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On Erb's uncertainty principle

by

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Abstract. We improve a result of Erb, concerning an uncertainty principle for orthogonal polynomials. The proof uses numerical range and a decomposition of some multiplication operators as a difference of orthogonal projections.

1. Introduction. The aim of this paper is to improve a result of Erb [Erb13] about an uncertainty principle for orthogonal polynomials. The uncertainty principle asserts that a non-zero function and its Fourier transform cannot be simultaneously small. There exist many mathematical interpretations of this assertion (see [HJ94]). One way to quantify this statement is to use the notion of annihilating pairs (see Definition 2.1). Slepian and Pollak [SP61] proved that a non-zero function $f \in L^2(\mathbb{R})$ and its Fourier transform cannot be both supported on intervals. Using numerical range, Lenard [Len72] proved that a non-zero function and its Fourier transform cannot have their support in some bounded measurable sets at the same time. Amrein and Berthier [AB77] also proved a similar result for sets of finite Lebesgue measure. Nazarov [Naz93] gave an explicit upper bound on the annihilating constant, and Jaming [Jam07] generalized this result in $L^2(\mathbb{R}^d)$. We invite the reader to consult the survey of Folland and Sitaram [FS97]. In [Kla14] some new connections between numerical range of orthogonal projections and uncertainty principles were obtained. For some background on numerical range, we refer the reader to [GR97].

In 2013, Erb [Erb13] found an analogue to Slepian and Pollak's uncertainty principle in the context of orthogonal polynomials. We will recall the results of Erb and briefly explain how they are related to the numerical range of some operators. Let $\omega : [-1,1] \to [0,\infty[$ be a positive weight such

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that $\operatorname{supp}(\omega) = [-1, 1]$ and for every $n \in \mathbb{N}$,

$$\int_{[-1,1]} x^n \omega(x)\, dx < \infty.$$

We consider the Hilbert space $L^2([-1, 1], \omega)$ with the inner product

$$\langle f, g \rangle = \int_{[-1,1]} f(x)\overline{g(x)}\omega(x)\, dx.$$

Denote by $\{p_l\}_{l=0}^{\infty}$ the family of orthogonal polynomials with positive leading coefficient such that p_l is of degree l and is normalized (i.e. $\|p_l\| = 1$). Those polynomials satisfy a three-term recurrence relationship

$$b_{l+1}p_{l+1}(x) = (x - a_l)p_l(x) - b_l p_{l-1}(x).$$

We say that the weight ω belongs to the *Nevai class* $\mathcal{M}(0, 1)$ if $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = 1/2$. We say that ω belongs to the *Nevai subclass* $\mathcal{M}^*(0, 1)$ if $\omega \in \mathcal{M}(0, 1)$, $\operatorname{supp}(\omega) = [-1, 1]$ and $\sum_{n\in\mathbb{N}} |a_n| + |b_n - 1/2| < \infty$. Let $f \in L^2([-1, 1], \omega)$ and define

$$\epsilon(f) = \langle M_x f, f \rangle = \int_{[-1,1]} x|f(x)|^2\omega(x)\, dx.$$

If $\|f\| = 1$, then $\epsilon(f)$ can be interpreted as the location of the average of the L^2 mass of f. Denote by $\mathit{\Pi}_n^m$ the orthogonal projection on the subspace $\overline{\operatorname{span}}\{p_l : m \le l \le n\}$. In [Erb13], Erb was interested in the following set:

$$\mathcal{W}_{\epsilon,\mathit{\Pi}_n^m} = \{(\epsilon(f), \|\mathit{\Pi}_n^m f\|^2) : f \in L^2([-1, 1], \omega),\ \|f\| = 1\}.$$

