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Connected generalized inverse limits over intervals

by

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Abstract. Suppose that for each $i \geq 0$, I_i is a closed interval and $f_{i+1} : I_{i+1} \to 2^{I_i}$ is a surjective upper semicontinuous function with a connected graph. We give a condition on the graphs called a CC-sequence, and show that $\varprojlim (I_i, f_i)$ is disconnected if and only if the system admits a CC-sequence. We also show that $\varprojlim (I_i, f_i)$ is disconnected if and only if there is a basic open proper subset of $\prod_{i\geq 0} I_i$ that contains a component of $\varprojlim (I_i, f_i)$.

1. Introduction. In this paper we characterize connectedness of generalized inverse limits on intervals where all the bonding maps are surjective upper semicontinuous and have connected graphs. While there are more general settings, we focus on the case in which we are interested: Suppose for each nonnegative integer i, I_i is a closed interval and $f_{i+1} : I_{i+1} \to 2^{I_i}$, called a *bonding map*, is an upper semicontinuous mapping of I_{i+1} into the closed subsets of I_i. Then the *generalized inverse limit*, or the *inverse limit with set-valued mappings*, associated with these mappings is the set

$$\varprojlim (I_i, f_i) := \Big\{ (x_i) \in \prod_{i\geq 0} I_i : \text{for each } i \geq 1,\ x_{i-1} \in f_i(x_i) \Big\},$$

a subspace of $\prod_{i\geq 0} I_i$ endowed with the product topology.

Generalized inverse limits represent a new topic of study in continuum theory. They first appeared in 2004 in a paper published by Bill Mahavier [M]. Soon thereafter, Tom Ingram began studying them (see [IM2], [In2], [In3] and [In4]). The topic caught the interest of a number of continuum theorists, and is currently an intensely studied area, with papers appearing from many different authors. See for example [B], [BCMM], [CM], [CR], [Il], [K], [Nal], [P], [V]. Tom Ingram and Bill Mahavier included a chapter on these spaces in their recent book [IM1], and since then Tom Ingram has written another book [In1] on the topic.

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These objects are intrinsically interesting, and unlike any other object we have encountered, they demand investigation as they present a number of opportunities for application in other areas:

- They represent a new way of studying multi-valued mappings. Multi-valued mappings arise in models from many areas of science (e.g. the Christiano–Harrison model from economics). Finding suitable tools for effectively studying these mappings has been problematic. Unlike many other approaches, generalized inverse limits do not "lose" information as the system is iterated. (For example, if one takes the classical Ingram example (see Figure 1 below), adds the vertical line segment $\{0\} \times [0,1]$, and calls the resulting bonding map g, then for each $x \in [0,1]$, $g^2(x) = [0,1]$, since $0 \in g(x)$ for each x, and $g(0) = [0,1]$.) Instead, the information is retained in an infinite sequence of data points. One does not see only the initial point and the eventual outcome while losing all intermediate events.
- Generalized inverse limits provide a new technique for building examples of topological spaces. Unlike standard inverse limits on intervals, generalized inverse limits on intervals need not be one-dimensional, nor chainable nor embeddable in the plane, for example. And they need not be connected, as Ingram's example (see below) shows.
- Generalized inverse limits provide a new tool for studying some old unsolved problems in topology. Perhaps even the venerable old fixed-point problem may fall. (The fixed-point problem is this question: does every nonseparating plane continuum have the fixed-point property? A continuum X has the *fixed-point property* if for every continuous mapping f from X into X there is some point $p \in X$ such that $f(p) = p$.)

Recall that a function $f : X \to 2^Y$ is *upper semicontinuous at a point* $x \in X$ if for each open set V in Y containing $f(x)$, there is an open set U in X containing x such that if y is in U, then $f(y) \subset V$; and f is *upper semicontinuous* if it is upper semicontinuous at each point $x \in X$. We say that f is *surjective* if for each y in Y there is some x in X such that $y \in f(x)$. The *graph* of f is the set of all points $\langle x, y\rangle$ such that y is in $f(x)$. If X and Y are compact Hausdorff spaces then f is upper semicontinuous if and only if the graph of f is a compact subset of $X \times Y$ [IM2, Theorem 2.1]. If for each $i \geq 0$, X_i is a topological space and $f_{i+1} : X_{i+1} \to 2^{X_i}$ is an upper semicontinuous function, then each function f_i is called a *bonding map* and $\varprojlim (X_i, f_i)$ is the *inverse limit space* defined by

$$\varprojlim (X_i, f_i) = \Big\{(x_i) \in \prod_{i\geq 0} X_i : \forall i \geq 0,\, x_i \in f_{i+1}(x_{i+1})\Big\}$$

(we write (x_i) for the point $\langle x_0, x_1, \ldots \rangle$). A space $\varprojlim (I_i, f_i)$ is a *generalized inverse limit.*

This paper answers the following problem posed by Ingram:

Problem 1.1 (Ingram [In1]). *Characterize connectedness of inverse limits with upper semicontinuous functions on* $[0,1]$.

Partial answers to this problem, to date, include the following two theorems:

Theorem 1.2 ([IM2, Theorem 4.7]). *Suppose that for each nonnegative integer i, X_i is a compact Hausdorff space and $f_{i+1} : X_{i+1} \to 2^{X_i}$ is an upper semicontinuous function. If for each $i \in \mathbb{N}$, X_i is connected and for each $x \in X_{i+1}$, $f_{i+1}(x)$ is connected, then the inverse limit is connected.*

Theorem 1.3 ([IM2, Theorem 4.8]). *Suppose that for each nonnegative integer i, X_i is a compact Hausdorff space and $f_{i+1} : X_{i+1} \to 2^{X_i}$ is an upper semicontinuous function. If for each $i \in \mathbb{N}$, X_i is connected and for each $x \in X_i$, $\{y \in X_{i+1} : x \in f_{i+1}(y)\}$ is connected, then the inverse limit is connected.*

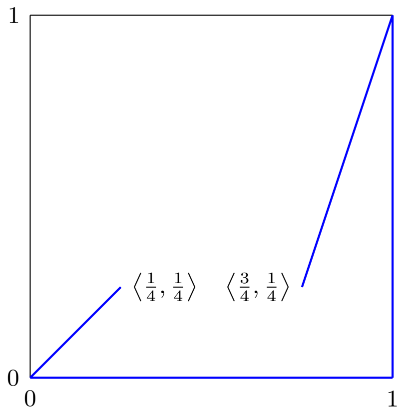

Fig. 1

Ingram showed that if for each $i \geq 0$, $I_i = [0,1]$ and $f_{i+1} : I_{i+1} \to 2^{I_i}$ is the bonding map whose graph is shown in Figure 1, then the inverse limit is disconnected. If we change the graph in this example by changing the line segment from $\left\langle \frac{3}{4}, \frac{1}{4} \right\rangle$ to $\langle 1, 1 \rangle$ to a line segment from $\left\langle \frac{3}{4}, \frac{1}{4} + \epsilon \right\rangle$ to $\langle 1, 1 \rangle$ for any ϵ, $0 < \epsilon < \frac{3}{4}$, then the inverse limit is connected. This small change does not affect any of the topological properties of the graph, nor does it affect any continuity property of the bonding functions. What it does do is destroy the alignment of the graph so that, for example, no point (x_i) in the inverse limit has first three coordinates $\frac{1}{4}, \frac{1}{4}, \frac{3}{4}$, which in the original example allows the set

$$\left\{ \left\langle \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, x_3, \ldots \right\rangle : \forall n \geq 3,\, x_n \in [0,1] \right\} \cap \varprojlim (I_i, f_i)$$

to be a nonempty clopen subset of $\varprojlim (I_i, f_i)$.

The following theorem from [GK] essentially selects a sequence of closed sets that are appropriately aligned, giving rise to a C-sequence. (The theorem is generalized to compact Hausdorff spaces in [GL]). We will not give the definition of a C-sequence here, but note that it is similar to that of a CC-sequence defined in the next section.

THEOREM 1.4. *Suppose that for each $i \geq 0$, I_i is an interval, and for each $i > 0$, $f_i : I_i \to 2^{I_{i-1}}$ is a surjective upper semicontinuous function and the graph G_i of f_i is connected. There exists $m \geq 0$ and $n > m+1$ such that if $m \leq i \leq n$ then there exists an open interval U_i and a closed interval A_i such that $A_i \subset U_i \subset I_i$, $U_i \neq I_i$, and*

$$\varprojlim (I_i, f_i) \cap \Big(\prod_{0 \leq i < m} I_i \times \prod_{m \leq i \leq n} U_i \times \prod_{i > n} I_i \Big) = \varprojlim (I_i, f_i) \cap \Big(\prod_{0 \leq i < m} I_i \times \prod_{m \leq i \leq n} A_i \times \prod_{i > n} I_i \Big) \neq \emptyset,$$

if and only if $\{f_i : i > 0\}$ has a C-sequence over $[m, n]$.

However, there may be an alignment that renders an inverse limit disconnected, but one that cannot be captured by a C-sequence.

EXAMPLE 1.5. For each $i \geq 0$, let $I_i = [0,1]$ and let $f_{i+1} : I_{i+1} \to 2^{I_i}$ be the bonding function whose graph G_{i+1} is as shown in Figure 2.

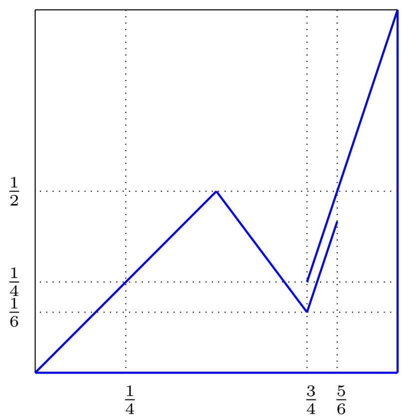

Fig. 2

Consider the basic open set

$$U = \left(\frac{1}{12}, \frac{7}{12}\right) \times \left(\frac{1}{12}, \frac{11}{12}\right) \times \left(\frac{4}{6}, \frac{35}{36}\right) \times \prod_{i \geq 3} I_i.$$

Observe that $G_1 \cap \left(\left(\frac{1}{12}, \frac{11}{12}\right) \times \left(\frac{1}{12}, \frac{7}{12}\right) \right)$ has two components. Let C be the component that contains the point $\langle \frac{1}{4}, \frac{1}{4} \rangle$ and let D be the other (see the first graph in Figure 3). The set $G_2 \cap \left(\left(\frac{4}{6}, \frac{35}{36}\right) \times \left(\frac{1}{12}, \frac{11}{12}\right) \right)$ also has two components. Let D' be the one containing $\langle \frac{3}{4}, \frac{1}{4} \rangle$ and C' the other (see the second graph in Figure 3).

Let K be a component of $U \cap \varprojlim (I_i, f_i)$. Either for each $(x_i) \in K$, $\langle x_1, x_0 \rangle \in C$, or for each $(x_i) \in K$, $\langle x_1, x_0 \rangle \in D$, and either for each $(x_i) \in K$, $\langle x_2, x_1 \rangle \in C'$, or for each $(x_i) \in K$, $\langle x_2, x_1 \rangle \in D'$.

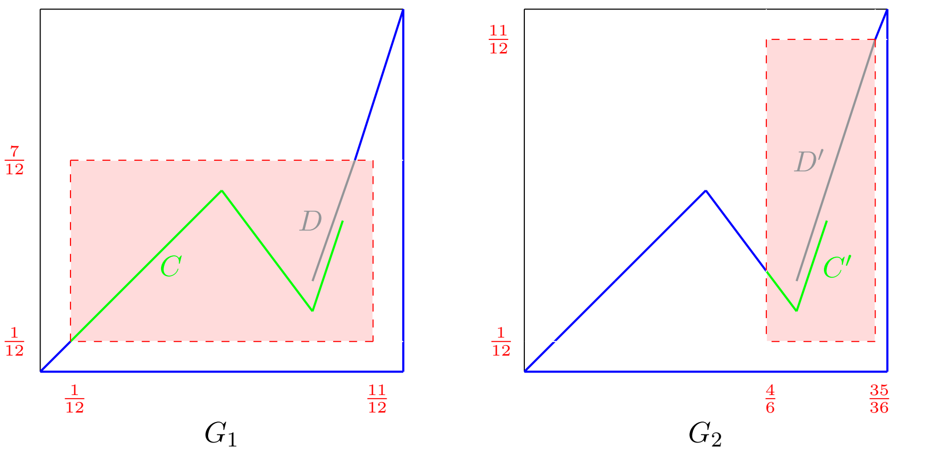

Fig. 3

Suppose that for each $(x_i) \in K$, $\langle x_1, x_0 \rangle \in C$ and $\langle x_2, x_1 \rangle \in D'$. Since $\langle x_1, x_0 \rangle \in C$, $x_0 \leq \frac{1}{2}$ and $x_1 \leq \frac{5}{6}$; since $\langle x_2, x_1 \rangle \in D'$, $\frac{1}{4} \leq x_1$. Thus $\frac{1}{6} \leq x_0 \leq \frac{1}{2}$, and since $x_1 \leq \frac{5}{6}$, $\frac{3}{4} \leq x_2 \leq \frac{17}{18}$. Thus we have a component K of $\varprojlim (f)$ such that

$$\begin{aligned} K &\subset \left[\frac{1}{6}, \frac{1}{2}\right] \times \left[\frac{1}{4}, \frac{5}{6}\right] \times \left[\frac{3}{4}, \frac{17}{18}\right] \times \prod_{i \geq 3} I_i \\ &\subset \left(\frac{1}{12}, \frac{7}{12}\right) \times \left(\frac{1}{12}, \frac{11}{12}\right) \times \left(\frac{4}{6}, \frac{35}{36}\right) \times \prod_{i \geq 3} I_i, \end{aligned}$$

hence K is clopen, and $K \neq \emptyset$ since $\langle \frac{1}{4}, \frac{1}{4}, \frac{3}{4} \rangle \in K$.

Thus, simply by considering f_1 and f_2 in Example 1.5, we have shown that $\varprojlim (I_i, f_i)$ is disconnected. Theorem 1.4 gives a condition (the existence of a C-sequence over some integer interval $[m, n]$) that implies disconnectedness of a generalized inverse limit. The bonding maps in this example do not admit a C-sequence over $[1, 2]$, as a C-sequence requires strong conditions which are not satisfied by the graphs in the example. Theorem 1.4 does not imply the disconnection observed in this example.

We prove the following theorem (a CC-sequence is defined in the next section).

THEOREM 1.6. *Suppose that for each $i \geq 0$, I_i is an interval, $f_{i+1} : I_{i+1} \to 2^{I_i}$ is a surjective upper semicontinuous function and the graph G_{i+1} of f_{i+1} is connected. The system admits a CC-sequence if and only if $\varprojlim (I_i, f_i)$ is disconnected.*

A CC-sequence picks out a connected subset in each product space $I_i \times I_{i-1}$ over some finite range $m < i \leq n$, and describes the possible behaviour of the graphs G_i within these subsets that will ensure that they trap a component of the inverse limit.

The following result is an immediate consequence of Theorem 1.6.

COROLLARY 1.7. *Suppose that for each $i \geq 0$, I_i is an interval, $f_{i+1} : I_{i+1} \to 2^{I_i}$ is a surjective upper semicontinuous function and the graph of f_{i+1} is connected. Then $\varprojlim (I_i, f_i)$ is disconnected if and only if there exist a component C of $\varprojlim (I_i, f_i)$ and numbers $m, n \geq 0$ such that $m < n - 1$ and $\pi_i(C) \neq I_i$ for each i, $m \leq i \leq n$.*

In Section 2 we establish basic notation and definitions, and present technical lemmas that are required to prove the main theorem, Theorem 1.6. In Section 3 we prove the forward direction of the theorem. In Section 4 we prove a number of results giving conditions that are necessary for an inverse limit to be disconnected. In Section 5 we complete the proof of Theorem 1.6.

2. Preliminaries.

This section presents definitions, notation and lemmas required for what follows. Definition 2.6 of a CC-sequence involves subsets (defined in Definition 2.5), with certain characteristics, of the graphs of bonding functions. Definition 2.20 introduces a property of a graph called the crossing property, and Proposition 2.23 establishes a relationship between sets defined in 2.5 and the crossing property. The remaining results are of a more technical nature.

NOTATION. $\mathbb{N}$ is the set of natural numbers $\{0, 1, 2, \ldots\}$. If $m, n \in \mathbb{N}$ and $m < n$, then we denote the sequence $\langle m, \ldots, n \rangle$ by $[m, n]$. (In each situation in which we require this notation it will be clear that $[m, n]$ does not refer to an interval of real numbers.)

The notation below is defined for general topological spaces denoted X, Y or X_i for each $i \in \mathbb{N}$. In most cases, and in particular in our main results,

X, Y and each X_i are a closed unit interval which we denote by I or I_i. Several of the preliminary results are more general and hold for all compact Hausdorff spaces.

Suppose that X and Y are topological spaces, and $f : X \to 2^Y$ is a function. Then:

- $\pi_V : X \times Y \to Y$ and $\pi_H : X \times Y \to X$ are the projection functions.
- $f^{-1} : Y \to X$ is the function defined by $f^{-1}(x) = \{y : x \in f(y)\}$. Observe that if X and Y are compact Hausdorff spaces, then f is upper semicontinuous if and only if f^{-1} is upper semicontinuous.
- If $S \subseteq X$, define $V_S = S \times Y$, and if $S \subseteq Y$, define $H_S = X \times S$. If $S = \{x\}$ write V_x and H_x.

Suppose that for each $i \in \mathbb{N}$, X_i is a topological space, and $f_{i+1} : X_{i+1} \to 2^{X_i}$ is a function. Then:

- For each $i \in \mathbb{N}$, $\pi_j : \prod_{i\in\mathbb{N}} X_i \to X_j$ is the projection onto X_j.
- If $m, n \in \mathbb{N}$ and $m < n$, then
$$\pi_{[m,n]} : \prod_{i\in\mathbb{N}} X_i \to \prod_{m\le i\le n} X_i$$
is the projection onto $\prod_{m\le i\le n} X_i$, and
$$\pi_{[m,\infty)} : \prod_{i\in\mathbb{N}} X_i \to \prod_{i\ge m} X_i$$
is the projection onto $\prod_{i\ge m} X_i$.
- If $m \in \mathbb{N}$ and $k_j \in \mathbb{N}$ for each $j \le m$, then
$$\pi_{k_0,\dots,k_m} : \prod_{i\in\mathbb{N}} X_i \to X_{k_0} \times \cdots \times X_{k_m}$$
is the projection onto $X_{k_0} \times \cdots \times X_{k_m}$. There is no assumption that the natural numbers $k_0, \dots, k_m$ are ordered. This notation is used, most frequently to project a subset of an inverse limit into a graph. For example if $\mathbf{x} = (x_i) \in \varprojlim (X_i, f_i)$, then $\pi_{1,0}(\mathbf{x}) = \langle x_1, x_0\rangle$, and $\langle x_1, x_0\rangle$ is an element of the graph of f_1.
- For each $i \in \mathbb{N}$,
$$\pi_{i+1,V} : X_{i+1} \times X_i \to X_i \quad \text{and} \quad \pi_{i+1,H} : X_{i+1} \times X_i \to X_{i+1}$$
are the projections onto X_i and X_{i+1} respectively.
- If $S \subseteq X_{i+1}$, define $V_S^{f_{i+1}} = S \times X_i$, and if $S \subseteq X_i$, define $H_S^{f_{i+1}} = X_{i+1} \times S$. If $S = \{x\}$ write $V_x^{f_{i+1}}$ and $H_x^{f_{i+1}}$. If it is clear which function is involved we will drop the superscripts and simply write V_S, H_S, V_x or H_x.

