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Spectral properties of weighted composition operators on the Bloch and Dirichlet spaces

by

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Abstract. The spectra of invertible weighted composition operators uC_φ on the Bloch and Dirichlet spaces are studied. In the Bloch case we obtain a complete description of the spectrum when φ is a parabolic or elliptic automorphism of the unit disc. In the case of a hyperbolic automorphism φ, exact expressions for the spectral radii of invertible weighted composition operators acting on the Bloch and Dirichlet spaces are derived.

1. Introduction. The space of analytic functions on the open unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$ is denoted by $H(\mathbb{D})$. Every analytic selfmap $\varphi\colon \mathbb{D} \to \mathbb{D}$ of the unit disc induces a *composition operator* $C_\varphi f = f \circ \varphi$ on $H(\mathbb{D})$. These operators have been studied for many decades starting from the papers of Littlewood, Hardy and Riesz in the beginning of the 20th century. For general information of composition operators on classical spaces of analytic functions the reader is referred to the excellent monographs by Cowen and MacCluer [CM] and Shapiro [Sh]. In recent years this well-recognized theory has received a new stimulus from the more general situation of linear *weighted composition operators* $uC_\varphi(f) = u \cdot (f \circ \varphi)$, where $u \in H(\mathbb{D})$. The main objective when studying the operators uC_φ is to relate operator-theoretic properties of uC_φ to function-theoretic properties of the inducing symbols φ and u.

This paper is devoted to the study of spectral properties of invertible weighted composition operators acting on the Bloch and Dirichlet spaces, defined in the preliminaries section below. The main references are the papers [CGP] by Chalendar, Gallardo-Gutiérrez and Partington, and [HLNS] by Hyvärinen et al. In [HLNS] the spectrum of weighted composition op-

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erators with automorphic symbols φ was extensively studied on spaces of analytic functions satisfying certain general conditions introduced in [HLNS, Section 2.2]. This class contains for example the weighted Bergman spaces and Hardy spaces. However, the Bloch and Dirichlet spaces are not in this class since the bounded analytic functions are not contained in the multipliers of these spaces. A new approach is thus needed, and found partly in [CGP, Section 3], where the spectra of invertible weighted composition operators induced by parabolic and elliptic automorphisms on the Dirichlet space are completely described and the hyperbolic case is left as an open problem.

The paper is organized as follows. In Section 3 we study the multipliers of the Bloch space, and obtain results similar to those in [CGP, Section 2]. Section 4 is devoted to the spectral theory of invertible weighted composition operators uC_φ acting on the Bloch space. In particular, we give a description of the spectrum when φ is a parabolic or elliptic automorphism of $\mathbb{D}$. In the case of hyperbolic φ, the spectral radius is computed and we obtain an inclusion of the spectrum in an annulus. Finally, in Section 5 we improve the estimates in [CGP, Theorem 3.3] of the spectrum of an invertible weighted composition operator on the Dirichlet space when φ is a hyperbolic automorphism of $\mathbb{D}$.

2. Preliminaries. We begin by recalling some Banach spaces of analytic functions on the unit disc $\mathbb{D}$. The *Bloch space* $\mathcal{B}$ is the set of functions $f \in H(\mathbb{D})$ such that $\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty$, and is equipped with the norm

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|, \quad f \in \mathcal{B}.$$

The *Dirichlet space* $\mathcal{D}$ consists of functions $f \in H(\mathbb{D})$ such that $f' \in A^2$, where the *Bergman space* A^2 is the set of analytic functions on $\mathbb{D}$ such that

$$\|f\|_{A^2}^2 = \int_{\mathbb{D}} |f(z)|^2 \, dA(z) < \infty,$$

with normalized Lebesgue measure $dA(\cdot)$ on $\mathbb{D}$. The *Dirichlet norm* is defined as

$$\|f\|_{\mathcal{D}}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z), \quad f \in \mathcal{D}.$$

Other spaces of analytic functions on the unit disc $\mathbb{D}$ used in this paper are the *disc algebra* $A(\mathbb{D})$, consisting of functions that are moreover continuous on the closed unit disc, the weighted Banach spaces of analytic functions

$$H_{v_s}^\infty = \Big\{ f \in H(\mathbb{D}) : \|f\|_{H_{v_s}^\infty} = \sup_{z\in\mathbb{D}} v_s(z)|f(z)| < \infty \Big\},$$

where $0 < s < \infty$ and $v_s(z) = (1-|z|^2)^s$ is the standard weight, and the space H^∞ of bounded analytic functions on $\mathbb{D}$ with supremum norm $\|\cdot\|_\infty$.