After identifying $\mathbb{R}^2$ with $\mathbb{C}$ we can see that $\mathcal{W}_{\epsilon,\mathit{\Pi}_n^m}$ is a numerical range. Indeed, if $f \in L^2([-1, 1], \omega)$ is such that $\|f\| = 1$, we have

$$\begin{aligned}
\epsilon(f) + \mathrm{i}\|\mathit{\Pi}_n^m f\|^2 &= \langle M_x f, f\rangle + \mathrm{i}\langle \mathit{\Pi}_n^m f, \mathit{\Pi}_n^m f\rangle = \langle M_x f, f\rangle + \mathrm{i}\langle (\mathit{\Pi}_n^m)^* \mathit{\Pi}_n^m f, f\rangle \\
&= \langle M_x f, f\rangle + \mathrm{i}\langle (\mathit{\Pi}_n^m)^2 f, f\rangle = \langle M_x f, f\rangle + \mathrm{i}\langle \mathit{\Pi}_n^m f, f\rangle \\
&= \langle (M_x + \mathrm{i}\mathit{\Pi}_n^m) f, f\rangle.
\end{aligned}$$

Therefore $\mathcal{W}_{\epsilon,\mathit{\Pi}_n^m} = W(M_x + \mathrm{i}\mathit{\Pi}_n^m)$. Write

$$x_{n,\max}^m = \sup \sigma(\mathit{\Pi}_n^m M_x \mathit{\Pi}_n^m) \quad \text{and} \quad x_{n,\min}^m = \inf \sigma(\mathit{\Pi}_n^m M_x \mathit{\Pi}_n^m).$$

We set

$$\gamma_1 : [x_{n,\max}^m, 1] \to \mathbb{R}, \qquad x \mapsto \frac{1}{2} + \frac{1}{2}(x x_{n,\max}^m + (1 - x^2)^{1/2}(1 - (x_{n,\max}^m)^2)^{1/2}),$$

and

$$\gamma_2 : [-1, x_{n,\min}^m] \to \mathbb{R}, \qquad x \mapsto \frac{1}{2} + \frac{1}{2}(x x_{n,\min}^m + (1 - x^2)^{1/2}(1 - (x_{n,\min}^m)^2)^{1/2}).$$

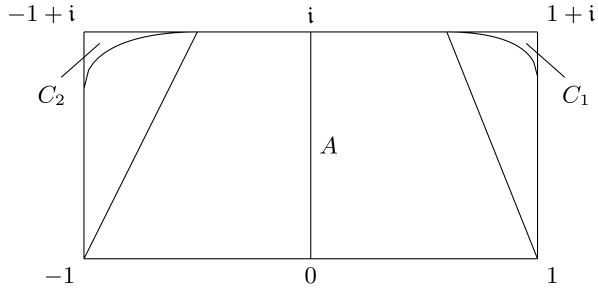

Fig. 1

Denote

$$A = \left\{ (x, y) \in [-1, 1] \times [0, 1] : y < \frac{1 - x}{1 - x^m_{n,\max}},\ y < \frac{1 + x}{1 + x^m_{n,\min}} \right\},$$

$$C_1 = \{(x, y) \in [-1, 1] \times [0, 1] : x \in]x^m_{n,\max}, 1[,\ y > \gamma_1(x)\},$$

$$C_2 = \{(x, y) \in [-1, 1] \times [0, 1] : x \in] - 1, x^m_{n,\min}[,\ y > \gamma_2(x)\}.$$

Now we can state Erb's result [Erb13].

THEOREM 1.1 ([Erb13]). *Let* $\omega \in \mathcal{M}^*(0, 1)$. *Then*

$$A \subset W(M_x + \mathrm{i}\Pi^m_n) \subset [-1, 1] \times [0, 1] \setminus (C_1 \cup C_2).$$

This uncertainty principle says that if a function has the average of its L^2 mass too close to 1, it cannot be well approximated by the polynomials generated by $p_n, \ldots, p_m$. The same phenomenon appears also at -1.

The goal of this paper is to generalize Erb's result (see Theorem 4.3). We will remove the condition that ω must belong to the Nevai subclass. Moreover we will replace the orthogonal projection Π^m_n on the subspace generated by the polynomials $p_n, \ldots, p_m$ with the orthogonal projection Π onto a subspace generated by finitely many arbitrary continuous functions. We will also replace multiplication by the independent variable M_x with multiplication by a continuous odd function M_ϕ.