Suppose $m', n' \in \mathbb{N}$, $m' < n' - 1$. Then:

- For each $j, m' \le j \le n'$, $\pi_j : \prod_{i\in[m',n']} X_i \to X_j$ is the projection onto X_j.
- If $m, n \in \mathbb{N}$ and $m' \le m < n \le n'$, then
$$\pi_{[m,n]} : \prod_{i\in[m',n']} X_i \to \prod_{m\le i\le n} X_i$$
is the projection onto $\prod_{m\le i\le n} X_i$.
- If $m \in \mathbb{N}$ and $m' \le k_j \le n'$ for each $j \le m$, then
$$\pi_{k_0,\dots,k_m} : \prod_{i\in[m',n']} X_i \to X_{k_0} \times \cdots \times X_{k_m}$$
is the projection onto $X_{k_0} \times \cdots \times X_{k_m}$. Again there is no assumption that $k_0, \dots, k_m$ are ordered.

Whenever a projection function is used, it will be clear which domain is intended.

Definition 2.1. If X is a set, $A \subseteq X$, and p and q are two points in A, we say that p *is connected to* q *in* A if there is a connected set $C \subseteq A$ such that $p, q \in C$. Similarly, if each of the sets $E \subseteq A$ and $F \subseteq A$ is connected, we say that E *is connected to* F *in* A if there exists a connected subset of A that contains $E \cup F$.

Unless otherwise stated, all continua and subcontinua are nondegenerate.

A CC-sequence is defined in terms of frames. We first define these notions for one function and its graph, since we require a number of lemmas that involve this case.

Definition 2.2. Suppose $A_1 = [a, b]$ and $A_0 = [c, d]$. Define

$$\begin{aligned}
J_0 &= [0, c), \\
J_1 &= [0, a), \\
K_0 &= (d, 1], \\
K_1 &= (b, 1], \\
Z &= A_1 \times A_0, \\
Z(\epsilon) &= ((a-\epsilon, b+\epsilon) \times (c-\epsilon, d+\epsilon)) \\
&\quad \cap ([0,1] \times [0,1]), \\
L &= (J_1 \times [0,1]) \cup Z,
\end{aligned}
\qquad
\begin{aligned}
R &= (K_1 \times [0,1]) \cup Z, \\
T &= ([0,1] \times K_0) \cup Z, \\
B &= ([0,1] \times J_0) \cup Z, \\
TL &= T \cup L, \\
TR &= T \cup R, \\
BL &= B \cup L, \\
BR &= B \cup R.
\end{aligned}$$

Although the sets listed in Definition 2.2 depend on the sets A_0 and A_1, there will be no cause for ambiguity as A_0 and A_1 will always be clear.

Definition 2.3. Suppose that $f : [0, 1] \to 2^{[0,1]}$ is a function with graph G, $A_0 = [c, d] \subsetneq [0, 1]$ and $A_1 = [a, b] \subsetneq [0, 1]$. If

- $S \in \{BL, BR, TL, TR\}$, or
- $A_1 \subset (0, 1)$ and $S \in \{L, R\}$, or
- $A_0 \subset (0, 1)$ and $S \in \{B, T\}$,

and there exists $\epsilon > 0$ and a component C' of the set $G \cap Z(\epsilon)$ such that $C' \subset S$, then any component C of $C' \cap Z$ is an *S-set in G framed by $A_i \times A_{i-1}$* and we write $G \sqsubset_C S$.

For example, Figure 4 shows allowable behaviour of G if $G \sqsubset_C TL$.

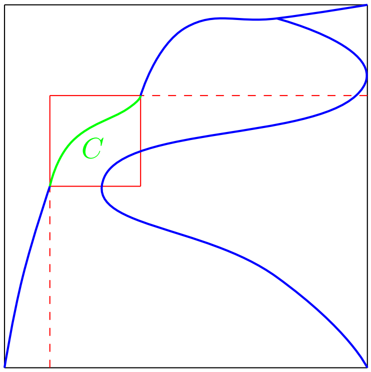

Fig. 4. $G \sqsubset_C TL$

We require the above terms for each member of an arbitrary sequence of functions.

Definition 2.4. Suppose $i > 0$, and I_i and I_{i-1} are closed intervals. Given closed intervals $A_j = [a_j, b_j] \subsetneq I_j$, $j \in \{i, i-1\}$, and $\epsilon > 0$, define

$$
\begin{aligned}
J_j &= [0, a_j), \\
K_j &= (b_j, 1], \\
Z_i &= A_i \times A_{i-1}, \\
Z_i(\epsilon) &= \big((a_i - \epsilon, b_i + \epsilon) \times (a_{i-1} - \epsilon, b_{i-1} + \epsilon)\big) \\
&\quad \cap (I_i \times I_{i-1}), \\
T_i &= (I_i \times K_{i-1}) \cup Z_i, \\
B_i &= (I_i \times J_{i-1}) \cup Z_i,
\end{aligned}
\qquad
\begin{aligned}
L_i &= (J_i \times I_{i-1}) \cup Z_i, \\
R_i &= (K_i \times I_{i-1}) \cup Z_i, \\
TL_i &= T_i \cup L_i, \\
TR_i &= T_i \cup R_i, \\
BL_i &= B_i \cup L_i, \\
BR_i &= B_i \cup R_i.
\end{aligned}
$$

Again the sets in Definition 2.4 depend on the sets A_i, and again there will be no cause for ambiguity as the A_i will always be clear.

Definition 2.5. Suppose $i > 0$, $I_i = I_{i-1} = [0, 1]$ and $f : I_i \to 2^{I_{i-1}}$ has graph G. If for each $j \in \{i, i-1\}$, $A_j = [a_j, b_j] \subsetneq I_j$, either

- $S \in \{BL_i, BR_i, TL_i, TR_i\}$, or
- $A_i \cap \{0, 1\} = \emptyset$ and $S \in \{L_i, R_i\}$, or
- $A_{i-1} \cap \{0, 1\} = \emptyset$ and $S \in \{B_i, T_i\}$,

and there exists $\epsilon > 0$ and a component C' of the set $G \cap Z_i(\epsilon)$ such that $C' \subset S$, then any component C of $C' \cap Z_i$ is an *S-set in G framed by $A_i \times A_{i-1}$*, written $G \sqsubset_C S$.

If the existence of a component C is clear from the context we will simply write $G \sqsubset S$.

Observe that in Example 1.5, if $A_0 = \left[\frac{1}{6}, \frac{1}{2}\right]$, $A_1 = \left[\frac{1}{4}, \frac{5}{6}\right]$ and $A_2 = \left[\frac{3}{4}, \frac{17}{18}\right]$, then $A_1 \times A_0$ frames an L-set which is a subset of C, and $A_2 \times A_1$ frames a T-set which is a subset D'.

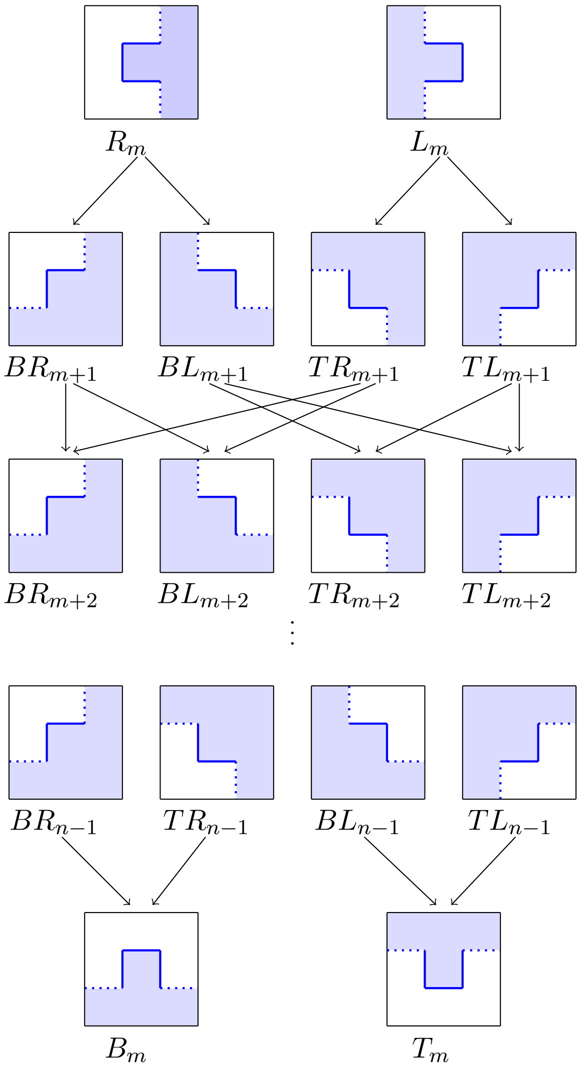

Fig. 5. CC-sequences

Definition 2.6. Suppose for each $i \in \mathbb{N}$, $I_i = [0,1]$, $f_{i+1} : I_{i+1} \to 2^{I_i}$ is a surjective upper semicontinuous function with a connected graph G_{i+1}, and $m, n \in \mathbb{N}$ are such that $m+1 < n$. Suppose that there exist

- a closed interval $A_i \subsetneq I_i$ for each i, $m \le i \le n$, and
- a point

$$(p_k) \in \varprojlim (I_i, f_i) \cap \Big(\prod_{i<m} I_i \times \prod_{m \le i \le n} A_i \times \prod_{i>n} I_i \Big).$$

For each $i > 0$ let C_i be the component of $G_i \cap Z_i$ containing $\langle p_i, p_{i-1} \rangle$ and suppose the following properties hold:

(1) $G_{m+1} \sqsubset_{C_{m+1}} R_{m+1}$ or $G_{m+1} \sqsubset_{C_{m+1}} L_{m+1}$;
(2) • if $n = m+2$, then $G_{m+2} \sqsubset_{C_{m+2}} T_{m+2}$ if $G_{m+1} \sqsubset_{C_{m+1}} L_{m+1}$, and $G_{m+2} \sqsubset_{C_{m+2}} B_{m+2}$ if $G_{m+1} \sqsubset_{C_{m+1}} R_{m+1}$;
 • if $n > m+2$, then $G_{m+2} \sqsubset_{C_{m+2}} BR_{m+2}$ or $G_{m+2} \sqsubset_{C_{m+2}} BL_{m+2}$ if $G_{m+1} \sqsubset_{C_{m+1}} R_{m+1}$, and $G_{m+2} \sqsubset_{C_{m+2}} TL_{m+2}$ or $G_{m+2} \sqsubset_{C_{m+2}} TR_{m+2}$ if $G_{m+1} \sqsubset_{C_{m+1}} L_{m+1}$;
(3) if $m+2 \le i < n-1$, then $G_{i+1} \sqsubset_{C_{i+1}} BL_{i+1}$ or $G_{i+1} \sqsubset_{C_{i+1}} BR_{i+1}$, if $G_i \sqsubset_{C_i} BR_i$ or $G_i \sqsubset_{C_i} TR_i$, and $G_{i+1} \sqsubset_{C_{i+1}} TL_{i+1}$ or $G_{i+1} \sqsubset_{C_{i+1}} TR_{i+1}$ if $G_i \sqsubset_{C_i} BL_i$ or $G_i \sqsubset_{C_i} TL_i$;
(4) if $n > m+2$, then $G_n \sqsubset_{C_n} B_n$ if $G_{n-1} \sqsubset_{C_{n-1}} BR_{n-1}$ or $G_{n-1} \sqsubset_{C_{n-1}} TR_{n-1}$, and $G_n \sqsubset_{C_n} T_n$ if $G_{n-1} \sqsubset_{C_{n-1}} BL_{n-1}$ or $G_{n-1} \sqsubset_{C_{n-1}} TL_{n-1}$.

Then $\{f_i : i > 0\}$ admits a *component cropping sequence*, or *CC-sequence*,

$$\{A_i : m \le i \le n\},$$

over $[m, n]$ with *pivot point* (p_k). The collection $\{f_i : i > 0\}$ of functions admits a *CC-sequence* if there exist $m, n \in \mathbb{N}$ such that $\{f_i : i > 0\}$ admits a CC-sequence over $[m, n]$ with some pivot point.

Figure 5 shows the forms of CC-sequences.

Comment. The main theorem states that a generalized inverse limit is disconnected if and only if the bonding maps admit a CC-sequence. If the length of the sequence is 2, it is reasonably easy to spot L-sets and R-sets in the first graph, T-sets and B-sets in the second graph and to find a pair that matches, forming a CC-sequence. A longer sequence is far more difficult to spot. Detecting L-, R-, T-, or B-sets at least indicates if a CC-sequence might be present. These are the sets that provide a disconnection. The role of the intermediary sets (the BL-sets etc.) is to bring an L-set or R-set into line with a T-set or B-set.

Definition 2.7. Suppose that for each $i \ge 0$, X_i is a topological space, and for each $i > 0$, $f_i : X_i \to 2^{X_{i-1}}$ is a function with graph G_i. For all

$m,n \in \mathbb{N}$, $0 < m \leq n$, define $\mathcal{G}(f_m,\ldots,f_n)$ to be the set

$$\Big\{\langle x_{m-1}, x_m, \ldots x_n\rangle \in \prod_{m-1\leq i\leq n} X_i : \forall i,\ m-1 \leq i < n,\ x_i \in f_{i+1}(x_{i+1})\Big\}.$$

Observe that

- $\mathcal{G}(f_m) = \{\langle y,x\rangle : \langle x,y\rangle \in G_m\}$, and
- if each bonding map in Definition 2.7 is surjective, then

$$\mathcal{G}(f_m,\ldots,f_n) = \pi_{[m-1,n]}\Big(\varprojlim (X_i,f_i)\Big).$$

Definition 2.8. Suppose that for each $i \geq 0$, X_i is a topological space and $f_{i+1} : X_{i+1} \to 2^{X_i}$ is a function, and $m \in \mathbb{N}$. For every $i \in \mathbb{N}$, let $X_i^m = X_{i+m}$, and $f_{i+1}^m = f_{i+1+m}$. Define $\varprojlim (X_i,f_i)_{i\geq m} = \varprojlim (X_i^m, f_i^m)$.

Lemma 2.9. *Suppose that for each $i \geq 0$, $I_i = [0,1]$ and $f_{i+1} : I_{i+1} \to 2^{I_i}$ is a surjective upper semicontinuous function, $m,n \in \mathbb{N}$, $m < n+1$, for each i, $m \leq i \leq n$, A_i is a closed subinterval of I_i, and $(p_i) \in \varprojlim (I_i,f_i)$. If C' is a component of $\varprojlim (I_i,f_i)_{i\geq m}$ and*

$$\langle p_m, p_{m+1},\ldots\rangle \in C' \subseteq \prod_{m\leq i\leq n} A_i \times \prod_{i>n} I_i,$$

then $\pi^{-1}_{[m,\infty)}(C') \cap \varprojlim (I_i,f_i)$ contains a component C of $\varprojlim (I_i,f_i)$ and

$$(p_i) \in C \subseteq \prod_{0\leq i<m} I_i \times \prod_{m\leq i\leq n} A_i \times \prod_{i>n} I_i.$$

Proof. Suppose C' is a component of $\varprojlim (I_i,f_i)_{i\geq m}$ and

$$\langle p_m, p_{m+1},\ldots\rangle \in C' \subseteq \prod_{m\leq i\leq n} A_i \times \prod_{i>n} I_i.$$

Let C be the component of $\varprojlim (I_i,f_i)$ such that $(p_i) \in C$. Then

- $\pi_{[m,\infty)}(C)$ is connected, and
- $\langle p_m, p_{m+1},\ldots\rangle \in \pi_{[m,\infty)}(C)$,

and since

- C' is a component of $\varprojlim (I_i,f_i)_{i\geq m}$, and
- $\langle p_m, p_{m+1},\ldots\rangle \subseteq C'$,

it follows that $\pi_{[m,\infty)}(C) \subseteq C'$ and hence $C \subseteq \prod_{0\leq i<m} I_i \times \prod_{m\leq i\leq n} A_i \times \prod_{i>n} I_i$. ■

Lemma 2.11(1) below gives a generalization of Nall's theorem:

Theorem 2.10 (Nall [Nal]). *If X is a Hausdorff continuum and $f : X \to 2^X$ is an upper semicontinuous function, then $\varprojlim (f)$ is connected if and only if $\mathcal{G}(f_1,\ldots,f_n)$ is connected for each $n > 0$.*

Lemma 2.11. *Suppose that for each $i \in \mathbb{N}$, X_i is a topological space, and for each $i > 0$, $f_i : X_i \to 2^{X_{i-1}}$ is a surjective upper semicontinuous function. Then*

(1) *$\varprojlim (I_i, f_i)$ is connected if and only if $\mathcal{G}(f_m, \ldots, f_n)$ is connected for all $m, n \in \mathbb{N}$, $1 \le m \le n$, and*
(2) *if $\mathcal{G}(f_1, \ldots, f_n)$ is disconnected then $\mathcal{G}(f_1, \ldots, f_{n'})$ is disconnected for all $n' \ge n$.*

Proof. (1) If there exist $m, n \in \mathbb{N}$, $1 \le m \le n$, such that $\mathcal{G}(f_m, \ldots, f_n)$ is disconnected, then $\mathcal{G}(f_1^m, \ldots, f_{n-m}^m)$ is disconnected, hence $\varprojlim (X_i, f_i)_{i \ge m}$ is disconnected by Theorem 2.10, and so $\varprojlim (I_i, f_i)$ is disconnected by Lemma 2.9.

If for every $m, n \in \mathbb{N}$, $0 \le m \le n$, $\mathcal{G}(f_m, \ldots, f_n)$ is connected, then $\mathcal{G}(f_1, \ldots, f_n)$ is connected for each $n > 0$ and hence $\varprojlim (I_i, f_i)$ is connected by Theorem 2.10.

(2) Straightforward. ∎

In some of the proofs in Sections 4 and 5 we take subcontinua of the graphs of given functions and form new functions whose graphs are the subcontinua. The following two propositions are required in cases where the domain or range of one or more of these new functions is a singleton.

Proposition 2.12. *Suppose $n > 0$, $A \subset \{0, \ldots, n\}$, $A' = \{0, \ldots, n\} \setminus A$ and the sequence $\langle i_0, \ldots, i_k \rangle$ lists the members of A' in increasing order. Suppose also that for each $i \le n$, X_i is a compact Hausdorff space and if $i \in A$ then X_i is a singleton, $\{a_i\}$ say, and for each $i < n$, $f_{i+1} : X_{i+1} \to 2^{X_i}$ is a surjective upper semicontinuous function. If for each j, $0 < j \le k$,*

- *whenever $i_j \in A'$ and $i_j - 1 \in A$, $g_j : X_{i_j} \to 2^{X_{i_{j-1}}}$ is the function whose graph is $X_{i_j} \times X_{i_{j-1}}$, and*
- *whenever $i_j \in A'$ and $i_j - 1 \in A'$, $g_j = f_{i_j}$,*

then $\mathcal{G}(f_1, \ldots, f_n)$ is homeomorphic to $\mathcal{G}(g_1, \ldots, g_k)$.

Proof. If $i_0 > 0$ then since $X_i = \{a_i\}$ for each $i < i_0$,

$$\mathcal{G}(f_1, \ldots, f_n) = \{\langle a_0, \ldots, a_{i_0-1}, x_{i_0}, \ldots, x_n \rangle : \langle x_{i_0}, \ldots, x_n \rangle \in \mathcal{G}(f_{i_0+1}, f_{i_0+2}, \ldots, f_n)\}.$$

Hence $\mathcal{G}(f_1, \ldots, f_n)$ is homeomorphic to

$$\{\langle a_0, \ldots, a_{i_0-1} \rangle\} \times \mathcal{G}(f_{i_0+1}, f_{i_0+2}, \ldots, f_n),$$

which is homeomorphic to $\mathcal{G}(f_{i_0+1}, f_{i_0+2}, \ldots, f_n)$.

If $i_k < n$ then since $f_{i_k+1} : \{a_{i_k+1}\} \to 2^{X_{i_k}}$ is surjective, the graph of $f_{i_{k+1}}$ is the set $\{\langle a_{i_{k+1}}, x \rangle : x \in X_{i_k}\}$. Thus

$$\mathcal{G}(f_1, \ldots, f_n) = \{\langle x_0, \ldots, x_{i_k}, a_{i_{k+1}}, \ldots, a_n \rangle : \langle x_0, \ldots, x_{i_k} \rangle \in \mathcal{G}(f_1, \ldots, f_{i_k})\}.$$

Hence $\mathcal{G}(f_1, \ldots, f_n)$ is homeomorphic to

$$\mathcal{G}(f_1, \ldots, f_{i_k}) \times \{\langle a_{i_{k+1}}, \ldots, a_n \rangle\},$$

which is homeomorphic to $\mathcal{G}(f_1, \ldots, f_{i_k})$. Thus $\mathcal{G}(f_1, \ldots, f_n)$ is homeomorphic to $\mathcal{G}(f_{i_0+1}, f_{i_0+2}, \ldots, f_{i_k})$.