The spectrum and spectral radius of an operator $T : \mathcal{X} \to \mathcal{X}$ on a space $\mathcal{X}$ are denoted respectively by $\sigma_{\mathcal{X}}(T)$ and $r_{\mathcal{X}}(T)$. A good reference for operator theory in function spaces is the monograph [Z] by Zhu.

When dealing with composition operators it is customary to denote the nth iterate of a selfmap φ of $\mathbb{D}$ by φ_n, that is,

$$\varphi_n := \underbrace{\varphi \circ \cdots \circ \varphi}_{n \,\text{times}}$$

with φ_0 representing the identity map, and it is easy to check that

$$(uC_\varphi)^n f(z) = u(z)u(\varphi(z)) \cdots u(\varphi_{n-1}(z))f(\varphi_n(z)), \quad f \in H(\mathbb{D}),\ z \in \mathbb{D}.$$

This can also be stated as $(uC_\varphi)^n = u_{(n)}C_{\varphi_n}$, where $u_{(0)} := 1$ and

$$u_{(n)} := \prod_{j=0}^{n-1} u \circ \varphi_j \in H(\mathbb{D}), \quad n \in \mathbb{N}.$$

It turns out that the spectral analysis of invertible weighted composition operators uC_φ strongly depends on the type of the (necessarily) automorphic symbol φ. Recall that a nontrivial automorphism φ of $\mathbb{D}$ is called *elliptic* if it has a unique fixed point in $\mathbb{D}$, *parabolic* if φ has a Denjoy–Wolff fixed point a in $\partial\mathbb{D}$ with $\varphi'(a) = 1$, and *hyperbolic* if φ has a Denjoy–Wolff fixed point $a \in \partial\mathbb{D}$ with $0 < \varphi'(a) < 1$ (called an *attractive fixed point*) and a *repulsive fixed point* $b \in \partial\mathbb{D}$ with $\varphi'(b) = 1/\varphi'(a)$ (see [CM, Section 2.3]). When computing the spectrum, we will make use of the formula

$$\lim_{n\to\infty} (1 - |\varphi_n(0)|)^{1/n} = \varphi'(a), \tag{2.1}$$

which is valid for parabolic and hyperbolic automorphisms φ of $\mathbb{D}$ with Denjoy–Wolff point a (see [CM, pp. 251–252]).

3. Multiplier spaces. In this section we consider the multiplier spaces

$$\mathcal{M}(\mathcal{X}) := \{u \in H(\mathbb{D}) : M_u : \mathcal{X} \to \mathcal{X} \text{ is bounded}\} \subset \mathcal{X},$$
$$\mathcal{M}(\mathcal{X}, \varphi) := \{u \in H(\mathbb{D}) : uC_\varphi : \mathcal{X} \to \mathcal{X} \text{ is bounded}\} \subset \mathcal{X},$$

where $\mathcal{X}$ is either the Bloch space $\mathcal{B}$ or the Dirichlet space $\mathcal{D}$, and the multiplication operator is defined in the obvious way: $M_u f = uf$. The main results of this section are Theorem 3.2, where we characterize those φ for which $\mathcal{M}(\mathcal{B}, \varphi) = \mathcal{B}$, and Theorem 3.3, where we show that $\mathcal{M}(\mathcal{B}, \varphi) = \mathcal{M}(\mathcal{B})$ whenever φ is a finite Blaschke product. The Dirichlet space versions of those results are given in [CGP, Theorems 2.2–2.3].

The multiplier space $\mathcal{M}(\mathcal{D})$ was characterized by Stegenga [St] as the set of functions $u \in H^\infty$ such that the multiplication operator $M_{u'} : \mathcal{D} \to A^2$ is bounded ($|u'(z)|^2 dA(z)$ being a Carleson measure for $\mathcal{D}$). For the Bloch space it is known from [OZ] that a weighted composition operator $uC_\varphi : \mathcal{B} \to \mathcal{B}$ is bounded if and only if the following two conditions hold:

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|u'(z)|\log\frac{e}{1-|\varphi(z)|^2}<\infty, \tag{3.1}$$

$$\sup_{z\in\mathbb{D}}\frac{1-|z|^2}{1-|\varphi(z)|^2}|u(z)\varphi'(z)|<\infty. \tag{3.2}$$

From this it follows that $u\in\mathcal{M}(\mathcal{B})$ if and only if

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|u'(z)|\log\frac{e}{1-|z|^2}<\infty, \tag{3.3}$$

$$u\in H^\infty. \tag{3.4}$$

Condition (3.3) can also be related to a multiplication operator in a similar fashion to the Dirichlet case:

Lemma 3.1. *If the function $u:\mathbb{D}\to\mathbb{C}$ is analytic, then the multiplication operator $M_{u'}:\mathcal{B}\to H^\infty_{v_1}$ is bounded if and only if condition* (3.3) *holds.*