In order to prove the main result, we need some information on the numerical range of $M_\phi + \mathrm{i}\Pi$. The idea is to write M_ϕ as a difference of orthogonal projections $P - Q$. Then we will prove that (P, Π) and (Q, Π) are strong annihilating pairs. As we will use numerical range, it will be easy to transfer those uncertainty principles to (M_ϕ, Π).

The paper is organized as follows. In Section 2, we recall some results that we will need. In Section 3, we write M_ϕ as a difference of orthogonal projections. In Section 4, we prove the main result (Theorem 4.3).

2. Preliminaries. In this section we recall the results that we will need. First we introduce the notion of annihilating pairs.

Definition 2.1. Let $P, Q \in \mathcal{B}(H)$ be orthogonal projections. We say that (P, Q) is an *annihilating pair* (a-pair) if $\operatorname{Ran} P \cap \operatorname{Ran} Q = \{0\}$.

Definition 2.2. Let $P, Q \in \mathcal{B}(H)$ be orthogonal projections. We say that (P, Q) is a *strong annihilating pair* (strong a-pair) if there exists a constant $C > 0$ such that for every $f \in H$, we have

$$\|f\|^2 \leq C\big(\|(I-P)f\|^2 + \|(I-Q)f\|^2\big).$$

It is easy to see that a strong a-pair is an a-pair, but in general the converse is not true. However, it is true under some additional assumptions.

Definition 2.3. Let $P, Q \in \mathcal{B}(H)$ be orthogonal projections. We say that (P, Q) is a *compact pair* if PQ is a compact operator.

Lemma 2.4. *Let $P, Q \in \mathcal{B}(H)$ be orthogonal projections. If (P, Q) is an a-pair and a compact pair, then (P, Q) is a strong a-pair.*

The following theorem is a direct corollary of a result of Lenard [Len72] and can also be found in [HJ94].

Theorem 2.5. *Let $P, Q \in \mathcal{B}(H)$ be orthogonal projections. Suppose that (P, Q) is an a-pair and a compact pair. Denote $x_M = \sup \sigma(PQP)$ and let*

$$\begin{aligned} &\gamma : [x_M, 1] \to \mathbb{R}, \\ &\qquad x \mapsto \frac{1}{2}\big(1 + (2x-1)(2x_M - 1) + 4\sqrt{(1-x)x}\sqrt{(1-x_M)x_M}\big). \end{aligned}$$

Then

$$W(P + \mathrm{i}Q) \subset \{x + \mathrm{i}y : 0 \leq x \leq 1,\ 0 \leq y \leq \min\{1, \gamma(x)\}\}.$$

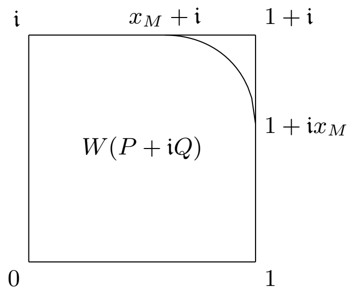

Fig. 2

3. Difference of orthogonal projections. Let $\phi : [-1, 1] \to [-1, 1]$ be an odd function. In this section, we will write the operator M_ϕ as a difference of orthogonal projections. Denote $Pos = L^2([0, 1], \omega)$ and $Neg = L^2([-1, 0], \omega)$. Let $\Psi : [0, 1] \to \mathbb{T}$ be a measurable function. We define an operator $V_\Psi : Pos \to Neg$ by setting, for every $h \in Pos$ and almost every $x \in [-1,0]$,

$$V_\Psi h(x) = h(-x)\sqrt{\frac{\omega(-x)}{\omega(x)}}\Psi(-x).$$

The operator V_Ψ is well defined because Ψ is bounded and measurable. For every $\tilde{h} \in Neg$ and almost every $x \in [0,1]$ we have

$$V_\Psi^* \tilde{h}(x) = \tilde{h}(-x)\sqrt{\frac{\omega(-x)}{\omega(x)}}\overline{\Psi(x)}.$$

So for almost every $x \in [-1,0]$,

$$V_\Psi V_\Psi^*(h)(x) = h(x)\sqrt{\frac{w(x)}{w(-x)}}\sqrt{\frac{\omega(-x)}{\omega(x)}}\Psi(-x)\overline{\Psi(-x)} = h(x).$$