If $i_1 - 1 \in A$, then

$$\mathcal{G}(f_{i_0+1}, f_{i_0+2}, \ldots, f_{i_1}) = \{\langle x, a_{i_0+1}, \ldots, a_{i_1-1}, y \rangle : x \in X_{i_0},\ y \in X_{i_1}\}.$$

Thus, since the range of f_{i_1} is $\{a_{i_1-1}\}$ and

$$f_{i_0+1} : \{a_{i_0+1}\} \to 2^{X_{i_0}}$$

is surjective, $\mathcal{G}(f_{i_0+1}, f_{i_0+2}, \ldots, f_{i_1})$ is homeomorphic to $X_{i_1} \times X_{i_0}$ which is the graph of g_1, and hence $\mathcal{G}(f_1, \ldots, f_n)$ is homeomorphic to

$$\mathcal{G}(g_1, f_{i_1+1}, \ldots, f_{i_k}).$$

If $i_1 - 1 \in A'$, then $i_1 = i_0 + 1$ and $g_1 = f_{i_1} = f_{i_0+1}$. Hence $\mathcal{G}(f_1, \ldots, f_n)$ is homeomorphic to

$$\mathcal{G}(g_1, f_{i_0+2}, \ldots, f_{i_k}) = \mathcal{G}(g_1, f_{i_1+1}, \ldots, f_{i_k}).$$

Suppose $\mathcal{G}(f_1, \ldots, f_n)$ is homeomorphic to

$$\mathcal{G}(g_1, \ldots, g_l, f_{i_l+1}, \ldots, f_{i_k}).$$

Then similarly, either $i_{l+1} = i_l + 1$, and hence $f_{i_{l+1}} = g_{l+1}$, or $i_{l+1} \neq i_l + 1$, and $\mathcal{G}(f_{i_l+1}, \ldots, f_{i_{l+1}})$ is homeomorphic to $X_{i_{l+1}} \times X_{i_l}$, which is the graph of g_{l+1}, and in either case we see that $\mathcal{G}(g_1, \ldots, g_l, f_{i_l+1}, \ldots, f_{i_k})$ is homeomorphic to

$$\mathcal{G}(g_1, \ldots, g_{l+1}, f_{i_{l+1}+1}, \ldots, f_{i_k}).$$

Thus the result follows by induction. ∎

PROPOSITION 2.13. *Suppose $m, n \in \mathbb{N}$, $0 < m < n$, and $A = \langle i_1, \ldots, i_m \rangle$ is an increasing sequence in $\mathbb{N} \setminus \{0\}$. Suppose also that for each $i < n$, $f_{i+1} : X_{i+1} \to 2^{X_i}$ is a surjective upper semicontinuous function whose graph is connected, and if $i_j \in A$ then $f_{i_j} : X_{i_j} \to 2^{X_{i_j-1}}$ is the function whose graph is $X_{i_j} \times X_{i_j-1}$. Then $\mathcal{G}(f_1, \ldots, f_n)$ is disconnected if and only if*

(1) *$i_1 > 1$ and $\mathcal{G}(f_1, \ldots, f_{i_1-1})$ is disconnected, or*
(2) *$i_m < n$ and $\mathcal{G}(f_{i_m+1}, \ldots, f_n)$ is disconnected, or*
(3) *there exists j such that $\mathcal{G}(f_{i_j+1}, \ldots, f_{i_{j+1}-1})$ is disconnected.*

Proof. This follows from the observation that

$$\mathcal{G}(f_1, \ldots, f_n) = \mathcal{G}(f_1, \ldots, f_{i_1-1}) \times \mathcal{G}(f_{i_1+1}, \ldots, f_{i_2-1}) \times \cdots \times \mathcal{G}(f_{i_m+1} \ldots f_n).$$ ∎

2.1. $G(\epsilon)$, fibre-connected subgraphs, and the crossing property

DEFINITION 2.14. Suppose that $f : [0,1] \to 2^{[0,1]}$ is an upper semicontinuous function with a connected graph G. If H is a subcontinuum of G and for each $x \in [0,1]$, either $\big(\{x\} \times f(x)\big) \cap H = \emptyset$, or $\big(\{x\} \times f(x)\big) \cap H$ is connected, then H is a *fibre-connected subgraph* of G.

PROPOSITION 2.15. *Suppose that $f : [0,1] \to 2^{[0,1]}$ is an upper semicontinuous function with a connected graph G. There exists an ordinal κ and a fibre-connected subgraph $G_\alpha \subseteq G$ for each $\alpha < \kappa$ such that*

(1) $G = \bigcup\{G_\alpha : \alpha \in \kappa\}$,
(2) *for each* $\alpha < \kappa$, $G \neq \bigcup\{G_\beta : \beta \in \kappa,\ \beta \neq \alpha\}$, *and*
(3) *if H is any fibre-connected subgraph of G such that $G_\alpha \subseteq H$ for some α, then $G_\alpha = H$.*

Proof. First, we prove

CLAIM. *For each point $p \in G$ there is a fibre-connected subgraph G_p of G such that if H is any fibre-connected subgraph of G and $G_p \subseteq H$, then $G_p = H$.*

Proof of Claim. Let $\mathcal{P}$ be the collection of all connected subsets S of G such that

- $p \in S$, and
- for every $x \in \pi_H(S)$, $f(x) \cap S$ is connected (π_H is defined on p. 7).

Clearly "$\subseteq$" is a partial order on $\mathcal{P}$, and the union of any chain in $\mathcal{P}$ is an upper bound of the chain. Hence by Zorn's Lemma, $\mathcal{P}$ has a maximal element G_p. Clearly $\overline{G_p} \in \mathcal{P}$, where $\overline{G_p}$ is the closure of G_p in G and hence in $[0,1] \times [0,1]$. Therefore G_p is closed (otherwise G_p is not maximal). Thus G_p is a subgraph and satisfies the requirements of the claim.

Let $\mathcal{Q}$ be the collection of all sets $Q \subseteq \{G_p : p \in G\}$ such that $G = \bigcup Q$ (and hence Q satisfies (1) and (3)). Let $\preccurlyeq$ be the partial order on $\mathcal{Q}$ where for all $P, Q \in \mathcal{Q}$, $P \preccurlyeq Q$ if and only if $Q \subseteq P$. Let $\langle Q_\alpha : \alpha < \lambda\rangle$ be a chain in $\mathcal{Q}$, λ an ordinal, and let $Q = \bigcap_{\alpha<\lambda} Q_\alpha$. Clearly $Q \subseteq \{G_p : p \in G\}$ and $G = \bigcup Q$, and hence Q is an upper bound for the sequence. Thus by Zorn's Lemma there exists a maximal element $M = \{G_{p_\alpha} : \alpha < \mu\}$ for some ordinal μ. This M satisfies (2): if there exists $\beta < \mu$ such that $G = \bigcup\{G_{p_\gamma} : \gamma < \mu \text{ and } \gamma \neq \beta\}$, then $M \smallsetminus \{G_{p_\beta}\} \in \mathcal{Q}$, $M \preccurlyeq M \smallsetminus \{G_{p_\beta}\}$ and $M \neq M \smallsetminus \{G_{p_\beta}\}$, which contradicts the maximality of M, and so M satisfies (1)–(3). ■

DEFINITION 2.16. Suppose that $f : [0,1] \to 2^{[0,1]}$ is an upper semicontinuous function with graph G, and

(1) κ is an ordinal and for each $\alpha < \kappa$, $G_\alpha \subseteq G$ is a fibre-connected subgraph,

(2) $G = \bigcup\{G_\alpha : \alpha \in \kappa\}$,
(3) for each $\alpha < \kappa$, $G \neq \bigcup\{G_\beta : \beta \in \kappa,\ \beta \neq \alpha\}$, and
(4) if H is any fibre-connected subgraph of G such that $G_\alpha \subseteq H$ for some α, then $G_\alpha = H$.

Then $\{G_\alpha : \alpha \in \kappa\}$ is a *decomposition of G into fibre-connected subgraphs.* If $\kappa = 1$ then G is *fibre-connected.*

Remarks. A maximal decomposition of a graph G into fibre-connected subgraphs need not be unique, and the members of the decomposition will not be pairwise disjoint if G is connected (and there is more than one member of the decomposition). It may happen that fibre-connected subgraphs are trivial. For example, consider a graph that is a pseudoarc. However, our results involving decompositions into fibre-connected subgraphs will have finite decompositions.

Definition 2.17. If G is the graph of a function $f : [0,1] \to 2^{[0,1]}$ then for each $\epsilon > 0$, let

$$G(\epsilon) = \bigcup\{([x-\epsilon, x+\epsilon] \times [y-\epsilon, y+\epsilon]) \cap [0,1]^2 : \langle x, y\rangle \in G\}.$$

Lemma 2.18. *Suppose G is the graph of an upper semicontinuous function $f : [0,1] \to 2^{[0,1]}$ and G is connected. For any $\epsilon > 0$ if $\{G_\alpha : \alpha \in \kappa\}$ is a decomposition of $G(\epsilon)$ into fibre-connected subgraphs, then*

(1) *$G(\epsilon)$ is closed in $[0,1] \times [0,1]$,*
(2) *for each α, $\pi_V(G_\alpha)$ are $\pi_H(G_\alpha)$ are nontrivial intervals, and*
(3) *κ is finite.*

Proof. (1) and (2) are straightforward.

(3) Suppose κ is infinite. By Definition 2.16(3),

(∗) for each $\alpha < \kappa$, there exists $\langle x_\alpha, y_\alpha\rangle \in G_\alpha$ such that for every $\beta < \kappa$, if $\beta \neq \alpha$, then $(\{x_\alpha\} \times f(x_\alpha)) \cap G_\alpha \cap G_\beta = \emptyset$.

For each α choose a point $\langle x'_\alpha, y'_\alpha\rangle \in G$ such that

$$\langle x_\alpha, y_\alpha\rangle \in S_\alpha := [x'_\alpha - \epsilon, x'_\alpha + \epsilon] \times [y'_\alpha - \epsilon, y'_\alpha + \epsilon].$$

Let $\mathcal{S} = \{S_\alpha : \alpha < \kappa\}$.

Claim. *There exists a point $a \in [0,1]$ such that $A := \{\alpha \in \kappa : V_a \cap S_\alpha \neq \emptyset\}$ is infinite.*

Proof. The set $E := \{V_{j\epsilon} : j \in \mathbb{N} \text{ and } 0 \leq j < 1/\epsilon\}$ is finite and each member of $\mathcal{S}$ meets some member of E. Since κ is infinite and hence $\mathcal{S}$ is infinite, there exists j' such that $\{\alpha : V_{j'\epsilon} \cap S_\alpha \neq \emptyset\}$ is infinite. Let $a = j'\epsilon$.

Let $m = \min\{x_\alpha : \alpha \in A\}$ and $M = \max\{x_\alpha : \alpha \in A\}$ (m and M exist since G is closed in $[0,1] \times [0,1]$). If $a \leq m \leq M$ then there exists a point $b \in [m, M]$ such that the set $A' := \{\alpha \in A : V_b \cap S_\alpha \neq \emptyset \text{ and } m \leq x_\alpha \leq b\}$ is infinite. Hence there exists a component E of $\bigcup\{S_\alpha : \alpha \in A'\} \cap V_{[m,b]}$ such that the set $\{\alpha \in A' : S_\alpha \cap E \neq \emptyset, V_m \cap S_\alpha \neq \emptyset$ and $V_b \cap S_\alpha \neq \emptyset\}$ is infinite. This contradicts $(*)$.

A similar argument holds if either $m \leq a \leq M$ or $m \leq M \leq a$. This proves the claim, and hence Lemma 2.18. ∎

THEOREM 2.19. *Suppose that for each $i \in \mathbb{N}$, $I_i = [0,1]$ and $f_{i+1} : I_{i+1} \to 2^{I_i}$ is an upper semicontinuous surjective function with graph G_i. For each $i \in \mathbb{N}$ and $\epsilon > 0$, let $f_{i+1,\epsilon} : I_{i+1} \to 2^{I_i}$ be the function whose graph is $G_{i+1}(\epsilon)$. If there exists $\delta > 0$ such that for every ϵ, $0 < \epsilon < \delta$, $\varprojlim (I_i, f_{i,\epsilon})$ is connected, then $\varprojlim (I_i, f_i)$ is connected.*

Proof. Suppose that there exists $\delta > 0$ such that for every ϵ, $0 < \epsilon < \delta$, $\varprojlim (I_i, f_{i,\epsilon})$ is connected. If $0 < \epsilon_0 < \epsilon_1 < \delta$ then clearly $\varprojlim (I_i, f_{i,\epsilon_0}) \subseteq \varprojlim (I_i, f_{i,\epsilon_1})$, so the collection of sets $\varprojlim (I_i, f_{i,\epsilon})$ is nested, and hence the set $\bigcap_{0<\epsilon<\delta} \varprojlim (I_i, f_{i,\epsilon})$ is connected.

Since for each i the graph $G_i(\epsilon)$ is closed, it follows that $\bigcap_{0<\epsilon<\delta} G_i(\epsilon) = G_i$.

Observe that $\mathbf{x} \in \varprojlim (I_i, f_{i,\epsilon})$ for each $\epsilon > 0$ if and only if $\pi_{i,i-1}(\mathbf{x}) \in G_i(\epsilon)$ for each $i > 0$ and $\epsilon > 0$, if and only if $\pi_{i,i-1}(\mathbf{x}) \in G_i$ for each $i > 0$. Hence $\bigcap_{0<\epsilon<\delta} \varprojlim (I_i, f_{i,\epsilon}) = \varprojlim (I_i, f_i)$ and so $\varprojlim (I_i, f_i)$ is connected. ∎

DEFINITION 2.20.

- Suppose that $f : [0,1] \to 2^{[0,1]}$ is a function with graph G. We say that G has the *crossing property* if for all a, b such that $0 \leq a < b \leq 1$, and for every component C of $G \cap V_{[a,b]}$, $C \cap V_a \neq \emptyset$ and $C \cap V_b \neq \emptyset$.
- Suppose S is a subcontinuum of G. If $\pi_H(S) = [s,t]$, then S has the *crossing property* if either $s = t$, or for all a, b such that $s \leq a < b \leq t$, and for every component C of $S \cap V_{[a,b]}$, $C \cap V_a \neq \emptyset$ and $C \cap V_b \neq \emptyset$.
- Suppose S is a subcontinuum of G. Let $S^{-1} = \{\langle x, y\rangle : \langle y, x\rangle \in S\}$. If $\pi_V(S) = [s,t]$, then S^{-1} has the *crossing property* if either $s = t$, or for all a, b such that $s \leq a < b \leq t$, and for every component C of $S \cap H_{[a,b]}$, $C \cap H_a \neq \emptyset$ and $C \cap H_b \neq \emptyset$.

In Section 4 we require the following definition.

DEFINITION 2.21. Suppose $f : [0,1] \to 2^{[0,1]}$ is a function with an upper semicontinuous graph G, and S is a subcontinuum of G. Then S is a *CP-subcontinuum* of G if both S and $S^{-1} = \{\langle x, y\rangle : \langle y, x\rangle \in S\}$ have the crossing property.

LEMMA 2.22. *Suppose that $f : [0,1] \to 2^{[0,1]}$ is a function with a connected graph G. If $\{G_\alpha : \alpha \in \kappa\}$ is a decomposition of G into fibre-connected subgraphs, then each subgraph G_α has the crossing property.*

The proof is straightforward, so we omit it.

Proposition 2.23. *Suppose that $f : [0,1] \to 2^{[0,1]}$ is a surjective upper semicontinuous function with a connected graph G. Then G has the crossing property if and only if G does not admit an L-set or an R-set, and the graph of f^{-1} has the crossing property if and only if G does not admit a B-set or a T-set.*

Proof. Suppose G does not have the crossing property. Then there exist $a, b \in [0,1]$, $a < b$, and a component C of $G \cap V_{[a,b]}$, such that either $C \cap V_a = \emptyset$ or $C \cap V_b = \emptyset$. Suppose $C \cap V_b = \emptyset$. Then C must intersect one of the sets V_a or V_b, so $C \cap V_a \neq \emptyset$. Let $M = \max(\pi_H(C))$, and hence $M < b$.

Let $\pi_V(C) = [c,d]$ and observe that $[c,d] \neq [0,1]$ since there must exist some component D of $G \cap V_{[a,b]}$ such that D meets both V_a and V_b, and if $[c,d] = [0,1]$ then $C \cap D \neq \emptyset$, which is a contradiction.

Thus

- $C \subseteq [a,M] \times [c,d]$,
- $[c,d]$ is a closed proper (possibly degenerate) subinterval of $[0,1]$,
- $M \neq 1$, and
- $a \neq 0$: otherwise, since C is a component of $G \cap V_{[a,b]}$ and $C \cap V_b = \emptyset$, if $a = 0$ then C is a component of G, a contradiction since G is connected.

Hence $[a,M] \times [c,d]$ frames an L-set in G.

Similarly, if $C \cap V_b \neq \emptyset$ but $C \cap V_a = \emptyset$, we get an R-set in G.

Suppose $[a,b] \times [c,d]$ frames an L-set C in G. Then there exists $\epsilon > 0$ such that the component of $[a, b+\epsilon] \times [c-\epsilon, d+\epsilon]$ containing C is C, and hence C is a component of $G \cap V_{[a,b+\epsilon]}$ and $C \cap V_{b+\epsilon} = \emptyset$. Thus G does not have the crossing property. Similarly, if G admits an R-set, G does not have the crossing property.

The remaining claim can be proved similarly. ∎

Lemma 2.24. *Suppose that $f : [0,1] \to 2^{[0,1]}$ is an upper semicontinuous function with a connected graph G, G has the crossing property, $[a,b] \subseteq [0,1]$, and K is a component of $G \cap V_{[a,b]}$. Then there exists a fibre-connected subcontinuum C of K such that C meets both V_a and V_b, and C has the crossing property.*

Proof. If either $a = b$ or $\pi_V(K)$ is a singleton, the result follows. If $a < b$ and $\pi_V(K)$ is not a singleton, then without loss of generality we can assume that $[a,b] = [0,1]$ since we can regard K as the graph of an upper semicontinuous function with domain $[a,b]$.

Let $C_0 = K$. Choose a component E of $V_{1/2} \cap K$. Since G has the crossing property, there are components $C_{1,0}$ of $K \cap V_{[0,1/2]}$ and $C_{1,1}$ of $K \cap V_{[1/2,1]}$ such that $C_{1,0} \cap C_{1,1} = E$, $C_{1,0} \cap V_0 \neq \emptyset$ and $C_{1,1} \cap V_1 \neq \emptyset$. Let $C_1 = C_{1,0} \cup C_{1,1}$.

By induction, for each $n \in \mathbb{N}$ and each $m < 2^n$, we can choose a continuum $C_{n,m}$ that is a component of $K \cap V_{[m/2^n,(m+1)/2^n]}$, $C_n := \bigcup\{C_{n,m} : m < 2^n\}$ is connected and $C_{n+1} \subseteq C_n$.

Let $C = \bigcap_{n\in\mathbb{N}} C_n$. Then C is a fibre-connected subcontinuum of K:

- The collection of sets C_n is nested, so C is a subcontinuum of K.
- For each $x \in [0,1]$ let $Y_x = C \cap V_x$. If $x = m/2^n$ for some m, n, then $Y_x = \bigcap\{C_{n+j,2^j m} : j \geq 0\}$. Since each $C_{n+j,2^j m}$ is a continuum, and the sets $C_{n+j,2^j m}$ are nested, Y_x is connected. If for each $n \in \mathbb{N}$ and $m \leq 2^n$, $x \neq m/2^n$, then for every n there exists m_n such that $m_n/2^n < x < (m_n+1)/2^n$, and hence $Y_x = \bigcap\{C_{n,m_n} : n \in \mathbb{N}\}$. Once again we see that Y_x is connected and hence C is fibre-connected.

