = (\kappa - 1)\varphi^2, \tag{2.14}$$

where $\kappa \leq 1$. Moreover, the constant scalar curvature r in that class is given by

$$r = 2n(2(n-1) + \kappa - n\mu). \tag{2.15}$$

3. K-contact manifolds. In this section, we consider K-contact metrics satisfying the Miao–Tam critical condition. First, we prove the following

LEMMA 3.1. *Let (M^n, g) be a Riemannian manifold without boundary. Suppose there exists a non-constant smooth function λ on M which is a solution of the equation*

$$-(\Delta_g\lambda)g + \nabla_g^2\lambda - \lambda S = g. \tag{3.1}$$

Then the curvature tensor R can be expressed as

$$R(X,Y)D\lambda = (X\lambda)QY - (Y\lambda)QX + \lambda(\nabla_X Q)Y - \lambda(\nabla_Y Q)X + (Xf)Y - (Yf)X. \tag{3.2}$$

Proof. First, we note that (3.1) implies $\Delta_g\lambda = -\frac{1}{n-1}(r\lambda + n)$. Thus, equation (3.2) can be written as

$$\nabla_X D\lambda = \lambda QX + fX, \quad \text{where} \quad f = -(r\lambda+1)/(n-1). \tag{3.3}$$

Taking covariant differentiation of (3.3) along an arbitrary vector field Y, we obtain

$$\nabla_Y(\nabla_X D\lambda) = (Y\lambda)QX + \lambda(\nabla_Y Q)X + \lambda Q(\nabla_Y X) + (Yf)X + f\nabla_Y X.$$

Interchanging X and Y in the above equation gives

$$\nabla_X(\nabla_Y D\lambda) = (X\lambda)QY + \lambda(\nabla_X Q)Y + \lambda Q(\nabla_X Y) + (Xf)Y + f\nabla_X Y.$$

Replacing X by $[X,Y]$ in (3.3) we obtain

$$\nabla_{[X,Y]} D\lambda = \lambda Q(\nabla_X Y) - \lambda Q(\nabla_Y X) + f\nabla_X Y - f\nabla_Y X.$$

Using the last three equations in the well known expression of the curvature tensor

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

gives the required result.

THEOREM 3.2. *Let $M(\varphi, \xi, \eta, g)$ be a complete K-contact manifold of dimension $2n+1$. If the metric g satisfies the Miao–Tam critical condition, then it is Einstein, Sasakian and isometric to the unit sphere S^{2n+1}.*

Proof. Taking covariant differentiation of (2.7) along an arbitrary vector field X and using (2.6), we get

$$(\nabla_X Q)\xi = Q\varphi X - 2n\varphi X. \tag{3.4}$$

As ξ is Killing for a K-contact manifold, we obtain $\mathcal{L}_\xi Q = 0$. Making use of (3.4) and (2.7) one can easily deduce that

$$\nabla_\xi Q = Q\varphi - \varphi Q. \tag{3.5}$$

Replacing X by ξ in (3.2) and using (3.4) and (3.5), we get

$$\begin{aligned} R(\xi, Y)D\lambda = (\xi\lambda)QY - 2n(Y\lambda)\xi - \lambda\varphi QY \\ + 2n\lambda\varphi Y + (\xi f)Y - (Yf)\xi. \end{aligned} \tag{3.6}$$

Taking the scalar product of the foregoing equation with an arbitrary vector field X and using (2.9) provides

$$\begin{aligned} & g((\nabla_Y\varphi)X, D\lambda) + (\xi\lambda)g(QY, X) + 2n\lambda g(\varphi Y, X) \\ & - \{2n(Y\lambda) + (Yf)\}\eta(X) - \lambda g(\varphi QY, X) + (\xi f)g(X,Y) = 0. \end{aligned} \tag{3.7}$$

Setting $X = \varphi X$ and $Y = \varphi Y$ in (3.7), adding the resulting equation to (3.7) and then using (2.5) (where $h = 0$, as M is K-contact) gives

$$\begin{aligned} & 2\xi(\lambda + f)g(X,Y) - Y\{(2n+1)\lambda + f\}\eta(X) \\ & - \xi(\lambda + f)\eta(X)\eta(Y) + (\xi\lambda)g(QY, X) + 4n\lambda g(\varphi Y, X) \\ & - \lambda g(Q\varphi Y + \varphi QY, X) + (\xi\lambda)g(Q\varphi Y, \varphi X) = 0. \end{aligned}$$