Therefore $V_\Psi V_\Psi^* = I_{Neg}$. In the same way, we can prove that $V_\Psi^* V_\Psi = I_{Pos}$. Moreover for every $h \in Pos$ we have

$$\begin{aligned}\|V_\Psi h\|_{Neg}^2 &= \int\limits_{[-1,0]} |h(-x)\sqrt{\frac{w(-x)}{w(x)}}\Psi(-x)|^2 w(x)\,dx \\ &= \int\limits_{[-1,0]} |h(-x)|^2 w(-x)\,dx = \|h\|_{Pos}^2.\end{aligned}$$

So V_Ψ is a surjective isometry, and V_Ψ^* is also a surjective isometry. In $H = Pos \oplus Neg$ we define the orthogonal projections P_Ψ and Q_Ψ by the formulas

$$P_\Psi = \frac{1}{2}\begin{pmatrix} M_{1+\phi} & M_{\sqrt{1-\phi^2}}V_\Psi^* \\ V_\Psi M_{\sqrt{1-\phi^2}} & V_\Psi M_{1-\phi}V_\Psi^* \end{pmatrix},$$

$$Q_\Psi = \frac{1}{2}\begin{pmatrix} M_{1-\phi} & M_{\sqrt{1-\phi^2}}V_\Psi^* \\ V_\Psi M_{\sqrt{1-\phi^2}} & V_\Psi M_{1+\phi}V_\Psi^* \end{pmatrix}.$$

The operators P_Ψ and Q_Ψ are indeed orthogonal projections, because they satisfy $P_\Psi^2 = P_\Psi = P_\Psi^*$ and $Q_\Psi^2 = Q_\Psi = Q_\Psi^*$. As ϕ is odd, the following diagram is commutative:

$$\begin{tikzcd} Neg \arrow[r, "M_\phi"] \arrow[d, "V_\Psi^*"'] & Neg \\ Pos \arrow[r, "-M_\phi"] & Pos \arrow[u, "V_\Psi"'] \end{tikzcd}$$

Therefore

$$M_\phi = \begin{pmatrix} M_\phi & 0 \\ 0 & -V_\Psi M_\phi V_\Psi^* \end{pmatrix} = P_\Psi - Q_\Psi.$$

We will now prove that $\mathrm{Ran}(P_\Psi)$ is the subspace of all functions in $L^2([-1,1],\omega)$ such that there exists $h \in Pos$ such that

$$P_{Pos} f = M_{\sqrt{1+\phi}} h \quad \text{and} \quad P_{Neg} f = V_\Psi M_{\sqrt{1-\phi}} h.$$

Denote this subspace by E_Ψ. First we will prove that $\mathrm{Ran}(P_\Psi) \subset E_\psi$. Let $g_{Pos} \in Pos$ and $g_{Neg} \in Neg$. Denote $g = g_{Pos} \oplus g_{Neg}$. We have

$$\begin{aligned} P_{Pos} P_\Psi g &= \frac{1}{2}\big(M_{1+\phi}(g_{Pos}) + M_{\sqrt{1-\phi^2}} V_\Psi^*(g_{Neg})\big) \\ &= \frac{1}{2} M_{\sqrt{1+\phi}}\big(M_{\sqrt{1+\phi}} g_{Pos} + M_{\sqrt{1-\phi}} V_\Psi^*(g_{Neg})\big). \end{aligned}$$

Let $h = \frac{1}{2}(M_{\sqrt{1+\phi}} g_{Pos} + M_{\sqrt{1-\phi}} V_\Psi^*(g_{Neg}))$. Then $P_{Pos} P_\Psi g = M_{\sqrt{1+\phi}} h$. In the same way, we get

$$\begin{aligned} P_{Neg} P_\Psi g &= \frac{1}{2}\big(V_\Psi M_{\sqrt{1-\phi^2}}(g_{Pos}) + V_\Psi M_{1-\phi} V_\Psi^*(g_{Neg})\big) \\ &= \frac{1}{2} V_\Psi M_{\sqrt{1-\phi}}\big(M_{\sqrt{1+\phi}}(g_{Pos}) + M_{\sqrt{1-\phi}} V_\Psi^*(g_{Neg})\big) \\ &= V_\Psi M_{\sqrt{1-\phi}} h. \end{aligned}$$

So $P_\Psi g \in E_\Psi$ and therefore $\mathrm{Ran}(P_\Psi) \subset E_\Psi$.