Since $\pi_H(C) = [0,1]$, C is fibre-connected and C is a continuum, it follows that C also has the crossing property. ■

3. Sufficiency

THEOREM 3.1. *Suppose that for each $i \in \mathbb{N}$, $I_i = [0,1]$ and $f_{i+1} : I_{i+1} \to 2^{I_i}$ is a surjective upper semicontinuous function with a connected graph G_{i+1}, and suppose there exist $m \geq 0$ and $n > m+1$ such that $\{f_i : i > 0\}$ has a CC-sequence over $[m,n]$. Then $\varprojlim (I_i, f_i)$ is not connected.*

Proof. Suppose $\{A_i = [a_i, b_i] : m \leq i \leq n\}$ is a CC-sequence admitted by the system, with pivot point (p_k) (recall that a CC-sequence, defined on p. 10, requires a pivot point). Clearly $\{f_i^m : i \in \mathbb{N}\}$ admits a CC-sequence over $[0, n-m]$, so by Lemmas 2.9 and 2.11 we can assume that $m = 0$ and we need only show that $\mathcal{G}(f_1, \ldots, f_n)$ is disconnected.

Since $\langle p_0, \ldots, p_n\rangle \in \mathcal{G}(f_1, \ldots, f_n) \cap \prod_{0\leq i\leq n} A_i$, $\mathcal{G}(f_1, \ldots, f_n) \cap \prod_{0\leq i\leq n} A_i \neq \emptyset$, and since $A_0 \neq I_0$ and each f_i is surjective, $\mathcal{G}(f_1, \ldots, f_n) \setminus \prod_{0\leq i\leq n} A_i \neq \emptyset$. We will show that $\prod_{0\leq i\leq n} A_i$ contains a component of $\mathcal{G}(f_1, \ldots, f_n)$ and hence $\mathcal{G}(f_1, \ldots, f_n)$ is not connected.

Let C be the component of $\mathcal{G}(f_1, \ldots, f_n)$ containing $\langle p_0, \ldots, p_n\rangle$. For each i, $0 < i \leq n$, let C_i be the component of $\pi_{i,i-1}(C) \cap Z_i$ containing $\langle p_i, p_{i-1}\rangle$, and let S_i be the member of

$$\{L_i, R_i, B_i, T_i, BL_i, BR_i, TL_i, TR_i\}$$

such that $G_i \sqsubset_{C_i} S_i$, as determined by the CC-sequence. For each i, by the definition of S_i there exists $\epsilon_i > 0$ such that the component D_i of $G_i \cap Z_i(\epsilon_i)$ that contains C_i is contained in S_i.

Let $D = \{\langle x_0, \ldots, x_n\rangle \in \mathcal{G}(f_1, \ldots, f_n) : \forall i < n,\ \langle x_{i+1}, x_i\rangle \in D_{i+1}\}$ and hence $D \neq \emptyset$ since $\langle p_0, \ldots, p_n\rangle \in D$.

Suppose $n = 2$. Let $G_1 \sqsubset_{C_1} L_1$, $G_2 \sqsubset_{C_2} T_2$ and $\langle x_0, x_1, x_2\rangle \in D$. If $\langle x_1, x_0\rangle \notin Z_1$ then $x_1 < a_1$. But since $\pi_{2,1}(D) \subset T_2$, we have $x_1 > a_1$, a contradiction, so $\langle x_1, x_0\rangle \in Z_1$. Thus $x_0 \in A_0$ and $x_1 \in A_1$, and since

$G_2 \sqsubset_{C_2} T_2$, it follows that $x_2 \in A_2$. Thus $D = C \subseteq A_0 \times A_1 \times A_2$ and hence $C \neq \mathcal{G}(f_1, \ldots, f_n)$.

A similar argument applies if $G_1 \sqsubset R_1$ and $G_n \sqsubset B_2$.

Suppose $n > 2$. Let $\langle x_0, \ldots, x_n \rangle \in D$.

If $G_1 \sqsubset L_1$ then either $G_2 \sqsubset TL_2$ or $G_2 \sqsubset TR_2$ and $x_1 \in I_1 \smallsetminus K_1$. It follows that

$$x_2 \in \begin{cases} I_2 \smallsetminus K_2 & \text{if } G_2 \sqsubset TL_2, \\ I_2 \smallsetminus J_2 & \text{if } G_2 \sqsubset TR_2. \end{cases}$$

If $G_1 \sqsubset R_1$ then either $G_2 \sqsubset BL_2$ or $G_2 \sqsubset BR_2$ and $x_1 \in I_1 \smallsetminus J_1$. Hence

$$x_2 \in \begin{cases} I_2 \smallsetminus K_2 & \text{if } G_2 \sqsubset BL_2, \\ I_2 \smallsetminus J_2 & \text{if } G_2 \sqsubset BR_2. \end{cases}$$

Consider the four conditions where $1 < i < n$:

(i) $x_i \in I_i \smallsetminus K_i$ and $G_i \sqsubset TL_i$,
(ii) $x_i \in I_i \smallsetminus J_i$ and $G_i \sqsubset TR_i$,
(iii) $x_i \in I_i \smallsetminus K_i$ and $G_i \sqsubset BL_i$,
(iv) $x_i \in I_i \smallsetminus J_i$ and $G_i \sqsubset BR_i$.

If (i) holds then either $G_{i+1} \sqsubset TL_{i+1}$ and hence $x_{i+1} \in I_{i+1} \smallsetminus K_{i+1}$, or $G_{i+1} \sqsubset TR_{i+1}$ and hence $x_{i+1} \in I_{i+1} \smallsetminus J_{i+1}$.

If (ii) holds then either $G_{i+1} \sqsubset BL_{i+1}$ and hence $x_{i+1} \in I_{i+1} \smallsetminus K_{i+1}$, or $G_{i+1} \sqsubset BR_{i+1}$ and hence $x_{i+1} \in I_{i+1} \smallsetminus J_{i+1}$.

If (iii) holds then either $G_{i+1} \sqsubset TL_{i+1}$ and hence $x_{i+1} \in I_{i+1} \smallsetminus K_{i+1}$, or $G_{i+1} \sqsubset TR_{i+1}$ and hence $x_{i+1} \in I_{i+1} \smallsetminus J_{i+1}$.

If (iv) holds then either $G_{i+1} \sqsubset BL_{i+1}$ and hence $x_{i+1} \in I_{i+1} \smallsetminus K_{i+1}$, or $G_{i+1} \sqsubset BR_{i+1}$ and hence $x_{i+1} \in I_{i+1} \smallsetminus J_{i+1}$.

Thus

$$x_{i+1} \in \begin{cases} I_{i+1} \smallsetminus K_{i+1} & \text{if } G_{i+1} \sqsubset TL_{i+1} \text{ or } G_{i+1} \sqsubset BL_{i+1}, \\ I_{i+1} \smallsetminus J_{i+1} & \text{if } G_{i+1} \sqsubset TR_{i+1} \text{ or } G_{i+1} \sqsubset BR_{i+1}. \end{cases}$$

It follows by induction that if $1 < i < n$ then one of conditions (i) to (iv) must be the case.

If $G_n \sqsubset B_n$ then either $G_{n-1} \sqsubset BR_{n-1}$ or $G_{n-1} \sqsubset TR_{n-1}$, and in either case $x_{n-1} \in I_{n-1} \smallsetminus J_{n-1}$, so $x_n \in A_n$. Since $x_n \in A_n$, $G_n \sqsubset B_n$ and $x_{n-1} \in I_{n-1} \smallsetminus J_{n-1}$, we have $x_{n-1} \in A_{n-1}$. Similarly, if $G_n \sqsubset T_n$, then $x_n \in A_n$ and $x_{n-1} \in A_{n-1}$.

Suppose $i > 1$ and $x_i \in A_i$. If condition (i) holds for i then $G_i \sqsubset TL_i$ and hence $x_{i-1} \in I_{i-1} \smallsetminus J_{i-1}$. Since $G_i \sqsubset TL_i$ it follows that either $G_{i-1} \sqsubset TL_{i-1}$ or $G_{i-1} \sqsubset BL_{i-1}$. In either case $x_{i-1} \in I_{i-1} \smallsetminus K_{i-1}$, and so $x_{i-1} \in A_{i-1}$. Similarly, if any of the remaining conditions occurs, then $x_{i-1} \in A_{i-1}$. Thus, by induction, $x_i \in A_i$ for each i, $0 < i \leq n$. Since either $G_1 \sqsubset L_1$ or $G_1 \sqsubset R_1$, it follows that $x_0 \in A_0$.

Thus, $D \setminus \prod_{i\le n} A_i = \emptyset$, so $C \subseteq D \subseteq \prod_{i\le n} A_i$ and therefore $\mathcal{G}(f_1, \dots, f_n)$ is disconnected. ∎

4. Necessary conditions. This section provides conditions that are necessary for an inverse limit to be connected. Some are more of a technical nature and are required for the proof of the main theorem in Section 5.

The following result implies both of Ingram's theorems [IM2, 4.3 and 4.5].

LEMMA 4.1. *Suppose that for each $i \ge 0$, $I_i = [0,1]$ and $f_{i+1} : I_{i+1} \to 2^{I_i}$ is a surjective upper semicontinuous function with a connected graph G_{i+1}. If either*

- *$\mathcal{G}(f_2, \dots, f_n)$ is connected, and*
- *G_1 is fibre-connected,*

or

- *$\mathcal{G}(f_1, \dots, f_{n-1})$ is connected, and*
- *the graph of f_n^{-1} is fibre-connected,*

then $\mathcal{G}(f_1, \dots, f_n)$ is connected.

Proof. Suppose that $\mathcal{G}(f_1, \dots, f_{n-1})$ is connected and for each $x \in X_{n-1}$, $f_n^{-1}(x)$ is connected. Suppose $\mathcal{G}(f_1, \dots, f_n) = A \cup B$ where A and B are nonempty closed subsets and $A \cap B = \emptyset$. Since $\mathcal{G}(f_1, \dots, f_{n-1})$ is connected, $\pi_{[0,n-1]}(A) \cap \pi_{[0,n-1]}(B) \ne \emptyset$. Choose $\langle p_0, \dots, p_{n-1}\rangle \in \pi_{[0,n-1]}(A) \cap \pi_{[0,n-1]}(B)$. Then both $\{\langle p_0, \dots, p_{n-1}, x\rangle \in A : x \in X_n\}$ and $\{\langle p_0, \dots, p_{n-1}, x\rangle \in B : x \in X_n\}$ are nonempty, closed and disjoint. But then $f_n^{-1}(p_{n-1}) \times \{p_{n-1}\}$ is disconnected, giving a contradiction. The proof is similar if $\mathcal{G}(f_2, \dots, f_n)$ is connected and for each $x \in X_1$, $f_1(x)$ is connected. ∎

THEOREM 4.2. *Suppose $n > 1$, for each $i \le n$, X_i is a connected compact Hausdorff space, and for each $i < n$, $f_{i+1} : X_{i+1} \to 2^{X_i}$ is a surjective upper semicontinuous function whose graph G_{i+1} is connected, $\mathcal{G}(f_1, \dots, f_{n-1})$ is connected and $\mathcal{G}(f_2, \dots, f_n)$ is connected. If either there exists a decomposition $\{G_{1\alpha} : \alpha < \kappa\}$ of G_1 into fibre-connected subgraphs such that for each α, $\pi_{1,H}(G_{1\alpha}) = X_1$, or there exists a decomposition $\{G_{n\alpha} : \alpha < \kappa\}$ of G_n^{-1}, the graph of f_n^{-1}, into fibre-connected sets such that for each α, $\pi_{n,V}(G_{n\alpha}) = X_{n-1}$, then $\mathcal{G}(f_1, \dots, f_n)$ is connected.*

Proof. Recall the definitions of $\pi_{i,V}$ and $\pi_{i,H}$ on p. 7.

Suppose $\{G_{1\alpha} : \alpha < \kappa\}$ is a decomposition of G_1 into fibre-connected subgraphs, and for each α, $\pi_{1,H}(G_{1\alpha}) = X_1$.

For each α let $D_\alpha = \pi_{1,V}(G_{1\alpha})$, and let $h_\alpha : X_1 \to 2^{D_\alpha}$ be the function whose graph is $G_{1\alpha}$. For each $\alpha < \kappa$, if D_α is a singleton, $\{s_\alpha\}$ say, then let $\mathcal{H}_\alpha = \{s_\alpha\} \times \mathcal{G}(f_2, \dots, f_n)$, and if D_α is not a singleton then let $\mathcal{H}_\alpha =$

$\mathcal{G}(h_\alpha, f_2, \dots, f_n)$. By Lemma 4.1, each $\mathcal{G}(h_\alpha, f_2, \dots, f_n)$ is connected, and since $\mathcal{G}(f_2, \dots, f_n)$ is connected, each $\{s_\alpha\} \times \mathcal{G}(f_2, \dots, f_n)$ is connected.

Suppose $\mathcal{G}(f_1, \dots, f_n) = A \cup B$ where A and B are nonempty closed subsets and $A \cap B = \emptyset$. For each α, since $\mathcal{H}_\alpha$ is connected, either $\mathcal{H}_\alpha \subseteq A$ or $\mathcal{H}_\alpha \subseteq B$. Let $X = \bigcup\{G_{1\alpha} : \mathcal{H}_\alpha \subseteq A\}$ and let $Y = \bigcup\{G_{1\alpha} : \mathcal{H}_\alpha \subseteq B\}$ and hence $G_1 = X \cup Y$. Since G_1 is connected, $X \cap Y \neq \emptyset$, so there exist α, β and $\langle y, x\rangle \in G_1$ such that $G_{1\alpha} \subset X$, $G_{1\beta} \subset Y$ and $\langle y, x\rangle \in G_{1\alpha} \cap G_{1\beta}$. Since each function f_i is surjective, there exists a point $\langle x, y, z_2, \dots, z_n\rangle \in \mathcal{G}(f_1, \dots, f_n)$. But if $\langle x, y, z_2, \dots, z_n\rangle \in \mathcal{G}(f_1, \dots, f_n)$, then $\langle x, y, z_2, \dots, z_n\rangle \in \mathcal{H}_\alpha \cap \mathcal{H}_\beta$ and hence $\langle x, y, z_2, \dots, z_n\rangle \in A \cap B$, giving a contradiction. Thus $\mathcal{G}(f_1, \dots, f_n)$ is connected.

A similar argument applies if $G_n^{-1} = \{\langle y, x\rangle : \langle x, y\rangle \in G_n\}$ has a decomposition into fibre-connected subgraphs, $\{G_{n\alpha} : \alpha < \kappa\}$, such that for each α, $\pi_{n,V}(G_{n\alpha}) = X_{n-1}$. ∎

Lemma 4.3. *Suppose that for each $i \in \mathbb{N}$, $I_i = [0,1]$ and $f_{i+1} : I_{i+1} \to 2^{I_i}$ is a surjective upper semicontinuous function with a connected graph G_{i+1}. If $\varprojlim (I_i, f_i)$ is disconnected, then for every set $\mathcal{G}(f_m, \dots, f_n)$ that is disconnected, where $0 < m < n$, and neither $\mathcal{G}(f_{m+1}, \dots, f_n)$ nor $\mathcal{G}(f_m, \dots, f_{n-1})$ is disconnected, G_m contains an L-set or an R-set, and G_n contains a T-set or a B-set.*

Proof. Suppose that G_m does not have an L-set or an R-set and both $\mathcal{G}(f_{m+1}, \dots, f_n)$ and $\mathcal{G}(f_m, \dots, f_{n-1})$ are connected. By Lemmas 2.9 and 2.11 we can assume that $m = 1$.

Let $\{G_{1\alpha} : \alpha < \kappa\}$, κ an ordinal, be a decomposition of G_1 into fibre-connected subgraphs. If $\kappa = 1$ then G_1 is fibre-connected, and so by Lemma 4.1, $\mathcal{G}(f_1, \dots, f_n)$ is connected. Hence $\kappa > 1$.

Claim. *For each α, $\pi_{1,H}(G_{1\alpha}) = I_1$.*

Proof. Suppose there exists α such that $\pi_{1,H}(G_{1\alpha}) = [c, d]$ and $d \neq 1$. Let $E = (\{d\} \times f_1(d)) \cap G_{1,\alpha}$ and hence E is a continuum contained in $V_{[d,1]} \cap G_1$. By Proposition 2.23, G_1 has the crossing property. Let C be the component of $V_{[d,1]} \cap G_1$ that contains E, and hence C meets V_1. By Lemma 2.24, there exists a subcontinuum D of C such that

(1) D meets both E and V_1,
(2) D has the crossing property, and
(3) for every $x \in [d, 1]$, $V_x \cap D$ is a component of $V_x \cap G_1$.

Without loss of generality assume that D is maximal, that is, if D' satisfies (1)–(3) and $D \subseteq D'$ then $D = D'$. Then $G_{1,\alpha} \cup D$ is connected since $G_{1,\alpha} \cap D = E$, and has the property that for every $x \in [c, 1]$,

$$(\{x\} \times f_1(x)) \cap (G_{1\alpha} \cup D)$$

is connected. It follows that $G_{1,\alpha}$ is not maximal, giving a contradiction, so $d = 1$. Similarly, $c = 0$ and the claim follows.

Thus, for each α, $\pi_{1,H}(G_{1\alpha}) = I_1$, and for each $x \in I_1$, $(\{x\}\times f_1(x))\cap G_{1\alpha}$ is connected. So by Theorem 4.2, $\mathcal{G}(f_1, \dots, f_n)$ is connected.

A similar argument applies to show that G_n does not have a T-set or a B-set. ∎

Recall the definition of a CP-subcontinuum on p. 17.

COROLLARY 4.4. *Suppose that for each* $i \geq 0$, $I_i = [0,1]$ *and* $f_{i+1} : I_{i+1} \to 2^{I_i}$ *is a surjective upper semicontinuous function with a connected graph* G_{i+1}, $n > 1$, $\mathcal{G}(f_2, \dots, f_n)$ *is connected and* $\mathcal{G}(f_1, \dots, f_{n-1})$ *is connected. Suppose that for each* $i \in \{1, \dots, n\}$, S_i *is a CP-subcontinuum in* G_i *such that* $\pi_{i,H}(S_i) = \pi_{i+1,V}(S_{i+1})$ *for* $i \neq n$. *Then the set*

$$\mathcal{S} = \{\langle x_0, \dots, x_n\rangle : \forall i < n,\ \langle x_{i+1}, x_i\rangle \in S_{i+1}\}$$

is connected.

Proof. Let $X_0 = \pi_{1,V}(S_1)$, $X_n = \pi_{n,H}(S_n)$ and if $0 < i < n$ let $X_i = \pi_{i,H}(S_i) = \pi_{i+1,V}(S_{i+1})$. For each $i < n$ let $g_{i+1} : X_{i+1} \to 2^{X_i}$ be the function whose graph is S_{i+1}. Observe that $\mathcal{S} = \mathcal{G}(g_1, \dots, g_n)$.

Suppose that each set X_i is nondegenerate. Since each S_i is a continuum, each X_i is a continuum, and since each S_i is a CP-continuum, by Proposition 2.23 none of the functions g_i admits an L-set or an R-set. Hence by Lemma 4.3, $\mathcal{S}$ is connected.

If any set X_i is a singleton, then by Proposition 2.12, $\mathcal{S}$ is homeomorphic to $\mathcal{G}(h_1, \dots, h_m)$ for some $m < n$ and functions $h_i : [0,1] \to 2^{[0,1]}$ such that either $h_i = f_j$ for some j or the graph of h_i is $[0,1]^2$. Hence for each i the graph of h_i does not admit an L-set or an R-set, and so $\mathcal{S}$ is connected. ∎

Many of the following proofs involve L-sets and R-sets. However, it is possible to define a number of different L-sets (infinitely many) that are essentially the same one; see the first two graphs in Figure 6 below, for example. The following definition introduces the notion of the *frame of an* S*-set* for each $S \in \{L, R, T, B\}$, in order to overcome this lack of uniqueness.