Antisymmetrizing the foregoing equation yields

$$\begin{aligned} & X\{(2n+1)\lambda + f\}\eta(Y) - Y\{(2n+1)\lambda + f\}\eta(X) \\ & - 8n\lambda g(\varphi X, Y) - 2\lambda g(Q\varphi Y + \varphi QY, X) = 0. \end{aligned} \tag{3.8}$$

At this point, we replace X by φX and Y by φY in (3.8) to achieve

$$\lambda[g(Q\varphi Y, X) + g(\varphi QY, X)] = 4n\lambda g(\varphi Y, X).$$

Since λ is non-zero in the interior of M, the last equation implies

$$(Q\varphi + \varphi Q)X = 4n\varphi X \tag{3.9}$$

for all vector fields X on M. Let $\{e_i, \varphi e_i, \xi\}$, $i = 1, \dots, n$, be a φ-basis of M such that $Qe_i = \rho_i e_i$. From this, we deduce $\varphi Qe_i = \rho_i \varphi e_i$. Substituting e_i for

X in (3.9) and using the foregoing equation, we obtain $Q\varphi e_i = (4n - \rho_i\varphi)e_i$. Using the φ-basis and (2.7) we now compute the scalar curvature

$$r = g(Q\xi, \xi) + \sum_{i=1}^{n}[g(Qe_i, e_i) + g(Q\varphi e_i, \varphi e_i)] = 2n(2n+1).$$

Since the dimension of M is $2n+1$, we have $f = -(r\lambda + 1)/(2n)$ (by Lemma 3.1). Hence from $r = 2n(2n+1)$ it follows that

$$(2n+1)\lambda + f = -\frac{1}{2n}. \tag{3.10}$$

Now, taking the inner product of (3.6) with $D\lambda$ and applying (3.10), we get

$$(\xi\lambda)\{QD\lambda - 2nD\lambda\} + \lambda\{Q\varphi D\lambda - 2n\varphi D\lambda\} = 0. \tag{3.11}$$

On the other hand, taking $D\lambda$ instead of X in (3.9), we obtain

$$Q\varphi D\lambda + \varphi QD\lambda - 4n\varphi D\lambda = 0. \tag{3.12}$$

Using the foregoing equation in (3.11), we find

$$(\xi\lambda)\{QD\lambda - 2nD\lambda\} + \lambda\{2n\varphi D\lambda - \varphi QD\lambda\} = 0. \tag{3.13}$$

Applying φ to (3.13) and using (2.7) gives

$$(\xi\lambda)\{\varphi QD\lambda - 2n\varphi D\lambda\} + \lambda\{QD\lambda - 2nD\lambda\} = 0. \tag{3.14}$$

Combining (3.13) with (3.14) provides

$$\{\lambda^2 + (\xi\lambda)^2\}(QD\lambda - 2nD\lambda) = 0. \tag{3.15}$$

If possible, let $\lambda^2 + (\xi\lambda)^2 = 0$ in some open set $\mathcal{O}$ in M. Then $\lambda = 0$ and $\xi\lambda = 0$ on $\mathcal{O}$. By the definition of the Miao–Tam critical condition, λ does not vanish in the interior on M. So $\lambda = 0$ on $\mathcal{O}$ is not possible. Further, $\xi\lambda = 0$ on $\mathcal{O}$ shows that $\lambda(QD\lambda - 2nD\lambda) = 0$ (by (3.15)) on $\mathcal{O}$. Since λ does not vanish in the interior on M, we must have $QD\lambda - 2nD\lambda = 0$. Now, the covariant differentiation of the foregoing equation along an arbitrary vector field X and the use of (3.3) gives

$$(\nabla_X Q)D\lambda + \lambda Q^2X + (f - 2n\lambda)QX - 2nfX = 0.$$

Contracting this over X with respect to an orthonormal field and noting that $r = 2n(2n+1)$, we obtain $|Q|^2 = 2nr$. Applying this and $r = 2n(2n+1)$, we compute

$$\begin{aligned}\left|Q - \frac{r}{2n+1}\right|^2 &= |Q|^2 - \frac{2r^2}{2n+1} + \frac{r^2}{2n+1} = 2nr - \frac{r^2}{2n+1}\\ &= 4n^2(2n+1) - 4n^2(2n+1) = 0.\end{aligned}$$

Since the length of the symmetric tensor $Q - \frac{r}{2n+1}I$ is zero, we must have $Q = \frac{r}{2n+1}I = 2nI$. This shows that M is Einstein with Einstein constant $2n$. Since M is complete, it is compact by Myers' theorem [M]. Using the result of Boyer–Galicki [BG] that any compact K-contact Einstein manifold is

Sasakian, we conclude that M is Sasakian. Now, by (3.10) equation (3.3) can be written as

$$\nabla_g^2\lambda = -\left(\lambda + \frac{1}{2n}\right)g.$$

Applying Tashiro's theorem [TY] we conclude that M is isometric to $S^{2n+1}(1)$.