Now we will prove that $E_\Psi \subset \mathrm{Ran}(P_\Psi)$. Let $g \in E_\Psi$, and let $h \in Pos$ be such that $P_{Pos} g = M_{\sqrt{1+\phi}} h$ and $P_{Neg} g = V_\Psi M_{\sqrt{1-\phi}} h$. We have

$$\begin{aligned} P_\Psi g &= \frac{1}{2}\begin{pmatrix} M_{1+\phi} & M_{\sqrt{1-\phi^2}} V_\Psi^* \\ V_\Psi M_{\sqrt{1-\phi^2}} & V_\Psi M_{1-\phi} V_\Psi^* \end{pmatrix}\begin{pmatrix} M_{\sqrt{1-\phi}} h \\ V_\Psi M_{\sqrt{1-\phi}} h \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} M_{\sqrt{1+\phi}} M_{1+\phi} h + M_{\sqrt{1+\phi}} M_{1-\phi} h \\ V_\Psi M_{\sqrt{1-\phi}} M_{1+\phi} h + V_\Psi M_{\sqrt{1-\phi}} M_{1-\phi} h \end{pmatrix} \\ &= \begin{pmatrix} M_{\sqrt{1+\phi}} h \\ V_\Psi M_{\sqrt{1-\phi}} h \end{pmatrix} = g. \end{aligned}$$

Therefore $g \in \mathrm{Ran}(P_\Psi)$ and $E_\Psi \subset \mathrm{Ran}(P_\Psi)$. We have just proved the following lemma.

Lemma 3.1. *Let $\phi : [-1,1] \to [-1,1]$ be an odd function and let $\Psi : [0,1] \to \mathbb{T}$ be measurable. Then there exist orthogonal projections P_Ψ and Q_Ψ such that $M_\phi = P_\Psi - Q_\Psi$ and $\mathrm{Ran}(P_\Psi) = E_\Psi$.*

The fact that Ψ is measurable and of modulus one is what makes V_Ψ well defined and gives the isometry between Pos and Neg.

Definition 3.2. Denote by $NC(\omega)$ the set of all measurable function $\Psi : [0,1] \to \mathbb{T}$ such that for every open subset $I \subset [0,1]$ and every $g : [0,1] \to \mathbb{C}$ such that for almost every $x \in [0,1]$, $g(x)\sqrt{\omega(-x)} = \sqrt{\omega(x)}\Psi(x)$, the restriction of g to I is discontinuous.

This class of functions is not empty. If we want to exhibit an element of $NC(1)$, we have to find a "highly discontinuous" function. The first candidate that comes to mind is $\exp(\mathbf{i}\mathbf{1}_{\mathbb{Q}}(x))$. However for almost every $x \in [0,1]$, $\exp(\mathbf{i}\mathbf{1}_{\mathbb{Q}}(x)) = 1$. As 1 is continuous, $\exp(\mathbf{i}\mathbf{1}_{\mathbb{Q}}(x)) \notin NC(1)$. The best strategy to build a function in $NC(1)$ is to create "a lot of jumps" in order to break the continuity, without any possibility to recover it by modifying the function on a set of measure zero. Let $\phi : \mathbb{N} \to \mathbb{Q} \cap [0,1]$ be a bijection. Denote by δ_a the Dirac mass at a and let $\mu = \sum_{n\in\mathbb{N}} 2^{-k}\delta_{\phi(k)}$. Let $f(x) = \int_{[0,1]} \mathbf{1}_{[0,x]}(t)\, d\mu(t)$. Then $\exp(\mathbf{i}f) \in NC(1)$.