DEFINITION 4.5. Let $I_0 = I_1 = [0,1]$ and $A_1 \times A_0 \subset I_1 \times I_0$, and let $f : I_1 \to 2^{I_0}$ be an upper semicontinuous function with graph G.

Suppose $A_1 \times A_0$ frames an L-set D such that $A_1 = [a_1, b_1] = \pi_{1,H}(D)$ and $A_0 = \pi_{1,V}(D)$. Let $M < a_1$ be the maximum value such that if E is the component of $V_{[M,b_1]} \cap G$ containing D, then for every x such that $M < x < a_1$, $E \cap V_{[x,b_1]}$ is disconnected (M exists since $b_1 \neq 1$). Let C' be the component of $V_{(M,b_1]} \cap G$ containing D, and let C be the closure of C' in G. Then C is an L-set framed by $\pi_H(C) \times \pi_V(C)$, and C is the *minimum* L*-set associated with* D (see the third graph in Figure 6, for example).

Suppose $A_1 \times A_0$ frames an R-set D such that $A_1 = [a_1, b_1] = \pi_{1,H}(D)$ and $A_0 = \pi_{1,V}(D)$. Let $M > b_1$ be the minimum value such that if E is the component of $V_{[a_1,M]} \cap G$ containing D, then for every x such that $M > x > b_1$, $E \cap V_{[a_1,x]}$ is disconnected (M exists since $a_1 \neq 0$). Let C' be the component of $V_{[a_1,M)} \cap G$ containing D, and let C be the closure of C' in G. Then C is an $\overline{R}$-set framed by $\pi_H(C) \times \pi_V(C)$, and C is the *minimum* $\overline{R}$*-set associated with* D.

Observe:

- Since $a_1 \neq 0$ and $b_1 \neq 1$, the component E in the definition is not G.
- a_1 and b_1 in the definition are not equal.
- Clearly, if D is an L-set and C is the minimum $\overline{L}$-set associated with D, then for any $\overline{L}$-set C' such that $D \subset C'$, we have $C \subseteq C'$. Similarly for an R-set.
- An $\overline{L}$-set is not an L-set.
- The graph in Figure 6 has three $\overline{L}$-sets.

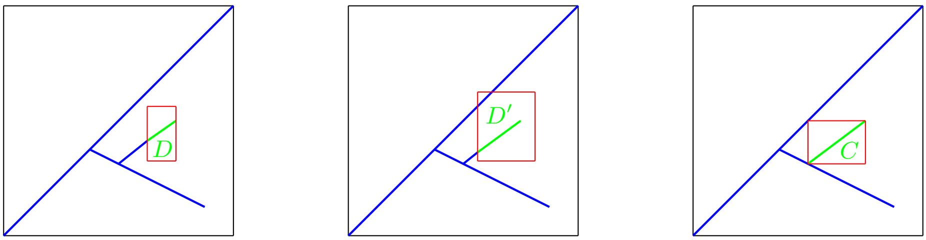

Fig. 6. L-sets D and D' and an $\overline{L}$-set C

By Lemma 4.3 if an inverse limit is disconnected, then one of the graphs must contain an L-set or R-set. Such a graph is necessary for the system to admit a CC-sequence. Lemma 4.6 below ensures that if an inverse limit $\varprojlim (I_i, f_i)$ is disconnected, then there is a component $K \subset \varprojlim (I_i, f_i)$ and an $\overline{L}$-set or $\overline{R}$-set $C \subset G_i$, for some i, such that $\pi_{i,i-1}(K) \subseteq C$. This provides a component around which to build a CC-sequence.

Lemma 4.6. *Suppose that $n > 1$, for each $i \leq n$, $I_i = [0, 1]$, for each $i < n$, $f_{i+1} : I_{i+1} \to 2^{I_i}$ is a surjective upper semicontinuous function with a connected graph G_{i+1}, and G_1 has a finite decomposition into fibre-connected subgraphs. If $\mathcal{G}(f_1, \ldots, f_n)$ is disconnected and neither $\mathcal{G}(f_2, \ldots, f_n)$ nor $\mathcal{G}(f_1, \ldots, f_{n-1})$ is disconnected, then there exist a set $A_1 \times A_0 \subset I_1 \times I_0$ that frames either an L-set or an R-set, and a component C of $\mathcal{G}(f_1, \ldots, f_n)$ such that $\pi_0(C) \subseteq A_0$ and $\pi_1(C) \subseteq A_1$.*

Proof. Since G_1 has a finite decomposition into fibre-connected subgraphs, the total number of $\overline{L}$-sets and $\overline{R}$-sets in G_1 is finite. We use induction on this number, proving both the base case and inductive case simultaneously, as the reasoning in each case is essentially the same.

By Lemma 4.3, since $\mathcal{G}(f_1, \dots, f_n)$ is disconnected and neither $\mathcal{G}(f_2, \dots, f_n)$ nor $\mathcal{G}(f_1, \dots, f_{n-1})$ is disconnected, G_1 contains at least one L-set or R-set. Suppose G_1 has a total of N $\overline{L}$-sets and $\overline{R}$-sets. Thus either

- $N = 1$, the base case, or
- $N > 1$, in which case assume that the result holds for any system that admits fewer than N $\overline{L}$-sets and $\overline{R}$-sets in its first graph.

Suppose there does not exist a component K of $\mathcal{G}(f_1, \dots, f_n)$ such that $\pi_{1,0}(K)$ is contained in either an L-set or an R-set. In other words, for any L-set or R-set C', if K is a component of $\mathcal{G}(f_1, \dots, f_n)$ then $\pi_{1,0}(K)$ is not a subset of C'. It follows that

(∗) if D is any $\overline{L}$-set and D is framed by $[a, b] \times [c, d]$, then there does not exist ϵ, $0 < \epsilon \leq b - a$, and a component K of $\mathcal{G}(f_1, \dots, f_n)$, such that $\pi_{1,0}(K) \subseteq D \cap (V_{[a+\epsilon,b]})$, and if D is any $\overline{R}$-set and D is framed by $[a, b] \times [c, d]$, then there does not exist ϵ, $0 < \epsilon \leq b - a$, and a component K of $\mathcal{G}(f_1, \dots, f_n)$, such that $\pi_{1,0}(K) \subseteq D \cap (V_{[a,b-\epsilon]})$.

Without loss of generality, suppose there is at least one $\overline{L}$-set (if not, the proof is analogous if based on an $\overline{R}$-set). Choose an $\overline{L}$-set C with frame $[a, b] \times A_0$ say, and such that a is minimal, that is, if D is an $\overline{L}$-set with frame $[c, d] \times A$, then $a \leq c$. Such a minimal value a exists since G_1 has a finite decomposition into fibre-connected subgraphs.

Let $M = \max\{x : \langle a, x\rangle \in C\}$ and let $m = \min\{x : \langle a, x\rangle \in C\}$. We require the following sets: let B be the union of all components of $V_a \cap G_1$ that meet C (and hence $C \cup B$ is connected), and

$$\begin{aligned}
\mathcal{C} &:= \{\langle x_0, \dots, x_n\rangle \in \mathcal{G}(f_1, \dots, f_n) : \langle x_1, x_0\rangle \in C\},\\
\mathcal{B} &:= \{\langle x_0, \dots, x_n\rangle \in \mathcal{G}(f_1, \dots, f_n) : \langle x_1, x_0\rangle \in B\},\\
W &:= \{a\} \times [m, M],\\
\mathcal{W} &:= \{\langle x_0, \dots, x_n\rangle \in \mathcal{G}(f_1, \dots, f_n) : \langle x_1, x_0\rangle \in W\}.
\end{aligned}$$

Claim.

(1) $W \cap B = W \cap G_1$.
(2) *Every point in $\mathcal{C}$ is connected in $\mathcal{G}(f_1, \dots, f_n)$ to a point in $\mathcal{B}$.*
(3) *Every point in $\mathcal{G}(f_1, \dots, f_n) \setminus \mathcal{C}$ is connected in $\mathcal{G}(f_1, \dots, f_n)$ to a point in $\mathcal{B}$.*
(4) *Any two points in $\mathcal{B}$ are connected in $\mathcal{G}(f_1, \dots, f_n)$.*

Proof of Claim. (1) Suppose $Y := \{a\} \times [c, d]$ is a component of $(W \cap G_1) \setminus B$. Then $m < c \leq d < M$, so the component Y_1 of $G_1 \cap ([a, b] \times A_0)$ containing Y is a subset of $[a, b) \times \mathrm{int}(A_0)$ since C is connected, $\langle a, m\rangle, \langle a, M\rangle \in C$ and $Y \cap C = \emptyset$. Furthermore, there exists $\epsilon > 0$ such that the component Y_2 of $G_1 \cap V_{[a-\epsilon,a]}$ containing Y is contained in $[a-\epsilon, a] \times [c-\epsilon, d+\epsilon]$, and $[c-\epsilon, d+\epsilon] \neq I_0$. But then $Y_1 \cup Y_2$ is an L-set contradicting the minimality of a.

(2) Let $\langle x_0, \ldots, x_n\rangle \in \mathcal{C} \setminus \mathcal{W}$ and let K be the component of $\mathcal{G}(f_1, \ldots, f_n)$ containing $\langle x_0, \ldots, x_n\rangle$. By $(*)$, $\pi_{1,0}(K) \not\subset C \setminus W$. Hence

$$K \cap \big(\mathcal{G}(f_1, \ldots, f_n) \setminus (\mathcal{C} \setminus \mathcal{W})\big) \neq \emptyset,$$

and so by the cut-wire theorem [Nad, Theorem 5.2, p. 72], $K \cap (\mathcal{W} \cap \mathcal{C}) \neq \emptyset$, and (2) follows since $\mathcal{W} \cap \mathcal{C} \subseteq \mathcal{B}$.

(3) Let $E = (G_1 \setminus C) \cup W$. Clearly E is connected and $\pi_{1,H}(E) = I_1$. Let $g : I_1 \to 2^{\pi_{1,V}(E)}$ be the function whose graph is E. Observe that

$$\mathcal{G}(g, f_2, \ldots, f_n) = \mathcal{W} \cup (\mathcal{G}(f_1, \ldots, f_n) \setminus \mathcal{C}).$$

Since C is not an $\overline{L}$-set admitted by g, E has fewer than N $\overline{L}$-sets and $\overline{R}$-sets. Then $\mathcal{G}(g, f_2, \ldots, f_n)$ is connected:

- Since $A_0 \neq I_0$ and $b < 1$, $\pi_{1,V}(E)$ is a nontrivial interval.
- If $N = 1$ then $\mathcal{G}(g, f_2, \ldots, f_n)$ is connected by Lemma 4.3, and if $N > 1$ it is connected by the inductive hypothesis and $(*)$.

If $\langle x_0, \ldots, x_n\rangle \in \mathcal{G}(f_1, \ldots, f_n) \setminus (\mathcal{B} \cup \mathcal{C})$ then

$$\langle x_0, \ldots, x_n\rangle \in \mathcal{G}(g, f_2, \ldots, f_n) \setminus \mathcal{W}.$$

Let K be the component of $\mathcal{G}(f_1, \ldots, f_n)$ containing $\langle x_0, \ldots, x_n\rangle$. Since $\mathcal{G}(g, f_2, \ldots, f_n)$ is connected, and by (1),

$$\overline{(\mathcal{G}(g, f_2, \ldots, f_n) \setminus \mathcal{W})} \setminus (\mathcal{G}(g, f_2, \ldots, f_n) \setminus \mathcal{W}) \subseteq \mathcal{B},$$

by the cut-wire theorem again, $K \cap \mathcal{B} \neq \emptyset$ and (3) follows.

(4) Suppose $\mathbf{x} = \langle x_0, \ldots, x_n\rangle$, $\mathbf{y} = \langle y_0, \ldots, y_n\rangle \in \mathcal{B}$ and hence $x_1 = y_1 = a$. If there exists a sequence $\langle \mathbf{z}_0, \ldots, \mathbf{z}_k\rangle$ in $\mathcal{G}(f_1, \ldots, f_n)$ such that

(I) $\mathbf{z}_0$ and $\mathbf{x}$ are connected in $\mathcal{G}(f_1, \ldots, f_n)$, $\mathbf{z}_k$ and $\mathbf{y}$ are connected in $\mathcal{G}(f_1, \ldots, f_n)$, and for each $i < k$, $\mathbf{z}_i$ and $\mathbf{z}_{i+1}$ are connected in $\mathcal{G}(f_1, \ldots, f_n)$,

then $\mathbf{x}$ and $\mathbf{y}$ are connected in $\mathcal{G}(f_1, \ldots, f_n)$.

If there does not exist such a sequence, that is, no sequence $\langle \mathbf{z}_0, \ldots, \mathbf{z}_k\rangle$ in $\mathcal{G}(f_1, \ldots, f_n)$ satisfies (I), then there exist two points $\mathbf{p} = \langle p_0, \ldots, p_n\rangle$ and $\mathbf{q} = \langle q_0, \ldots, q_n\rangle$ in $\mathcal{B}$ (hence $p_1 = q_1 = a$) and a subcontinuum $\mathcal{S} \subseteq \mathcal{W}$ containing $\mathbf{p}$ and $\mathbf{q}$ such that $\pi_{1,0}(\mathcal{S}) \subset W$ and $\pi_{2,1}(\mathcal{S}) \subseteq H_a^{f_2}$.

For each i, $1 < i \leq n$, let $S_i = \pi_{i,i-1}(\mathcal{S})$. Observe that $S_2 \subseteq H_a \cap G_2$.

For each $x \le a$ let K_x be the component of $V_{[x,1]} \cap G_1$ containing C. If $\pi_{1,H}(K_a) \ne [a,b]$, then let $c = a$, otherwise let

$$c = \inf\{x \le a : \pi_{1,H}(K_x) = [x,b]\}.$$

Thus $c \le a$. Let D_1 be the component of $V_{[c,b]} \cap G_1$ containing C and hence D_1 contains $\langle p_1, p_0\rangle$ (and $\langle q_1, q_0\rangle$), and let D_2 be the component of $H_{[c,b]} \cap G_2$ containing S_2. See Figure 7 for an example: C is the subgraph contained within the rectangle shown by the solid lines, and D_1 is the subgraph contained within the rectangle shown by the dashed lines in the first graph; D_2 is the subgraph contained within the rectangle shown by the dashed lines in the second graph.

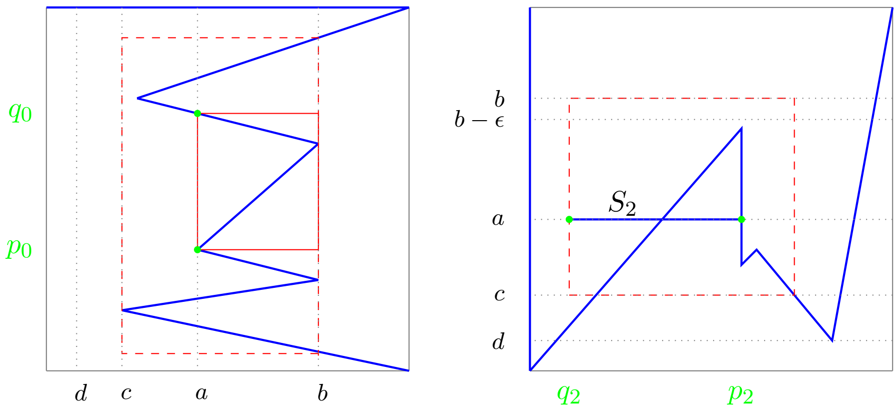

Fig. 7

For each $x \le a$ let K'_x be the component of $H_{[x,b]} \cap G_2$ containing S_2 and let $d = \sup\{x \le a : [x,b] = \pi_{1,V}(K'_x)\}$.

Clearly $d \le a$. Suppose $d < c$. Then $d < a$ and so by the definition of d as a supremum, there exists $\epsilon > 0$ such that $D_2 \subset H_{[c,b-\epsilon]} \cap G_2$ and hence D_2 is a B-set framed by $\pi_{2,H}(D_2) \times [c,b]$; see the second graph in Figure 7. Since $d < c$, we have $c \ne 0$. Thus by the definition of c as an infimum, D_1 is an R-set framed by $[c,b] \times \pi_{1,V}(D_1)$; see the first graph in Figure 7. It follows that $\{\pi_{1,V}(D_1), [c,b], \pi_{2,H}(D_2)\}$ is a CC-sequence with pivot point $\langle p_0, p_1, p_2\rangle$ (recall from Definition 2.6 that a CC-sequence requires a pivot point), and hence by Theorem 3.1 there exists a component K of $\mathcal{G}(f_1, \dots, f_n)$ containing $\mathbf{p}$ such that $\pi_{1,0}(K)$ is contained in the L-set D_1, giving a contradiction.

Thus $c \le d$. Let $X = \pi_{[1,n]}(\pi_1^{-1}([d,b]) \cap \mathcal{G}(f_1, \dots, f_n))$ and let K be the component of X containing $\pi_{[1,n]}(\mathcal{S})$; it exists since

$$\mathcal{S} \subset \pi_1^{-1}(a) \cap \mathcal{G}(g, f_2, \dots, f_n)$$

and $a \in [d,b]$. If $b \notin \pi_1(K)$ then $K \subset [d,b) \times \prod_{2\le i\le n} I_i$. Either $d = 0$ or there exists $\epsilon > 0$ such that $K \subset (d-\epsilon, b)\times\prod_{2\le i\le n} I_i$. It follows that K is a component of $\mathcal{G}(f_2,\dots,f_n)$, but as all graphs are surjective, $\pi_1(\mathcal{G}(f_2,\dots,f_n)) = I_1$ and hence $K \ne \mathcal{G}(f_2,\dots,f_n)$. Since $\mathcal{G}(f_2,\dots,f_n)$ is connected we have a contradiction. Thus $b \in \pi_1(K)$ and hence $[a,b] \subseteq \pi_1(K)$. Let $[e,b] = \pi_1(K)$ and hence $a \in [e,b]$.

Let C_1 be the component of $V_{[e,b]} \cap G_1$ containing C, $X_0 = \pi_{1,V}(C_1)$ and $X_1 = [e,b]$. For each i, $1 < i \le n$, let $X_i = \pi_i(K)$ and $C_i = \pi_{i,i-1}(K)$ and let $g_i : X_i \to 2^{X_{i-1}}$ be the function whose graph is the continuum C_i. Since $p_0 \ne q_0$ and $\langle p_1, p_0\rangle, \langle q_1, q_0\rangle \in C \subset C_1$, X_0 is nondegenerate. So is X_1 since $a < b$ and $[a,b] \subset X_1$.

By Proposition 2.12, there exist $k \le n$ and for each i, $0 < i \le k$, a function h_i whose domain and range are intervals, and $\mathcal{G}(g_1,\dots,g_n)$ is homeomorphic to $\mathcal{G}(h_1,\dots,h_k)$. Since the graph of h_1 is C_1 (this follows from the definition of h_1 according to Proposition 2.12), it has fewer than N $\overline{L}$-sets and $\overline{R}$-sets, so $\mathcal{G}(h_1,\dots,h_k)$ is connected and hence $\mathcal{G}(g_1,\dots,g_n)$ is connected. Clearly $\langle p_0,\dots,p_n\rangle\langle q_0,\dots,q_n\rangle \in \mathcal{G}(g_1,\dots,g_n) \subset \mathcal{G}(f_1,\dots,f_n)$ and hence $\langle p_0,\dots,p_n\rangle$ and $\langle q_0,\dots,q_n\rangle$ are connected in $\mathcal{G}(f_1,\dots,f_n)$.