COROLLARY 3.3. *Let $M(\varphi,\xi,\eta,g)$ be a complete and simply connected Sasakian manifold. If the metric g satisfies the Miao–Tam critical condition, then M is compact and isometric to $S^{2n+1}(1)$.*

Proof. On a Sasakian manifold the Ricci operator Q and φ commute, i.e., $Q\varphi = \varphi Q$ (see [Bl]). Using this in (3.9) implies $Q\varphi X = 2n\varphi X$. Replacing X by φX in the last equation and using (2.7) gives $QX = 2nX$. This shows that M is Einstein with Einstein constant $2n$. The rest follows from the last theorem.

4. (κ,μ)-contact spaces

THEOREM 4.1. *Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be a non-Sasakian (κ,μ)-contact manifold. If the metric g satisfies the Miao–Tam critical condition, then M is flat for $n=1$, while for $n>1$, it is locally isometric to $E^{n+1}\times S^n(4)$.*

Proof. Taking $Y=\xi$ in (2.12) gives

$$l = -\kappa\varphi^2 + \mu h. \tag{4.1}$$

Using (2.14) and (4.1) in (2.3), we obtain

$$\nabla_\xi h = \mu h\varphi. \tag{4.2}$$

Differentiating (2.13) along ξ and recalling (4.2) yields

$$(\nabla_\xi Q)X = \mu(2(n-1)+\mu)h\varphi X. \tag{4.3}$$

On the other hand, from (2.9) we have $Q\xi = 2n\kappa\xi$. Differentiating this along an arbitrary vector field X and using (2.1) shows that

$$(\nabla_X Q)\xi = Q(\varphi+\varphi h)X - 2n\kappa(\varphi+\varphi h)X. \tag{4.4}$$

Taking the scalar product of (3.2) with ξ and using (4.4) gives

$$\begin{aligned} g(R(X,Y)D\lambda,\xi) &= 2n\kappa[(X\lambda)-(Y\lambda)\eta(X)] + \lambda g(Q\varphi X+\varphi QX,Y) \\ &\quad + \lambda g(Q\varphi hX + h\varphi QX,Y) - 4n\kappa\lambda g(\varphi X,Y) \\ &\quad + (Xf)\eta(Y) - (Yf)\eta(X). \end{aligned} \tag{4.5}$$

Replacing X by φX, Y by φY in (4.5) and noting that $R(\varphi X,\varphi Y)\xi = 0$ (by (2.12)), we obtain

$$Q\varphi X + \varphi QX - \varphi QhX - hQ\varphi X - 4n\kappa\varphi X = 0. \tag{4.6}$$

Replacing X by φX in (2.13) gives

$$Q\varphi X = [2(n-1)-n\mu]\varphi X + [2(n-1)+\mu]h\varphi X.$$

The action of h on the foregoing equation and using (2.14) implies that

$$hQ\varphi X = [2(n-1) - n\mu]h\varphi X - (\kappa - 1)[2(n-1) + \mu]\varphi X.$$

Operating by φ on (2.13) gives

$$\varphi QX = [2(n-1) - n\mu]\varphi X + [2(n-1) + \mu]\varphi hX.$$

Taking hX instead of X and using (2.14) reduces the last equation to

$$\varphi QhX = [2(n-1) - n\mu]\varphi hX - (\kappa - 1)[2(n-1) + \mu]\varphi X.$$

Next, we use the last four equations in (4.6) to obtain

$$\kappa(\mu - 2) = \mu(n+1). \tag{4.7}$$

Substituting ξ instead of X in (4.5) and using $Q\xi = 2n\kappa\xi$ and (4.1) we obtain

$$\kappa D\lambda + \mu hD\lambda - 2n\kappa((\xi\lambda)\xi - D\lambda) - (\kappa(\xi\lambda) + \xi f)\xi + Df = 0. \tag{4.8}$$

Contracting (3.2) over X, noting that the scalar curvature is constant, we obtain

$$rD\lambda + 2nDf = 0. \tag{4.9}$$

Using (4.9) in (4.8) yields

$$\begin{aligned} 0 = {} & 2n\kappa D\lambda + 2n\mu hD\lambda - 4n^2\kappa((\xi\lambda)\xi - D\lambda) \\ & + 2n(\kappa(\xi\lambda) + \xi f)\xi + rD\lambda. \end{aligned} \tag{4.10}$$