LEMMA 3.3. *Suppose that $\phi : [-1,1] \to [-1,1]$ is odd, continuous on $[-1,1]$, and such that $-1 < \phi(x) < 1$ for every $x \in]-1,1[$. Let $\Psi \in NC(\omega)$. If $f \in \operatorname{Ran} P_\Psi$, then either $f = 0$ or f is discontinuous.*

Proof. We will prove the lemma by contradiction. Suppose that there exists $f \in \operatorname{Ran} P_\Psi$ such that f is a non-zero continuous function. According to Lemma 3.1, there exists a function $h \in Pos$ such that for almost every $x \in [0,1]$ we have

$$f(x) = \sqrt{1+\phi(x)}h(x),$$
$$f(-x) = \sqrt{1-\phi(x)}h(x)\sqrt{\frac{\omega(x)}{\omega(-x)}}\Psi(x).$$

As $\sqrt{1+\phi(x)}$ does not vanish on $[0,1[$, for almost every $x \in [0,1]$ we have

$$h(x) = \frac{f(x)}{\sqrt{1+\phi(x)}},$$

and thus

$$f(-x) = \frac{\sqrt{1-\phi(x)}}{\sqrt{1+\phi(x)}}f(x)\sqrt{\frac{\omega(x)}{\omega(-x)}}\Psi(x).$$

As f is continuous, the set $\{x \in [-1,1] : f(x) \neq 0\}$ is open. Moreover f is non-zero, so at least one of the following two assertions is true:

(1) $m(\{x \in [0,1] : f(x) \neq 0\}) \neq 0$,
(2) $m(\{x \in [-1,0] : f(x) \neq 0\}) \neq 0$.

Suppose that (1) is satisfied (the other case can be handled in the same way). Denote $I = \{x \in]0,1[: f(x) \neq 0\}$. Then I is an open set. Moreover, for almost every $x \in I$,

$$\frac{f(-x)}{f(x)}\frac{\sqrt{1+\phi(x)}}{\sqrt{1-\phi(x)}}\sqrt{\omega(-x)} = \sqrt{\omega(x)}\Psi(x).$$

This contradicts $\Psi \in NC(\omega)$. ■

We summarize this section in the following corollary.

Corollary 3.4. *Suppose that* $\phi : [-1,1] \to [-1,1]$ *is odd, continuous on* $[-1,1]$ *and such that* $-1 < \phi(x) < 1$ *for every* $x \in\ ]-1,1[$*. Then there exist orthogonal projections* P *and* Q *such that* $M_\phi = P - Q$ *and* $\operatorname{Ran} P$ *does not contain any non-zero continuous function.*

Proof. Take a function $\Psi \in NC(\omega)$ and apply Lemma 3.3. ∎

The readers interested in pairs of orthogonal projections and differences of orthogonal projections can consult [Hal69] and [Dav58].

4. Proof of the main result. Now we are ready to prove the main theorem. First we will take care of what happens around $1+\mathfrak{i}$. The situation around $-1+\mathfrak{i}$ can be handled in a similar way. Let $x_M \in [0,1]$. Denote

$$\gamma_{x_M} : [x_M, 1] \to \mathbb{R},$$
$$x \mapsto \frac{1}{2}\big(1 + (2x-1)(2x_M - 1) + 4\sqrt{(1-x)x}\sqrt{(1-x_M)x_M}\big).$$

Theorem 4.1. *Let* $\phi : [-1,1] \to [-1,1]$ *be odd, continuous on* $[-1,1]$*, and such that* $-1 < \phi(x) < 1$ *for every* $x \in\ ]-1,1[$*. Let* Π *be the orthogonal projection on a subspace of* $L^2([-1,1],\omega)$ *generated by finitely many continuous functions. Denote* $x_M = \inf_{\Psi \in NC(\omega)} \sup \sigma(P_\Psi \Pi P_\Psi)$*. Then*

$$W(M_\phi + \mathfrak{i}\Pi) \subset \{x + \mathfrak{i}y : -1 \le x \le 1,\ 0 \le y \le \min\{1, \gamma_{x_M}(x)\}\}.$$