It follows from the claim that $\mathcal{G}(f_1,\dots,f_n)$ is connected. ■

5. The main theorem. This section completes the proof of the main theorem. The theorem is first proved for the case where graphs have finite decompositions into fibre-connected subgraphs (Lemmas 5.1 and 5.3). Given a system (I_i, f_i), if we “fatten” the graphs of the functions by ϵ as in Definition 2.17, then by Lemma 2.18 the resulting graphs have finite decompositions into fibre-connected subgraphs. By Theorem 2.19, if the inverse limit $\varprojlim (I_i, f_i)$ is disconnected then there exists $\epsilon > 0$ such that $\varprojlim (I_i, f_{i,\epsilon})$ is disconnected, where each $f_{i,\epsilon}$ is the function whose graph is f_i fattened by ϵ. Hence $\{f_{i,\epsilon} : i > 0\}$ admits a CC-sequence. The final part of the proof demonstrates how to trim a fattened graph so as to destroy a CC-sequence. The trimmed graphs contain the original graphs, and if the system (I_i, f_i) does not admit a CC-sequence then we obtain a contradiction.

5.1. Finite decompositions into fibre-connected subgraphs

LEMMA 5.1. *Suppose that $I_0 = I_1 = I_2 = [0,1]$, $f_1 : I_1 \to 2^{I_0}$ and $f_2 : I_2 \to 2^{I_1}$ are surjective upper semicontinuous functions, the graphs G_1 and G_2 of the functions f_1 and f_2 are connected, and G_1 and G_2^{-1} have finite decompositions into fibre-connected subgraphs. Then $\mathcal{G}(f_1, f_2)$ is disconnected if and only if $\{f_1, f_2\}$ admits a CC-sequence.*

Proof. If $\{f_1, f_2\}$ admits a CC-sequence then by Theorem 3.1, $\mathcal{G}(f_1, f_2)$ is disconnected.

Conversely, suppose that $\mathcal{G}(f_1, f_2)$ is disconnected.

Since G_1 has a finite decomposition into fibre-connected subgraphs, the number of $\overline{L}$-sets and $\overline{R}$-sets is finite. We prove by induction on the total number of $\overline{L}$-sets and $\overline{R}$-sets that $\{f_1, f_2\}$ admits a CC-sequence. Let m be the number of $\overline{L}$-sets and $\overline{R}$-sets in G_1. By Lemma 4.6, if $m = 0$ then $\mathcal{G}(f_1, f_2)$ is connected.

Suppose that $m > 0$ and whenever $f_1' : I_1 \to 2^{I_0}$ and $f_2' : I_2 \to 2^{I_1}$ are surjective upper semicontinuous functions with connected graphs, and the graphs of f_1' and $(f_2')^{-1}$ have finite decompositions into fibre-connected subgraphs, if the total number of $\overline{L}$-sets and $\overline{R}$-sets in the graph of f_1' is less than m and $\mathcal{G}(f_1', f_2')$ is disconnected, then $\{f_1', f_2'\}$ admits a CC-sequence.

By Lemma 4.6 there exists a component K of $\mathcal{G}(f_1, f_2)$ and either an L-set or an R-set C such that $\pi_{1,0}(K) \subseteq C$. Without loss of generality suppose that C is an L-set. Let D be the minimal $\overline{L}$-set associated with C and suppose that D is framed by $[a, b] \times A_0$. Let $C_1 = \pi_{1,0}(K)$, $C_2 = \pi_{2,1}(K)$ and let D' be the component of $H_{[a,b]} \cap G_2$ containing C_2.

If $H_a \cap D' = \emptyset$ then there exists $\epsilon > 0$ such that $a + \epsilon < b$ and

$$\pi_1(C_2) \subseteq \pi_1(D') = [a + \epsilon, b].$$

Let E be the component of $V_{[a+\epsilon,b]}$ containing C_1. It follows that D' is a T-set framed by $\pi_{2,H}(D') \times [a + \epsilon, b]$ (here $\pi_{2,H}(D') \neq I_2$ since f_2 is surjective), and since D is an $\overline{L}$-set, E is an L-set framed by $[a + \epsilon, b] \times \pi_{1,V}(E)$, and since $K \subseteq \pi_{1,V}(E) \times [a + \epsilon, b] \times \pi_{2,H}(D')$, $\{\pi_{1,V}(E), [a + \epsilon, b], \pi_{2,H}(D')\}$ is a CC-sequence and contains a pivot point. Recall the definition (2.6) of a CC-sequence, and in particular the requirement that a CC-sequence has a pivot point.

Suppose $H_a \cap D' \neq \emptyset$.

If $b \notin \pi_{2,V}(D')$ then let E be the component of $V^{f_1}_{\pi_{2,V}(C_2)} \cap G_1$ containing C_1. If $a \notin \pi_{1,H}(E) := [a', b']$, then E is an R-set framed by $\pi_{1,H}(E) \times \pi_{1,V}(E)$. Furthermore, D' is a B-set framed by $\pi_{2,H}(C_2) \times [a', b']$. Thus, since

$$K \subseteq \pi_{1,V}(E) \times [a', b'] \times \pi_{2,H}(C_2),$$

$\{\pi_{1,V}(E), [a', b'], \pi_{2,H}(C_2)\}$ is a CC-sequence and contains a pivot point.

If $b \in \pi_{2,V}(D')$ then let $g_1 : [a, b] \to 2^{A_0}$ be the function whose graph is D, and let $g_2 : \pi_{2,H}(D') \to 2^{[a,b]}$ be the function whose graph is D'. Observe that g_1 and g_2 are both surjective and have connected graphs. If either A_0 or $\pi_{2,H}(D')$ is a singleton then $\mathcal{G}(g_1, g_2)$ is connected.

Suppose neither set is a singleton. Now $K \subset \mathcal{G}(g_1, g_2)$ since $\pi_{1,0}(K) = C_1 \subseteq D$ and $\pi_{2,1}(K) = C_2 \subseteq D'$, and $a \notin \pi_1(K)$ since $\pi_{1,0}(K) \subset C \subsetneq D$. So $K \neq \mathcal{G}(g_1, g_2)$ and hence $\mathcal{G}(g_1, g_2)$ is disconnected. The graph of g_1 has fewer than m $\overline{L}$-sets and $\overline{R}$-sets. Thus by the inductive hypothesis, since $\mathcal{G}(g_1, g_2)$ is disconnected, $\{g_1, g_2\}$ admits a CC-sequence $\{B_0, B_1, B_2\}$. Clearly $\{B_0, B_1, B_2\}$ is a CC-sequence admitted by $\{f_1, f_2\}$. ∎

Definition 5.2. If $a, b \in [0,1]$ are such that $a \le b$, and $G \subseteq [0,1] \times [0,1]$, then A is a subset of G *about* H_a *and* H_b if $\pi_V(A) = [a,b]$, and A is a subset of G *about* V_a *and* V_b if $\pi_H(A) = [a,b]$.

Lemma 5.3. *Suppose that for each* $i \ge 0$, $I_i = [0,1]$, $f_{i+1} : I_{i+1} \to 2^{I_i}$ *is a surjective upper semicontinuous function, the graph* G_{i+1} *of* f_{i+1} *is connected and has a finite decomposition into fibre-connected subgraphs, and the graph of* f_{i+1}^{-1} *has a finite decomposition into fibre-connected subgraphs. Suppose* $n > 1$, $\mathcal{G}(f_1, \ldots, f_n)$ *is disconnected and for all* k, l *such that* $1 \le k < l \le n$, $\mathcal{G}(f_k, \ldots, f_l)$ *is disconnected if and only if* $k = 1$ *and* $l = n$. *Then the system admits a CC-sequence.*

Proof. Suppose $\mathcal{G}(f_1, \ldots, f_n)$ is disconnected. We prove the result by induction on the number n of functions such that $\mathcal{G}(f_1, \ldots, f_n)$ is disconnected. By Lemma 5.1 the result holds for $n = 2$. Suppose $n > 2$ and the result holds for all systems where the number of functions is less than n. We prove the inductive case by induction on N, the total number of $\overline{L}$-sets and $\overline{R}$-sets in G_1.

For our proof by induction on N we consider the base case and the inductive case simultaneously, as the arguments are similar. Thus either

- $N = 1$ for the base case, or
- $N > 1$ and we assume that the result holds for any system with fewer than N $\overline{L}$-sets and $\overline{R}$-sets in the graph of the first function.

Observe that N is at least 1 by Lemma 4.6.

Thus, $I_1 \times I_0$ contains an L-set or an R-set C_1 framed by $A_1 \times A_0$ say, such that there exists a component C of $\mathcal{G}(f_1, \ldots, f_n)$ such that $\pi_{1,0}(C) \subseteq C_1$. Suppose that C_1 is an L-set.

Let $A_0 = [a_0, b_0]$ and $A_1 = [a_1, b_1]$. Without loss of generality assume that $A_1 = \pi_{1,H}(C_1)$ and $a_1 \in \pi_1(C)$. For each i, $1 < i \le n$, let $A_i = [a_i, b_i] = \pi_i(C)$ and let $C_i = \pi_{i,i-1}(C_i)$. Since C_1 is an L-set, $A_1 \cap \{0,1\} = \emptyset$ and $A_0 \ne I_0$.

Let $\mathcal{C} = \{\langle x_0, \ldots, x_n \rangle : \forall i < n,\ \langle x_{i+1}, x_i \rangle \in C_i\}$. Clearly $\mathcal{C} \subset \mathcal{G}(f_1, \ldots, f_n)$.

Claim. *If* $\mathcal{C}$ *is disconnected then* $\{f_1, \ldots, f_n\}$ *admits a CC-sequence with a pivot point.*

Proof of Claim. For each $i < n$ let $g_{i+1} : [a_{i+1}, b_{i+1}] \to 2^{[a_i.b_i]}$ be the functions whose graph is C_i. Suppose that for each $i \le n$, $a_i < b_i$. By our inductive hypothesis with respect to n, either $\mathcal{G}(g_2, \ldots, g_n)$ is connected or $\{g_2, \ldots, g_n\}$ admits a CC-sequence with a pivot point. Clearly if $\{B_1, \ldots, B_n\}$ is a CC-sequence admitted by $\{g_2, \ldots, g_n\}$, then $\{B_1, \ldots, B_n\}$ is a CC-sequence admitted by $\mathcal{G}(f_1, \ldots, f_n)$.

Suppose $\mathcal{G}(g_2, \dots, g_n)$ is connected. Either

- $N = 1$ and therefore $[a_1, b_1] \times [a_0, b_0]$ does not contain an $\overline{L}$-set and $\overline{R}$-set, thus by Lemma 4.6, $\mathcal{C} = \mathcal{G}(g_1, \dots, g_n)$ is connected, or
- $N > 1$ and therefore $[a_1, b_1] \times [a_0, b_0]$ has fewer than N $\overline{L}$-sets and $\overline{R}$-sets, thus by the inductive hypothesis with respect to N, $\mathcal{G}(g_1, \dots, g_n)$ is connected or $\{g_1, \dots, g_n\}$ admits a CC-sequence with a pivot point.

Again, if $\{B_0, \dots, B_n\}$ is a CC-sequence admitted by $\{g_1, \dots, g_n\}$, then $\{B_0, \dots, B_n\}$ is a CC-sequence admitted by $\mathcal{G}(f_1, \dots, f_n)$.

Suppose that there exists $i \le n$ such that $a_i = b_i$. By Proposition 2.12 there exist $k < n$ and functions $h_1, \dots, h_k$ such that $\mathcal{G}(g_1, \dots, g_n)$ is homeomorphic to $\mathcal{G}(h_1, \dots, h_k)$. If $\mathcal{G}(h_1, \dots, h_k)$ is disconnected then by Proposition 2.13 there exist j, j' such that $1 \le j < j' \le n$ and $\mathcal{G}(g_j, \dots, g_{j'})$ is disconnected, and hence by the inductive hypothesis with respect to N, $\{g_j, \dots, g_{j'}\}$ admits a CC-sequence with a pivot point. Since any CC-sequence admitted by $\{g_j, \dots, g_{j'}\}$ must also be a CC-sequence admitted by $\{f_j, \dots, f_{j'}\}$, by Theorem 3.1, $\mathcal{G}(f_j, \dots, f_{j'})$ is disconnected, contradicting one of our assumptions. Thus $\mathcal{G}(h_1, \dots, h_k)$ is connected and hence $\mathcal{G}(g_1, \dots, g_n)$ is connected.

Thus if $\mathcal{C}$ is disconnected, $\{f_1, \dots, f_n\}$ admits a CC-sequence with a pivot point, and the claim is proved.

Suppose $\mathcal{C}$ is connected and hence $\mathcal{C} = C$. Then since for each i, $0 < i \le n$, G_i is connected, G_i and G_i^{-1} have finite decompositions into fibre-connected subgraphs, $\pi_i(C_i) = [a_i, b_i]$, $\pi_{i-1}(C_{i-1}) = [a_{i-1}, b_{i-1}]$ and $\mathcal{C}$ is connected, the following hold:

(F1) For each i, if $b_i - a_i \neq 0$ then there exist $\epsilon > 0$ and CP-subcontinua S_{a_i} and S_{b_i} in C_i such that $\pi_{i,H}(S_{a_i}) = [a_i, a_i + \epsilon]$ and $\pi_{i,H}(S_{b_i}) = [b_i - \epsilon, b_i]$, and if $b_{i-1} - a_{i-1} \neq 0$ then there exist $\epsilon > 0$ and CP-subcontinua $S_{a_{i-1}}$ and $S_{b_{i-1}}$ in C_i such that $\pi_{i,V}(S_{a_{i-1}}) = [a_{i-1}, a_{i-1} + \epsilon]$ and $\pi_{i,V}(S_{b_{i-1}}) = [b_{i-1} - \epsilon, b_{i-1}]$.
For each $c \in (a_i, b_i)$ there exist $\epsilon > 0$ and a CP-subcontinuum $S \subseteq C_i$ such that $\pi_{i,H}(S) = [c-\epsilon, c+\epsilon]$, and for each $c \in (a_{i-1}, b_{i-1})$ there exist $\epsilon > 0$ and a CP-subcontinuum $S \subseteq C_i$ such that $\pi_{i,V}(S) = [c - \epsilon, c + \epsilon]$.

(F2) Any subcontinuum of G_i that connects a point in C_i to a point in $G_i \setminus (A_i \times A_{i-1})$, contains a CP-subcontinuum that meets C_i and is a subset of one of the following sets:

$$
\begin{aligned}
R_{TL_i} &= ([0, a_i) \times (b_{i-1}, 1]) \cup \{\langle a_i, b_{i-1}\rangle\}, & R_{T_i} &= A_i \times [b_{i-1}, 1],\\
R_{TR_i} &= ((b_i, 1] \times (b_{i-1}, 1]) \cup \{\langle b_i, b_{i-1}\rangle\}, & R_{L_i} &= [0, a_i] \times A_{i-1},\\
R_{BL_i} &= ([0, a_i) \times [0, a_{i-1})) \cup \{\langle a_i, a_{i-1}\rangle\}, & R_{R_i} &= [b_i, 1] \times A_{i-1},\\
R_{BR_i} &= ((b_i, 1] \times [0, a_{i-1})) \cup \{\langle b_i, a_{i-1}\rangle\}, & R_{B_i} &= A_i \times [0, a_{i-1}].
\end{aligned}
$$

See Figure 8. Observe that some of the sets will be empty if $a_i = 0$ or $b_i = 1$.

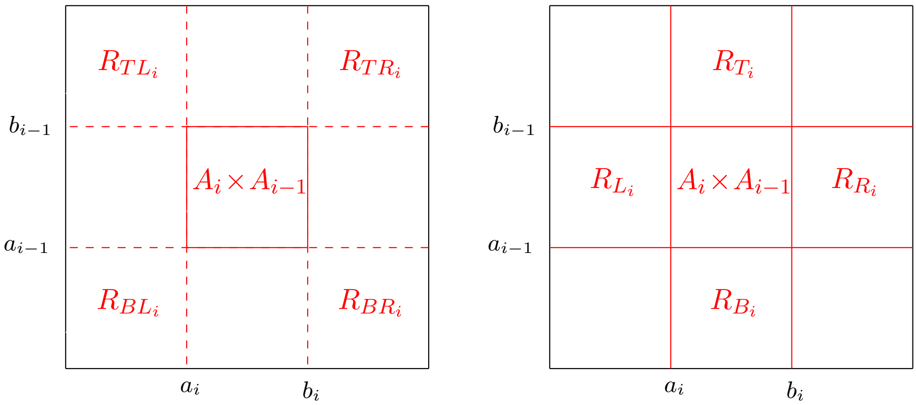

Fig. 8

For each $0 < i \leq n$, let $\mathcal{K}_i$ be the collection of all CP-subcontinua $T \subset G_i$ such that $T \subset C_i$ (including degenerate subcontinua), let $\mathcal{H}_i$ be the collection of all CP-subcontinua $T \subset G_i$ such that

- T is contained in exactly one of the sets $R_{TL_i}, R_{TR_i}, R_{L_i}, R_{R_i}, R_{BL_i}$ and R_{BR_i},
- T meets C_i, and
- $\pi_{i,H}(T)$ is nondegenerate,

let $\mathcal{V}_i$ be the collection of all CP-subcontinua $T \subset G_i$ such that

- T is contained in exactly one of the sets $R_{TL_i}, R_{T_i}, R_{TR_i}, R_{BL_i}$, R_{B_i} and R_{BR_i},
- T meets C_i, and
- $\pi_{i,V}(T)$ is nondegenerate,

and let $\mathcal{A}_i = \mathcal{H}_i \cup \mathcal{V}_i$ and $\mathcal{F}_i = \mathcal{A}_i \cup \mathcal{K}_i$.

Observe that if $T \in \mathcal{A}_i$ then $T \cap C_i$ is a subset of the boundary of $A_i \times A_{i-1}$.

Since C_1 is an L-set, there exists $Y_1 \in \mathcal{H}_1$ about $V^{f_1}_{a_1}$ and $V^{f_1}_{a_1-\gamma_1}$, for some $\gamma_1 > 0$, and hence $\pi_{1,H}(Y_1 \cap C_1) = \{a_1\}$ (and $Y_1 \subset ([0,a_1) \times I_0) \cup (\{a_1\} \times A_0)$). Suppose there exist $\gamma > 0$ and $S_2 \in \mathcal{V}_2$ about $H^{f_2}_{a_1}$ and $H^{f_2}_{a_1-\gamma}$ such that $S_2 \subset R_{B_2}$ and hence $\pi_{2,V}(S_2 \cap C_2) = \{a_1\}$. We can assume that $\gamma_1 = \gamma$. Let $[c,d] = \pi_{2,H}(S_2)$ and hence $[c,d] \subset [a_2, b_2]$. Then

($\star$) there exists a point $\langle x, a_1 \rangle \in S_2 \cap C_2$ and we can assume that for some $\epsilon \geq 0$, $\pi_{2,H}(S_2) = [c,d] = [x-\epsilon, x]$ or $[x, x+\epsilon]$ (if $\epsilon = 0$ then S_2 is a vertical line segment), and hence by (F1), if ϵ is small enough, there exists $S_3 \in \mathcal{K}_3$ (possibly degenerate) about $H^{f_3}_c$ and $H^{f_3}_d$.

By induction we can choose $S_i \in \mathcal{K}_i$ for each i, $2 < i \leq n$, such that $\pi_{i,H}(S_i) = \pi_{i+1,V}(S_{i+1})$ for each $i < n$. Let $S_1 = Y_1$. Then

$$S := \{\langle x_0, \dots, x_n\rangle : \forall i < n, \langle x_{i+1}, x_i\rangle \in S_{i+1}\}$$

is connected by Corollary 4.4, meets C by the connectedness of $\mathcal{C}$ $(= C)$, and is nondegenrate. But $\pi_1(S) \not\subset A_1$ giving a contradiction.

Thus for every $\gamma > 0$ there does not exist $S_2 \in \mathcal{V}_2$ about $H^{f_2}_{a_1}$ and $H^{f_2}_{a_1-\gamma}$ such that $S_2 \subset R_{B_2}$.

The comment $(\star)$ illustrates a technique that will be used throughout the rest of this proof. We will not provide as much detail again.

Since G_2 is connected and surjective and $0 \notin A_1$ (since C_1 is an L-set), it follows that $A_2 \neq I_2$.