From (3.3), we have

$$\nabla_\xi D\lambda = (2n\kappa\lambda - f)\xi. \tag{4.11}$$

Differentiating (4.10) along ξ and using (4.2), (4.3) and (4.11), we obtain

$$\begin{aligned} 0 = {} & (2n\kappa + 4n^2\kappa - r)(2n\kappa\lambda - f)\xi + 2n\mu^2 h\varphi D\lambda \\ & - 4n^2\kappa\xi(\xi\lambda)\xi - 2n\xi(\kappa(\xi\lambda) + \xi f)\xi. \end{aligned} \tag{4.12}$$

Operating by φ on the foregoing equation we obtain $\mu^2 hD\lambda = 0$. Again, operating on this equation by h, and recalling (2.14), shows that

$$\mu^2(\kappa - 1)\varphi^2 D\lambda = 0.$$

Since M is non-Sasakian, we have either (i) $\mu = 0$, or (ii) $\varphi^2 D\lambda = 0$.

Case (i): In this case it follows from (4.7) that $\kappa = 0$. Hence $R(X,Y)\xi = 0$ and therefore M is locally flat in dimension 3, and in higher dimensions it is locally isometric to the trivial bundle $E^{n+1} \times S^n(4)$ (see [Bl]).

Case (ii): This case yields $D\lambda = (\xi\lambda)\xi$. Differentiating this along an arbitrary vector field X together with (2.1) entails that $\nabla_X D\lambda = X(\xi\lambda)\xi - (\xi\lambda)(\varphi X - \varphi hX)$. By making use of the Poincaré lemma: $g(\nabla_X Df, Y) = g(\nabla_Y D\lambda, X)$, the foregoing equation implies $X(\xi\lambda)\eta(Y) - Y(\xi\lambda)\eta(X) + (\xi\lambda)d\eta(X,Y) = 0$. Replacing X by φX and Y by φY, since $d\eta$ is non-zero for any contact metric structure, it follows that $\xi\lambda = 0$. Hence $D\lambda = 0$, i.e., λ is constant and consequently (3.3) implies that M is Einstein, i.e., $QX = \lambda X = (2n\kappa)X$. Taking traces we find the scalar curvature $r = 2n\kappa(2n+1)$. Comparing this with (2.15) it follows that $n\mu = 2(n-1) - 2n\kappa$. On the other hand, using the last equation and $QX = (2n\kappa)X$ in (2.13) we obtain $(2(n-1)+\mu)h = 0$. Since M is non-Sasakian, we have $2(n-1)+\mu = 0$. Then for $n = 1$, $\mu = 0 = \kappa$, and therefore $R(X,Y)\xi = 0)$ and hence M is locally flat. Again, for $n > 1$, using $\mu = 2(1-n)$ in (4.7), we have $\kappa = n - 1/n > 1$, a contradiction. This completes the proof of Theorem 4.1.

Acknowledgements. D. S. Patra is financially supported by the Council of Scientific and Industrial Research, India (grant no. 17-06/2012(i)EU-V).

References

[BDR] A. Barros, R. Diogenes and E. Ribeiro, Jr., *Bach-flat critical metrics of the volume functional on 4-dimensional compact manifolds with boundary*, J. Geom. Anal. 25 (2015), 2698–2715.

[Bl] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Birkhäuser, Boston, 2002.

[BKP] D. E. Blair, T. Koufogiorgos and B. J. Papantoniou, *Contact metric manifolds satisfying a nullity condition*, Israel J. Math. 91 (1995), 189–214.

[BG] C. P. Boyer and K. Galicki, *Einstein manifolds and contact geometry*, Proc. Amer. Math. Soc. 129 (2001), 2419–2430.

[MT1] P. Miao and L.-F. Tam, *On the volume functional of compact manifolds with boundary with constant scalar curvature*, Calc. Var. Partial Differential Equations 36 (2009), 141–171.

[MT2] P. Miao and L.-F. Tam, *Einstein and conformally flat critical metrics of the volume functional*, Trans. Amer. Math. Soc. 363 (2011), 2907–2937.

[M] S. B. Myers, *Connections between differential geometry and topology*, Duke Math. J. 1 (1935), 376–391.

[T] S. Tanno, *The topology of contact Riemannian manifolds*, Illinois J. Math. 12 (1968), 700–717.

[TY] Y. Tashiro, *Complete Riemannian manifolds and some vector fields*, Trans. Amer. Math. Soc. 117 (1965), 251–275.

Dhriti Sundar Patra
Department of Mathematics
Jadavpur University
Kolkata 700 032, India
E-mail: dhritimath@gmail.com

Amalendu Ghosh
Department of Mathematics
Chandernagore College
Chandannagar 712 136, W.B., India
E-mail: aghosh_70@yahoo.com