Proof. Let $\Psi \in NC(\omega)$. According to the last section, there exist orthogonal projections P_Ψ and Q_Ψ such that $M_\phi = P_\Psi - Q_\Psi$ and the range of P_Ψ contains no non-zero continuous function, i.e. $\operatorname{Ran} P_\Psi \cap \operatorname{Ran} \Pi = \{0\}$. In other words (P_ψ, Π) is an a-pair. As Π is a finite rank projection, ΠP_Ψ is a compact operator. So (P_ψ, Π) is a compact a-pair. We have

$$W(M_\phi + \mathfrak{i}\Pi) = W(P - Q + \mathfrak{i}\Pi) \subset W(-Q) + W(P + \mathfrak{i}\Pi).$$

Denote $x_\Psi = \sup \sigma(P_\Psi \Pi P_\Psi)$. Then by using Theorem 2.5, we get

$$W(M_\phi + \mathfrak{i}\Pi) \subset \big\{x + \mathfrak{i}y : -1 \le x \le 1,\ 0 \le y \le \min\{1, \gamma_{x_\Psi}(x)\}\big\}.$$

The theorem follows by taking the infimum over all functions in $NC(\omega)$. ∎

In a similar way, we can prove the following theorem.

Theorem 4.2. *Let* $\phi : [-1,1] \to [-1,1]$ *be odd, continuous on* $[-1,1]$*, and such that* $-1 < \phi(x) < 1$ *for every* $x \in\ ]-1,1[$*. Let* Π *be the orthogonal projection on a subspace of* $L^2([-1,1],w)$ *generated by finitely many continuous functions. Denote* $x_m = \inf_{\Psi \in NC(\omega)} \sup \sigma(Q_\Psi \Pi Q_\Psi)$*. Then*

$$W(M_\phi + \mathfrak{i}\Pi) \subset \big\{x + \mathfrak{i}y : -1 \le x \le 1,\ 0 \le y \le \min\{1, \gamma_{x_m}(-x)\}\big\}.$$

Now we can state the main result which is just the combination of Theorems 4.1 and 4.2.

THEOREM 4.3. *Let $\phi : [-1,1] \to [-1,1]$ be odd, continuous on $[-1,1]$ and such that $-1 < \phi(x) < 1$ for every $x \in \,]-1,1[$. Let Π be the orthogonal projection on a subspace of $L^2([-1,1],w)$ generated by finitely many continuous functions. Denote $x_M = \inf_{\Psi \in NC(\omega)} \sup \sigma(P_\Psi \Pi P_\Psi)$ and $x_m = \inf_{\Psi \in NC(\omega)} \sup \sigma(Q_\Psi \Pi Q_\Psi)$. Then*

$$W(M_\phi + \mathrm{i}\Pi) \subset \big\{x + \mathrm{i}y : -1 \le x \le 1,\ 0 \le y \le \min\{1, \gamma_{x_M}(x), \gamma_{x_m}(-x)\}\big\}.$$

This improves the result of Erb because now we can consider a weight ω which is not in the Nevai subclass. This uncertainty principle is now available for continuous functions (and not just for some consecutive orthogonal polynomials).

REMARK 4.4. Let $a = \max \sigma(\Pi M_\phi \Pi)$, $b = \min \sigma(\Pi M_\phi \Pi)$, $c = \max \sigma((I-\Pi)M_\phi(I-\Pi))$ and $d = \min \sigma((I-\Pi)M_\phi(I-\Pi))$. Recall that $\Pi M_\phi \Pi$ is compact. Let $x_a \in \operatorname{Ran}(\Pi)$ be such that $\|x_a\| = 1$ and $M_\phi x_a = a x_a$. Then $\langle (M_\phi + \mathrm{i}\Pi)x_a, x_a\rangle = a + \mathrm{i} \in W(M_\phi + \mathrm{i}\Pi)$. Let $x_{c,n} \in \operatorname{Ran}(I - \Pi)$ be such that $\|x_{c,n}\| = 1$ for every $n \in \mathbb{N}$, and $\lim_{n\to\infty} \langle M_\phi x_{c,n}, x_{c,n}\rangle = c$. Then

$$\lim_{n\to\infty} \langle (M_\phi + \mathrm{i}\Pi)x_{c,n}, x_{c,n}\rangle = c \in \overline{W(M_\phi + \mathrm{i}\Pi)}.$$