Suppose $a_2 \neq 0$ and there exists $S_2 \in \mathcal{H}_2$ such that $S_2 \subset R_{L_2} \cup R_{BL_2}$. If $S_2 \subset R_{BL_2}$ then let $S_1 = Y_1$ as defined above such that $\pi_{1,H}(S_1) = \pi_{2,V}(S_2)$, and if $S_2 \subset R_{L_2}$ then choose $S_1 \in \mathcal{K}_1$ such that $\pi_{1,H}(S_1) = \pi_{2,V}(S_2)$, which is possible by (F1).

If there exists $S_i \in \mathcal{F}_i$ for each i, $2 < i \leq n$, such that $\pi_{i,V}(S_i) = \pi_{i-1,H}(S_{i-1})$ for $1 < i \leq n$, then

$$S = \{\langle x_0, \dots, x_n\rangle : \forall i < n, \langle x_{i+1}, x_i\rangle \in S_{i+1}\} \neq \emptyset,$$

S is connected by Corollary 4.4 and meets C, but $\pi_2(S) \not\subset A_2$. Thus either there does not exist $S_2 \in \mathcal{H}_2$ such that $S_2 \subset R_{L_2} \cup R_{BL_2}$ (this includes the case $a_2 = 0$), or it is not the case that for each i, $2 < i \leq n$, there exists $S_i \in \mathcal{F}_i$ such that $\pi_{i-1,H}(S_{i-1}) = \pi_{i,V}(S_i)$ for $1 < i \leq n$.

Similarly, either there does not exist $S'_2 \in \mathcal{H}_2$ such that $S'_2 \subset R_{R_2} \cup R_{BR_2}$, or it is not the case that for each $2 < i \leq n$, there exists $S'_i \in \mathcal{F}_i$ such that $\pi_{i-1,H}(S'_{i-1}) = \pi_{i,V}(S'_i)$ for $1 < i \leq n$.

Thus there are four possibilities to consider.

(P1) If there does not exist $S_2 \in \mathcal{H}_2$ such that $S_2 \subset R_{L_2} \cup R_{BL_2}$, and there does not exist $S'_2 \in \mathcal{H}_2$ such that $S'_2 \subset R_{R_2} \cup R_{BR_2}$, then C_2 is a T-set, and hence $\{A_0, A_1, A_2\}$ is a CC-sequence. If $n = 2$ we are done. If $n > 2$ we have a contradiction, and hence either there exists a set S_2, or there exists a set S'_2.

(P2) Suppose both sets S_2 and S'_2 exist. Then it is not the case that there exists $S_i \in \mathcal{F}_i$ for each i, $2 < i \leq n$, such that $\pi_{i-1,H}(S_{i-1}) = \pi_{i,V}(S_i)$ for $2 < i \leq n$, and neither is it the case that there exists $S'_i \in \mathcal{F}_i$ for each i, $2 < i \leq n$, such that $\pi_{i-1,H}(S'_{i-1}) = \pi_{i,V}(S'_i)$ for $2 < i \leq n$.

Let L be the set of natural numbers greater than 1 where $j \in L$ if and only if

(i) for each i, $2 < i \leq j$, there exist $S_i, S'_i \in \mathcal{H}_i$ such that $S_i \subset [0, a_i] \times I_{i-1}$ and $S'_i \subset [b_i, 1] \times I_{i-1}$ (so $a_i \neq 0$ and $b_i \neq 1$), or

(ii) for each $i \leq j$, $\pi_{i-1,H}(S_{i-1}) = \pi_{i,V}(S_i)$ and $\pi_{i-1,H}(S'_{i-1}) = \pi_{i,V}(S'_i)$ where $S_1 = Y_1$ or $S_1 \in \mathcal{K}_1$ chosen appropriately, or

(iii) each of the sets

$$S := \{\langle x_0, \ldots, x_j\rangle : \forall i < j,\ \langle x_{i+1}, x_i\rangle \in S_{i+1}\},$$
$$S' := \{\langle x_0, \ldots, x_j\rangle : \forall i < j,\ \langle x_{i+1}, x_i\rangle \in S'_{i+1}\}$$

meets $\mathcal{G}(f_1, \ldots, f_j) \setminus \pi_{[0,j]}(C)$.

Observe that by the definition of $\mathcal{H}_2$ and the sets A_i, and by (ii), both S and S' meet $\pi_{[0,j]}(C)$. Let $l = \max(L)$.

Suppose $l = 2$. If there exist $S_3 \in \mathcal{V}_3$ such that either $S_3 \subset R_{T_3}$ or $S_3 \subset R_{B_3}$ then there exist $S_i \in \mathcal{K}_i$ for each i, $3 < i \leq n$, such that $\pi_{i,H}(S_i) = \pi_{i+1,V}(S_{i+1})$ for $i < n$, and the set

$$\{\langle x_0, \ldots, x_n\rangle : \forall i < n,\ \langle x_{i+1}, x_i\rangle \in S_{i+1} \cup S'_{i+1}\}$$

is nonempty and meets both C and $\mathcal{G}(f_1, \ldots, f_n) \setminus C$ (where S_1 is chosen appropriately), giving a contradiction.

Thus either C_3 is an L-set or an R-set, or there exist $S_3, S'_3 \in \mathcal{H}_3$ such that $S_3 \subset [0, a_3] \times I_2$ and $S'_3 \subset [b_3, 1] \times I_2$. Suppose the latter is the case. Then since (ii) and (iii) hold for $i = 2$, once again there does not exist $S_i \in \mathcal{A}_i$ for each i, $3 < i \leq n$, such that $\pi_{i-1,H}(S_{i-1}) = \pi_{i,V}(S_i)$, and neither is it the case that there exists $S'_i \in \mathcal{A}_i$ for each i, $3 < i \leq n$, such that $\pi_{i-1,H}(S'_{i-1}) = \pi_{i,V}(S'_i)$.

Suppose $l > 2$. By the maximality of l we can present a similar argument to that above to show that there does not exist $T \in \mathcal{A}_{l+1}$ such that $T \subset R_{T_{l+1}} \cup R_{B_{l+1}}$, and either C_{l+1} is an L-set or an R-set, or there exist $S_{l+1}, S'_{l+1} \in \mathcal{H}_{l+1}$ such that $S_{l+1} \in [0, a_{l+1}] \times I_l$ and $S'_{l+1} \in [b_{l+1}, 1] \times I_l$.

By induction there is a maximum $r \geq 2$ such that there exist $S_j, S'_j \in \mathcal{H}_r$ where $S_j \in [0, a_j] \times I_{j-1}$ and $S'_j \in [b_j, 1] \times I_{j-1}$ for each $j \leq r$ (so clearly $r \neq n$), and for any $T \in \mathcal{H}_{r+1}$, either $T \not\subset [0, a_{r+1}] \times I_r$ or $T \not\subset [b_{r+1}, 1] \times I_r$. We can also show that $T \not\subset R_{T_{r+1}} \cup R_{B_{r+1}}$ by a similar argument to that above, and hence C_{r+1} is either an L-set or an R-set.

If C_{r+1} is an L-set say, then

($\star\star$) there exists $S_{r+1} \in \mathcal{H}_{r+1}$ such that $S_{r+1} \subset [0, a_{r+1}] \times I_r$, and by the maximality of r there exist $S_i \in \mathcal{F}_i$ for each i, $0 < i \leq r$, such that

$$S = \{\langle x_0, \ldots, x_r\rangle : \forall i \leq r,\ \langle x_{i+1}, x_i\rangle \in S_{i+1}\} \neq \emptyset,$$

S is connected, meets $\pi_{[0,r+1]}(C)$, and $\pi_{r+1}(S) \not\subset A_{r+1}$.

Similarly if C_{r+1} is an R-set. In either case it follows that $n \neq r+1$, otherwise $G_n \subseteq A_n \times A_{n-1}$, which is impossible.

If C_j is neither an L-set nor an R-set for every $j \leq r + 1$, then the argument we develop to show that there is a CC-sequence relies only on the existence of Y_1 and the properties of C_1. It will be clear that we can present exactly the same argument starting with the graph G_{r+1} instead of G_1, the role of Y_1 can be filled by S_{r+1}, through ($\star\star$), and that of C_1 by C_{r+1}, to obtain a CC-sequence with a pivot point admitted by $\{f_{r+1}, \dots, f_n\}$. We would then deduce that $\mathcal{G}(f_{r+1}, \dots, f_n)$ is disconnected, getting a contradiction since $r+1 > 1$. This implies that there does not exist $j > 2$ such that C_j is either an L-set or an R-set.

(P3)&(P4) It follows from (P1) and (P2) that exactly one of the sets S_2 and S_2' exists. Thus, either there exists a set S_2 and $G_2 \sqsubset TL_2$, or there exists a set S_2' and $G_2 \sqsubset TR_2$.

By a similar argument it follows that if C_1 is an R-set in G_1, then $\{f_1, f_2\}$ admits a CC-sequence $\{A_0, A_1, A_2\}$ if $n = 2$, and if $n > 2$, then either $G_2 \sqsubset BR_2$ or $G_2 \sqsubset BL_2$.

Thus, if $n > 2$, there exists a CP-subcontinuum Y_1 of G_1, and:

- $G_2 \sqsubset TL_2$ and there exists $Y_2 \in \mathcal{H}_2$ such that $Y_2 \subset R_{L_2} \cup R_{BL_2}$, or
- $G_2 \sqsubset BL_2$ and there exists $Y_2 \in \mathcal{H}_2$ such that $Y_2 \subset R_{TL_2} \cup R_{L_2}$, or
- $G_2 \sqsubset TR_2$ and there exists $Y_2 \in \mathcal{H}_2$ such that $Y_2 \subset R_{R_2} \cup R_{BR_2}$, or
- $G_2 \sqsubset BR_2$ and there exists $Y_2 \in \mathcal{H}_2$ such that $Y_2 \subset R_{TR_2} \cup R_{R_2}$.

Note:

(a) In each case there exists $S_1 \in \mathcal{F}_1$ such that the set

$$\{\langle x_0, x_1, x_2\rangle : \langle x_0, x_1\rangle \in S_1,\ \langle x_2, x_1\rangle \in Y_2\}$$

is a connected nondegenerate subset of $\mathcal{G}(f_1, f_2)$ ($S_1 = Y_1$ if $Y_2 \subset I_2 \times [0, a_1]$, otherwise $S_1 \in \mathcal{K}_1$).

(b) If $G_2 \sqsubset TL_2$ or $G_2 \sqsubset TR_2$ and hence $G_1 \sqsubset L_1$, since there does not exist $S_1 \in \mathcal{H}_1$ such that $S_1 \subset V^{f_1}_{[b_1,1]}$, if $S_2 \in \mathcal{V}_2$, $S_2 \subset H^{f_2}_{[b_1,1]}$ and S is the component of $\{\langle x_0, x_1, x_2\rangle \in \mathcal{G}(f_1, f_2) : \langle x_2, x_1\rangle \in S_2\}$ that meets $\pi_{[0,2]}(C)$, then $S \subset \pi_{[0,2]}(C)$ and $\pi_1(\mathcal{S}_2) = \{b_1\}$.

(c) Similarly if $G_2 \sqsubset BL_2$ or $G_2 \sqsubset BR_2$ and $G_1 \sqsubset R_1$, since there does not exist $S_1 \in \mathcal{H}_1$ such that $S_1 \subset V^{f_1}_{[0,a_1]}$, if $S_2 \in \mathcal{V}_2$, $S_2 \subset H^{f_2}_{[0,a_1]}$ and S is the component of $\{\langle x_0, x_1, x_2\rangle \in \mathcal{G}(f_1, f_2) : \langle x_2, x_1\rangle \in S_2\}$ that meets $\pi_{[0,2]}(C)$, then $S \subset \pi_{[0,2]}(C)$ and $\pi_1(\mathcal{S}_2) = \{a_1\}$.

Suppose $2 \le k < n-1$ and the following conditions hold:

(IH1) For each i, $1 < i \le k$:

- $G_i \sqsubset TL_i$ and there exists $Y_i \in \mathcal{H}_i$ such that $Y_i \subset R_{L_i} \cup R_{BL_i}$, or
- $G_i \sqsubset BL_i$ and there exists $Y_i \in \mathcal{H}_i$ such that $Y_i \subset R_{TL_i} \cup R_{L_i}$, or
- $G_i \sqsubset TR_i$ and there exists $Y_i \in \mathcal{H}_i$ such that $Y_i \subset R_{R_i} \cup R_{BR_i}$, or
- $G_i \sqsubset BR_i$ and there exists $Y_i \in \mathcal{H}_i$ such that $Y_i \subset R_{TR_i} \cup R_{R_i}$.

(IH2) For each i, $1 \le i \le k$, there exists $S_i \in \mathcal{F}_i$ such that the set

$$\{\langle x_0,\dots,x_k\rangle \in \mathcal{G}(f_1\dots,f_k) : \langle x_k,x_{k-1}\rangle \in Y_k \text{ and } \forall j < k-1,\ \langle x_{j+1},x_j\rangle \in S_{j+1}\}$$

is a connected nondegenerate subset of $\mathcal{G}(f_1\dots,f_k)$ that meets both $\pi_{[0,k]}(C)$ and $\mathcal{G}(f_1\dots,f_k) \setminus \pi_{[0,k]}(C)$.

(IH3)
- If $G_k \sqsubset TL_k$ or $G_k \sqsubset TR_k$, $S_k \in \mathcal{V}_k$ and $S_k \subset V^{f_k}_{[b_k,1]}$, then
$$\{\langle x_0,\dots,x_k\rangle \in \mathcal{G}(f_1,\dots,f_k) : \langle x_k,x_{k+1}\rangle \in S_k\} \subset \pi_{[0,k]}(C).$$
- If $G_k \sqsubset BL_k$ or $G_k \sqsubset BR_k$, $S_k \in \mathcal{V}_k$ and $S_k \subset V^{f_k}_{[0,a_k]}$, then
$$\{\langle x_0,\dots,x_k\rangle \in \mathcal{G}(f_1,\dots,f_k) : \langle x_k,x_{k+1}\rangle \in S_k\} \subset \pi_{[0,k]}(C).$$

By (IH1), $A_k \ne I_k$. In fact, since if $G_k \sqsubset TL_k$ there exists $S_i \in \mathcal{H}_i$ such that $S_i \subset (R_{L_i} \cup R_{BL_i})$, it follows that $a_k \ne 0$ (moreover, if $a_k = 0$ we would have a T-set). Similarly, if $G_i \sqsubset BL_i$ then $a_k \ne 0$, and if either $G_i \sqsubset TR_i$ or $G_i \sqsubset BR_i$ then $b_k \ne 1$.

(I) Suppose $A_k \times A_{k-1}$ frames a TL-set.

Suppose there exists $S_{k+1} \in \mathcal{V}_{k+1}$ such that $S_{k+1} \subset R_{B_{k+1}}$.

Since $A_k \times A_{k-1}$ frames a TL-set, $Y_k \subset R_{L_k} \cup R_{BL_k}$ by (IH1). Without loss of generality we can assume that $\pi_{k+1,V}(S_{k+1}) = \pi_{k,H}(Y_k)$.

By (IH2) it follows that for each i, $1 \le i \le k$, there exists $S_i \in \mathcal{F}_i$ such that the set

$$\{\langle x_0,\dots,x_{k+1}\rangle \in \mathcal{G}(f_1\dots,f_{k+1}) : \forall j < k+1,\ \langle x_{j+1},x_j\rangle \in S_{j+1}\}$$

is a connected nondegenerate subset of $\mathcal{G}(f_1\dots,f_{k+1})$ that meets both $\pi_{[0,k+1]}(C)$ and $\mathcal{G}(f_1\dots,f_{k+1}) \setminus \pi_{[0,k+1]}(C)$.

Since $\pi_{k+1,H}(S_{k+1}) \subset A_{k+1}$, by (F1) and by induction for each i, $k+1 < i \le n$, we can choose $S_i \in \mathcal{K}_i$ such that $\pi_{i,V}(S_i) = \pi_{i-1,H}(S_{i-1})$ for $i > k+1$, and hence the set $\{\langle x_0,\dots,x_n\rangle : \forall i < n,\ \langle x_{i+1},x_i\rangle \in S_{i+1}\}$ contains a connected subset of $\mathcal{G}(f_1,\dots,f_n)$ that meets C, but $\pi_k(S) \not\subseteq A_k$. Hence if $S_{k+1} \in \mathcal{V}_{k+1}$ then $S_{k+1} \not\subset R_{B_{k+1}}$.

Suppose $S_{k+1} \in \mathcal{H}_{k+1}$ and $S_{k+1} \subset R_{L_{k+1}} \cup R_{BL_{k+1}}$. If

($*$) for each i, $k+1 < i \le n$, there exist $S_i \in \mathcal{F}_i$ such that $\pi_{i-1,H}(S_{i-1}) = \pi_{i,V}(S_i)$,

then

$$S = \{\langle x_0,\dots,x_n\rangle : \forall i < n,\ \langle x_{i+1},x_i\rangle \in S_{i+1}\}$$

is a connected subset of $\mathcal{G}(f_1,\dots,f_n)$ that meets C, but is not a subset of C since $\pi_{k+1}(S_{i+1}) \not\subseteq A_{k+1}$. Thus either no such S_{k+1} exists, or ($*$) does not hold.

Similarly, either there does not exist $S'_{k+1} \in \mathcal{H}_{k+1}$ with $S'_{k+1} \subset R_{R_{k+1}} \cup R_{BR_{k+1}}$, or it is not the case that

(∗∗) for each i, $k+1 < i \le n$, there exist $S'_i \in \mathcal{F}_i$ such that $\pi_{i-1,H}(S'_{i-1}) = \pi_{i,V}(S'_i)$.

Suppose neither of the subcontinua S_{k+1} and S'_{k+1} exist. Then $S_{k+1} \sqsubset T_{k+1}$, and since $S_k \sqsubset TL_k$, $\{A_0, \ldots, A_{k+1}\}$ is a CC-sequence, giving a contradiction since $k < n-1$.

If neither (∗) nor (∗∗) holds, then by a similar argument to the one in (P2) we obtain a contradiction.

Thus, either $G_{k+1} \sqsubset TL_{k+1}$ or $G_{k+1} \sqsubset TR_{k+1}$.

(II) All other cases are similar.

By induction we see that the sets $A_0, \ldots, A_{n-1}$ are consistent with the first n members of a CC-sequence.

Suppose $G_{n-1} \sqsubset TL_{n-1}$. If there exists $S_n \in \mathcal{H}_n$ with $S_n \subset R_{L_n} \cup R_{R_n}$, then by (F1) for each $i \le n$ there exists $S_i \in \mathcal{K}_i$ such that $\pi_{i,H}(S_i) = \pi_{i+1,V}(S_{i+1})$, and so the set $S = \{\langle x_0, \ldots, x_n\rangle : \forall i < n,\ \langle x_{i+1}, x_i\rangle \in S_{i+1}\}$ is a connected subset of $\mathcal{G}(f_1, \ldots, f_n)$ that meets C, but $\pi_n(S) \not\subset A_n$. Thus such a CP-subcontinuum does not exist.

If there exists $S_n \in \mathcal{H}_n$ such that $S_n \subset I_n \times [0, a_{n-1}]$, then we know that for each i, $1 \le i < n$, there exists $S_i \in \mathcal{F}_i$ such that $\pi_{i,H}(S_i) = \pi_{i+1,V}(S_{i+1})$, where $S_{n-1} = Y_{n-1}$, and hence the set $S = \{\langle x_0, \ldots, x_n\rangle : \forall i < n,\ \langle x_{i+1}, x_i\rangle \in S_{i+1}\}$ is a connected subset of $\mathcal{G}(f_1, \ldots, f_n)$ that meets C, but $\pi_{n-1}(S) \not\subset A_{n-1}$.

Thus $a_{n-1} \ne 0$ (by (IH1) since $G_{n-1} \sqsubset TL_{n-1}$), and as G_n is connected, $b_{n-1} \ne 1$ and $A_n \ne I_n$. Thus $G_n \sqsubset T_n$ and hence $\{A_0, \ldots, A_n\}$ is a CC-sequence and any point in C is a pivot point.