In the same way, we see that $b + \mathrm{i} \in W(M_\phi + \mathrm{i}\Pi)$ and $d \in \overline{W(M_\phi + \mathrm{i}\Pi)}$. As the numerical range of an operator is always convex, $\overline{W(M_\phi + \mathrm{i}\Pi)}$ is convex and we have

$$\operatorname{conv}\{a + \mathrm{i}, b + \mathrm{i}, c, d\} \subset \overline{W(M_\phi + \mathrm{i}\Pi)}.$$

REMARK 4.5. Let $k \le n \le M$ be positive integers, $\delta \in \,]0,1[$ and $A \in \mathcal{M}_{n,M}(\mathbb{C})$ be an n by M matrix. We say that A has (k,δ) RIP (see [GJ11] or [T]) if for every k-sparse vector $x \in \mathbb{C}^M$, we have

$$(1-\delta)\|x\|^2 \le \|Ax\|^2 \le (1+\delta)\|x\|^2.$$

Denote by $\mathcal{P}_k$ the set of all orthogonal projections on k-sparse subspaces of $\mathbb{C}^M$. Then A has (k,δ) RIP if and only if for every $x \in \mathbb{C}^M$ and for every $P \in \mathcal{P}_k$,

$$(1-\delta)\|Px\|^2 \le \|APx\|^2 \le (1+\delta)\|Px\|^2,$$

that is,

$$(1-\delta)\langle Px, Px\rangle \le \langle A^*APx, Px\rangle \le (1+\delta)\langle Px, Px\rangle.$$

If $Px = 0$, this is obviously true. Suppose that $Px \ne 0$, and denote $y = Px/\|Px\|$. Then $\|y\| = 1$, $y \in \operatorname{Ran}(P)$ and

$$(1-\delta)\langle y, y\rangle \le \langle A^*Ay, y\rangle \le (1+\delta)\langle y, y\rangle,$$

or equivalently

$$(1-\delta)\langle Py, y\rangle \le \langle A^*Ay, y\rangle \le (1+\delta)\langle Py, y\rangle.$$

Therefore, the matrix $A \in \mathcal{M}_{n,M}(\mathbb{C})$ has (k,δ) RIP if and only if for every $P \in \mathcal{P}_k$,

$$[\mathfrak{i}, (1-\delta)+\mathfrak{i}[\cap W(A^*A+\mathfrak{i}P)=\emptyset, \quad](1+\delta)+\mathfrak{i}, 2+\mathfrak{i}] \cap W(A^*A+\mathfrak{i}P)=\emptyset.$$

Using the fact that the numerical range of a matrix is always closed, we deduce that if for every $P \in \mathcal{P}_k$, we have $\mathfrak{i} \notin W(A^*A+\mathfrak{i}P)$ and $2+\mathfrak{i} \notin W(A^*A+\mathfrak{i}P)$, then there exists $\delta \in]0,1[$ such that A has (k,δ) RIP.

Denote $B = A^*A$. Then B is selfadjoint, and therefore by the spectral theorem, there exists a measure μ on $\mathbb{R}$, a function $\Phi : \mathbb{R} \to \mathbb{R}$ and a surjective isometry $U : \mathbb{C}^M \to L^2(\mathbb{R},\mu)$ such that $B = UM_\Phi U^*$. Recall that the numerical range is invariant under conjugation by a surjective isometry. In this language, if for every $P \in \mathcal{P}_k$, we have $\mathfrak{i} \notin W(M_\Phi + \mathfrak{i}UPU^*)$ and $2+\mathfrak{i} \notin W(M_\Phi + \mathfrak{i}UPU^*)$, then there exists $\delta \in]0,1[$ such that A has (k,δ) RIP.

Recall that Theorem 4.3 says that $\mathfrak{i} \notin W(M_\phi + I + \mathfrak{i}\Pi) = W(M_{1+\phi} + \mathfrak{i}\Pi)$ and $2+\mathfrak{i} \notin W(M_{1+\phi} + \mathfrak{i}\Pi)$ for the orthogonal projection Π on any subspace of finite rank generated by finitely many continuous functions. Therefore we can interpret Theorem 4.3 as an "infinite-dimensional" manifestation of a uniform uncertainty principle.

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