In every other case, by a similar argument we conclude that $\{A_0, \ldots, A_n\}$ is a CC-sequence with a pivot point. ∎

5.2. The general case

LEMMA 5.4. *Suppose that $I_0 = I_1 = I_2 = [0,1]$, $f_1 : I_1 \to 2^{I_0}$, $f_2 : I_2 \to 2^{I_1}$ are surjective upper semicontinuous functions, and the graphs G_1 and G_2 of f_1 and f_2 are connected. Then $\mathcal{G}(f_1, f_2)$ is disconnected if and only if $\{f_1, f_2\}$ admits a CC-sequence.*

Proof. Suppose $\mathcal{G}(f_1, f_2)$ is disconnected and $\{f_1, f_2\}$ does not admit a CC-sequence.

For each $\epsilon > 0$ and $i \in \{1,2\}$, let $g_{i,\epsilon}$ be the upper semicontinuous function whose graph is $G_i(\epsilon)$. Then there exists $\epsilon' > 0$ such that for each $\epsilon < \epsilon'$, $\mathcal{G}(g_{1,\epsilon}, g_{2,\epsilon})$ is disconnected.

Suppose $\epsilon < \epsilon'$. Both $g_{1,\epsilon}$ and $g_{2,\epsilon}^{-1}$ have finite decompositions into fibre-connected subgraphs, and hence by Lemma 5.1, $\{g_{1,\epsilon}, g_{2,\epsilon}\}$ admits a CC-sequence.

Let $\mathcal{S} = \{(S_{i,1} \times S_{i,0}, C_i) : i < m_L\}$ be the collection of all pairs such that C_i is an $\overline{L}$-set in $G_1(\epsilon)$ framed by $S_{i,1} \times S_{i,0}$. Clearly $\mathcal{S}$ is finite since $G_1(\epsilon)$ has a finite decomposition into fibre-connected subgraphs. In the same way that L-sets extend to $\overline{L}$-sets we want to extend T-sets to $\overline{T}$-sets. Let $\mathcal{T} = \{(T_{i,2} \times T_{i,1}, D_i) : i < n_T\}$ be the collection of all pairs such that D_i is an $\overline{R}$-set in the graph of f_2^{-1} framed by $T_{i,1} \times T_{i,2}$; call D_i a *$\overline{T}$-set in $G_2(\epsilon)$ framed by $T_{i,2} \times T_{i,1}$.*

For each $i < m_L$ let $S_{i,1} = [a_{i,1}, b_{i,1}]$. If $b_{i,1} - a_{i,1} \le \epsilon$ then let

$$\mathrm{left}(i) = C_i \cap ((a_{i,1}, b_{i,1}] \times A_0),$$

otherwise let $\mathrm{left}(i) = C_i \cap ((b_{i,1} - \epsilon, b_{i,1}] \times A_0)$. Let

$$G_1^{-L}(\epsilon) = G_1(\epsilon) \setminus \bigcup\{\mathrm{left}(i) : i \le m_L\}.$$

For each $i < m_R$ let $T_{i,1} = [a'_{i,1}, b'_{i,1}]$. If $b'_{i,1} - a'_{i,1} \le \epsilon$ then let

$$\mathrm{top}(i) = C_i \cap (A_2 \times [a'_{i,1}, b'_{i,1})),$$

otherwise let $\mathrm{top}(i) = C_i \cap (A_2 \times [a'_{i,1}, a'_{i,1} + \epsilon))$. Let

$$G_2^{-T}(\epsilon) = G_2(\epsilon) \setminus \bigcup\{\mathrm{top}(i) : i \le n_T\}.$$

We are effectively removing a left portion of each $\overline{L}$-set in $G_1(\epsilon)$ and a bottom portion of each $\overline{T}$-set in $G_2(\epsilon)$. Observe that since C_i is an $\overline{L}$-set, we have $a_{i,1} \ne b_{i,1}$.

Clearly $G_1 \subset G_1^{-L}(\epsilon) \subset G_1(\epsilon)$, $G_2 \subset G_2^{-T}(\epsilon) \subset G_2(\epsilon)$ and each $G_1^{-L}(\epsilon)$ and each $G_2^{-T}(\epsilon)$ is connected and closed (we have removed from each graph a finite union of sets open in $G_1(\epsilon)$ or in $G_2(\epsilon)$, which do not disconnect the graph).

If $\epsilon \ge b_{i,1} - a_{i,1}$ then there is no L-set in $G_i^{-L}(\epsilon)$ that is a subset of C_i.

If $\epsilon < b_{i,1} - a_{i,1}$ and $[c,d] \times B$ frames an L-set $E \subset C_i$ in $G_1^{-L}(\epsilon)$, then by the definition of an $\overline{L}$-set, $E \cap (\{b_{i,1}\} \times B) \ne \emptyset$, hence $(\{c\} \times B) \cap G_1 \ne \emptyset$ and therefore $[c,d] \times B$ frames an L-set $E' \subset E$ in G_1. Similarly if $B_2 \times B_1$ frames a T-set in $G_2^{-L}(\epsilon)$ then $B_2 \times B_1$ frames a T-set in G_2.

Let $f_{1,\epsilon} : I_1 \to 2^{I_0}$ be the function whose graph is $G_1^{-L}(\epsilon)$, and let $f_{2,\epsilon} : I_2 \to 2^{I_1}$ be the function whose graph is $G_2^{-T}(\epsilon)$. Suppose $\{A_0, A_1, A_2\}$ is a CC-sequence admitted by $\{f_{1,\epsilon}, f_{2,\epsilon}\}$, and suppose $A_1 \times A_0$ frames an L-set E and $A_2 \times A_1$ frames a T-set E'. Let $p_1 \in \pi_{1,H}(E) \cap \pi_{2,V}(E')$. Then there exist a point $\langle p_1, p_0\rangle \in E \cap G_1$ and a point $\langle p_2, p_1\rangle \in E' \cap G_2$, and hence $\langle p_0, p_1, p_2\rangle \in \mathcal{G}(f_1, f_2)$.

Thus $\{A_0, A_1, A_2\}$ is a CC-sequence admitted by $\{f_1, f_2\}$ with pivot point $\langle p_0, p_1, p_2\rangle$, giving a contradiction. Hence $\{f_{1,\epsilon}, f_{2,\epsilon}\}$ does not admit a CC-sequence that involves an L-set followed by a T-set.

We have destroyed each of the CC-sequences $\{A_0, A_1, A_2\}$ admitted by $\{g_{1,\epsilon}, g_{2,\epsilon}\}$, where $A_1 \times A_0$ frames an L-set and $A_2 \times A_1$ frames a T-set, by trimming by ϵ the right hand side of each of the L-sets in $G_1(\epsilon)$, and trimming by ϵ the bottom of each T-set in $G_2(\epsilon)$. We can similarly trim R-sets in $G_1^{-L}(\epsilon)$, calling the remaining set $G_1^-(\epsilon)$, and the B-sets in $G_2^{-L}(\epsilon)$, calling the remaining set $G_2^-(\epsilon)$, to remove all CC-sequences. That is, if $f_{1,\epsilon}^- : I_1 \to 2^{I_0}$ is the function whose graph is $G_1^-(\epsilon)$ and $f_{2,\epsilon}^- : I_2 \to 2^{I_1}$ is the function whose graph is $G_2^-(\epsilon)$, then $\{f_{1,\epsilon}^-, f_{2,\epsilon}^-\}$ does not admit a CC-sequence. Furthermore, for each $i = 1, 2$, $G_i^-(\epsilon)$ is connected and closed and $G_i \subset G_i^-(\epsilon)$.

Thus $\mathcal{G}(f_1, f_2) = \bigcap_{0<\epsilon<\epsilon'} \mathcal{G}(f_{1,\epsilon}^-, f_{2,\epsilon}^-)$ is connected, giving a contradiction. ∎

5.3. End of proof of Theorem 1.6. Suppose that for each $i \geq 0$, $I_i = [0, 1]$, $f_{i+1} : I_{i+1} \to 2^{I_i}$ is a surjective upper semicontinuous function and the graph G_{i+1} of f_{i+1} is connected. Suppose that $\mathcal{G}(f_1, \ldots, f_n)$ is disconnected and for all k, l such that $1 \leq k < l \leq n$, $\mathcal{G}(f_k, \ldots, f_l)$ is disconnected if and only if $k = 1$ and $l = n$.

Suppose $\{f_1, \ldots, f_n\}$ does not admit a CC-sequence.

For every $\epsilon > 0$ and i, $0 < i \leq n$, let $f_{i,\epsilon} : I_i \to 2^{I_{i-1}}$ be the function whose graph is $G_i(\epsilon)$. Since $\mathcal{G}(f_1, \ldots, f_n)$ is disconnected, there exists $\mu > 0$ such that for all $\epsilon < \mu$, $\mathcal{G}(f_{1,\epsilon}, \ldots, f_{n,\epsilon})$ is disconnected. Choose $\epsilon < \mu$.

Define $G_1^-(\epsilon)$ and $f_{1,\epsilon}^-$ as in the proof of Lemma 5.4, and $G_n^-(\epsilon)$ and $f_{n,\epsilon}^-$ similarly to $G_2^-(\epsilon)$ and $f_{2,\epsilon}^-$ in the proof of Lemma 5.4.

We now define a $\overline{TL}$-set in $G_i(\epsilon)$. Suppose $0 < i \leq n$ and $A_i \times A_{i-1} \subset I_i \times I_{i-1}$ frames a TL-set D in $G_i(\epsilon)$ such that $A_{i-1} = [a_{i-1}, b_{i-1}] = \pi_{i,V}(D)$ and $A_i = [a_i, b_i] = \pi_{i,H}(D)$. Let $M < a_i$ be the maximum value, if it exists, such that if E is the component of $V_{[M,b_i]} \cap G_i(\epsilon)$ containing D, then for every x such that $M < x < a_1$, $E \cap V_{[x,b_i]}$ is disconnected. If the value does not exist then let $M = 0$. Let $M' > b_{i-1}$ be the minimum value, if it exists, such that if E is the component of $H_{[a_{i-1},M']} \cap G_i(\epsilon)$ containing D, then for every x such that $b_{i-1} < x < M'$, $E \cap H_{[a_{i-1},x]}$ is disconnected. If such a value does not exist then let $M' = 1$. Let D' be the component of $((M, b_i] \times [a_{i-1}, M')) \cap G_i$ containing D, and D'' the closure of D' in $G_i(\epsilon)$. Then D'' is a $\overline{TL}$-set framed by $\pi_{i,H}(D'') \times \pi_{i,V}(D'')$, and D'' is the *minimal* $\overline{TL}$*-set associated with* D.

It is possible that $D'' = G_i(\epsilon)$.

$\overline{TR}$-sets, $\overline{BL}$-sets and $\overline{BR}$-sets in $G_i(\epsilon)$ are defined analogously.

Suppose $A_i = [c, d]$, $A_{i-1} = [c', d']$ and $[c, d] \times [c', d']$ frames a $\overline{TL}$-set D in $G_i(\epsilon)$. Observe that

$$G_i \cap D \cap \big(((c, d] \cap (d - \epsilon, d]) \times [c', d')\big) = \emptyset,$$
$$G_i \cap D \cap ((c, d] \times ([c', d'] \cap [c', c' + \epsilon))) = \emptyset,$$

and since D is a $\overline{TL}$-set, $c \neq d$ and $c' \neq d'$.

For each $\langle x, y\rangle \in G_i \cap D$, if $((x, x+\epsilon] \times [y-\epsilon, y)) \cap G_i \cap D = \emptyset$ let

$$R_{x,y} = ((x, x+\epsilon] \times [y-\epsilon, y)) \cap D,$$

otherwise let $R_{x,y} = \emptyset$. Let $Y'_D := G_i(\epsilon) \setminus \bigcup\{R_{x,y} : \langle x, y\rangle \in G_i \cap D\}$ and let $Y_D = \overline{Y'_D}$.

(i) If $[c, d] \times [c', d']$ frames an R-set, B-set, TR-set, BL-set or BR-set E in Y_D, then E is an R-set, T-set, B-set, TR-set, BL-set or BR-set (respectively) in $G_i(\epsilon)$. If $[c, d] \times [c', d']$ frames an L-set or TL-set $E \subset D$ in Y_D, then $[c, d+\epsilon] \times [c'-\epsilon, d']$ (or $[c, 1] \times [c'-\epsilon, d']$ if $1-d < \epsilon$, or $[c, d+\epsilon] \times [c', 1]$ if $1-d' < \epsilon$) frames an L-set or TL-set (respectively) $E' \subset D$ in $G_i(\epsilon)$.

(ii) Both Y_D and $Y_D^{-1} := \{\langle y, x\rangle : \langle x, y\rangle \in Y_D\}$ have finite decompositions into fibre-connected subgraphs, since removing sets that are unions of $\epsilon \times \epsilon$ squares can only create finitely many new fibre-connected subgraphs.

For each $\overline{TL}$-set E in $G_i(\epsilon)$, define Y_E similarly and let

$$G_i^{-TL}(\epsilon) = \bigcap\{Y_E : E \text{ is a } \overline{TL}\text{-set in } G_i(\epsilon)\}.$$

Thus we have trimmed all $\overline{TL}$-sets in $G_i(\epsilon)$ to obtain $G_i^{-TL}(\epsilon)$. Since there are finitely many $\overline{TL}$-sets, it follows that $G_i^{-TL}(\epsilon)$ is a closed connected subgraph of $G_i(\epsilon)$, (i) and (ii) hold if we replace Y_D with $G_i^{-TL}(\epsilon)$ and D is replaced by any $\overline{TL}$-set, and

(iii) if $[r, s] \times [r', s']$ frames a TL-set $E \subset D$ such that $\pi_{i,H}(E) = [r, s]$ and $\pi_{i,V}(E) = [r', s']$, then there exist $x, y \in [0, 1]$ such that $\langle s, x\rangle, \langle y, r'\rangle \in E \cap G_i$. If there does not exist x such that $\langle s, x\rangle \in E \cap G_i$, then let E' be the minimal $\overline{TL}$-set in $\mathcal{G}_i(\epsilon)$ associated with E, framed by $[q, s] \times [r', q']$. Either $E \cap G_i = \emptyset$, in which case we have a contradiction by the definition of Y_E, or there exists $\langle u, v\rangle \in G_i \cap E'$ such that u is maximal (if $\langle u', v'\rangle \in G_i \cap E'$ then $u' \le u$). But then $(u, s] \times [r', q') \cap G_i^{-TL}(\epsilon) = \emptyset$. So x exists and similarly y exists.

Similarly define and trim all $\overline{TR}$-sets in $G_i^{-TL}(\epsilon)$, then all $\overline{BL}$-sets, then all $\overline{BR}$-sets. Write $G_i^{-X}(\epsilon)$ for the remaining subset of $G_i(\epsilon)$.

At each step we obtain a closed connected subgraph of $G_i(\epsilon)$, and the analogous statements to (i) and (ii) hold with respect to $G_i^{-X}(\epsilon)$.

Finally, trim all $\overline{L}$-sets, $\overline{R}$-sets, $\overline{T}$-sets and $\overline{B}$-sets as described earlier. Write $G_i^{-}(\epsilon)$ for the remaining subset of $G_i(\epsilon)$. For each i, $1 < i < n$, let $f_{i,\epsilon}^{-}$ be the function whose graph is $G_i^{-}(\epsilon)$. Recall that $f_{1,\epsilon}$ and $f_{n,\epsilon}$ have been defined above.

Let $Z = \{L, R, T, B, TL, TR, BL, BR\}$. Again the following observations can be readily established for each i:

(1) $G_i^-(\epsilon)$ is a closed connected subset of $I_i \times I_{i-1}$.
(2) $G_i^-(\epsilon)$ has a finite decomposition into fibre-connected subgraphs, and $\{\langle y,x\rangle : \langle x,y\rangle \in G_i^-(\epsilon)\}$ has a finite decomposition into fibre-connected subgraphs.
(3) $G_i \subset G_i^-(\epsilon)$.
(4) If $B_i \times B_{i-1}$ frames a TL-set E in $G_i^-(\epsilon)$, then for every $\langle x,y\rangle \in E$ there exist $\langle x',y'\rangle$ and $\langle x'',y''\rangle$ in $G_i \cap E$ such that $y' \le y$ and $x'' \ge x$. Similarly for each $V \in Z$.
(5) If $B_i \times B_{i-1}$ frames a V-set in $G_i^-(\epsilon)$ for any $V \in Z$, then by (4), $B_i \times B_{i-1}$ frames a V-set in G_i.

Suppose $\mathcal{G}(f_{1,\epsilon}^-, \ldots, f_{n,\epsilon}^-)$ is disconnected; then by Lemma 5.3, there exist l, m, $0 < l < m \le n$, and a CC-sequence $\{A_{l-1}, \ldots, A_m\}$ admitted by $\{f_{l,\epsilon}^-, \ldots, f_{m,\epsilon}^-\}$ with pivot point $\mathbf{p}$.

It follows from (5) that $\{A_{l-1}, \ldots, A_m\}$ is a CC-sequence admitted by $\{f_l, \ldots, f_m\}$, a contradiction, so $\mathcal{G}(f_{1,\epsilon}^-, \ldots, f_{n,\epsilon}^-)$ is connected.

Thus for each $\epsilon < \mu$, $\mathcal{G}(f_{1,\epsilon}^-, \ldots, f_{n,\epsilon}^-)$ is connected and hence

$$\mathcal{G}(f_1, \ldots, f_n) = \bigcap_{0<\epsilon<\mu} \mathcal{G}(f_{1,\epsilon}^-, \ldots, f_{n,\epsilon}^-)$$

is connected. Again we have a contradiction, so if $\mathcal{G}(f_1, \ldots, f_n)$ is disconnected then $\{f_1, \ldots, f_n\}$ admits a CC-sequence.

Thus, by Theorem 3.1 and Lemmas 2.9 and 2.11, $\varprojlim (I_i, f_i)$ is disconnected if and only if $\{f_i : i \in \mathbb{N}\}$ admits a CC-sequence. ∎

6. Concluding remarks. We conclude by observing that many of the results to date follow from Theorem 1.6. If the graph of a function $f : [0,1] \to 2^{[0,1]}$ has an L-set or an R-set, then there exists $x \in [0,1]$ such that $f(x)$ is disconnected, and Theorem 1.2 follows. Similarly Theorem 1.3 follows from the necessity of a B-set or T-set if $\varprojlim (I_i, f_i)$ is disconnected. Classical inverse limits cannot admit L-sets or R-sets, and hence it follows that they must be connected.

We can generalize Theorems 1.2 and 1.3 further:

THEOREM 6.1. *Suppose that $I_0 = I_1 = I_2 = [0,1]$, $f_1, g_1 : I_1 \to 2^{I_0}$ and $f_2, g_2 : I_2 \to 2^{I_1}$ are surjective upper semicontinuous functions, and the graphs G_i of the functions f_i and G_i' of g_i are connected, $i = 1,2$. If $\mathcal{G}(g_1, g_2)$ is connected, $G_i' \subseteq G_i$, and for every $x \in I_1$ each component of $\{x\} \times f_i(x)$ meets G_i' and each component of $f_2^{-1}(x) \times \{x\}$ meets G_2', then $\mathcal{G}(f_1, f_2)$ is connected.*

Proof. If $\mathcal{G}(g_1, g_2)$ is connected then $\{g_1, g_2\}$ does not admit a CC-sequence. If $\{A_0, A_1, A_2\}$ is a CC-sequence admitted by $\{f_1, f_2\}$ with pivot point $\langle p_0, p_1, p_2\rangle$, then there exists $x \in A_0$ such that $\langle p_1, x\rangle \in G'_1$ and $\langle p_1, x\rangle$ and $\langle p_1, p_0\rangle$ are connected in $V_{p_1} \cap G'_1$, and there exists $y \in A_2$ such that $\langle y, p_1\rangle \in G'_2$ and $\langle y, p_1\rangle$ and $\langle p_2, p_1\rangle$ are connected in $H_{p_1} \cap G'_2$. Hence $\langle x, p_1, y\rangle \in (A_0 \times A_1 \times A_2) \cap \mathcal{G}(g_1, g_2)$, and so $\{A_0, A_1, A_2\}$ is a CC-sequence admitted by $\{g_1, g_2\}$. ■

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