Proof. If condition (3.3) holds then

$$\begin{aligned}\|M_{u'}\|_{\mathcal{B}\to H^\infty_{v_1}} &= \sup_{\|f\|_{\mathcal{B}}=1}\|M_{u'}f\|_{H^\infty_{v_1}} = \sup_{\|f\|_{\mathcal{B}}=1}\sup_{z\in\mathbb{D}}(1-|z|^2)|u'(z)f(z)|\\ &\le \sup_{\|f\|_{\mathcal{B}}=1}\sup_{z\in\mathbb{D}}(1-|z|^2)\big|u'(z)\big|\alpha\|f\|_{\mathcal{B}}\log\frac{e}{1-|z|^2}\\ &=\alpha\sup_{z\in\mathbb{D}}(1-|z|^2)|u'(z)|\log\frac{e}{1-|z|^2}<\infty,\end{aligned}$$

where we have used the fact that every Bloch function f satisfies

$$\sup_{z\in\mathbb{D}}\frac{|f(z)|}{\log\frac{e}{1-|z|^2}}\le\alpha\|f\|_{\mathcal{B}} \tag{3.5}$$

for some positive constant α independent of f.

Conversely, if the operator $M_{u'}:\mathcal{B}\to H^\infty_{v_1}$ is bounded then there is a constant $c>0$ such that for every $f\in\mathcal{B}$,

$$\|M_{u'}(f)\|_{H^\infty_{v_1}}\le c\|f\|_{\mathcal{B}}.$$

Applying this to the Bloch functions $f_a(z):=\log\frac{e}{1-\bar{a}z}$ for $a\in\mathbb{D}$, we obtain

$$(1-|z|^2)|u'(z)|\left|\log\frac{e}{1-\bar{a}z}\right|\le\|M_{u'}(f_a)\|_{H^\infty_{v_1}}\le c\|f_a\|_{\mathcal{B}}\le 2c$$

for all $z,a\in\mathbb{D}$, since $\|f_a\|_{\mathcal{B}}\le 2$ for every $a\in\mathbb{D}$. Now choose $a=z$ and take the supremum over $z\in\mathbb{D}$ to get (3.3). ∎

Theorem 3.2. *Let φ be an analytic selfmap of $\mathbb{D}$. Then $\mathcal{M}(\mathcal{B},\varphi)=\mathcal{B}$ if and only if*

(1) $\|\varphi\|_\infty<1$ *and*
(2) $\varphi\in\mathcal{M}(\mathcal{B})$.

Proof. Assume first that (1) and (2) hold and choose $u \in \mathcal{B}$. Then
$$\sup_{z\in\mathbb{D}}(1-|z|^2)|u'(z)|\log\frac{e}{1-|\varphi(z)|^2} \le \|u\|_{\mathcal{B}}\log\frac{e}{1-\|\varphi\|_\infty^2} < \infty,$$
so (3.1) holds. We also have
$$\begin{aligned}&\sup_{z\in\mathbb{D}}\frac{1-|z|^2}{1-|\varphi(z)|^2}|u(z)\varphi'(z)|\\&\quad\le\frac{1}{1-\|\varphi\|_\infty^2}\Big(\sup_{z\in\mathbb{D}}(1-|z|^2)|\varphi'(z)|\log\frac{e}{1-|z|^2}\Big)\cdot\Big(\sup_{z\in\mathbb{D}}\frac{|u(z)|}{\log\frac{e}{1-|z|^2}}\Big),\end{aligned}$$
which is finite because u is a Bloch function (see (3.5)) and $\varphi \in \mathcal{M}(\mathcal{B})$ (replace u with φ in (3.3)). Thus (3.2) also holds, and so $u \in \mathcal{M}(\mathcal{B},\varphi)$, which shows that $\mathcal{M}(\mathcal{B},\varphi) = \mathcal{B}$ (the inclusion $\mathcal{M}(\mathcal{B},\varphi) \subset \mathcal{B}$ being trivial).

Conversely, assume that $\mathcal{M}(\mathcal{B},\varphi) = \mathcal{B}$, so that $uC_\varphi : \mathcal{B} \to \mathcal{B}$ is bounded for every $u \in \mathcal{B}$. This ensures that if $f \in \mathcal{B}$ then $(f\circ\varphi)\cdot u \in \mathcal{B}$ for every $u \in \mathcal{B}$, which means that the multiplication operator $M_{f\circ\varphi} : \mathcal{B} \to \mathcal{B}$ is well defined and hence bounded by the Closed Graph Theorem. Thus $C_\varphi f = f\circ\varphi \in \mathcal{M}(\mathcal{B}) \subset H^\infty$, so that the composition operator $C_\varphi : \mathcal{B} \to H^\infty$ is well defined and bounded.

To reach a contradiction assume that $\|\varphi\|_\infty = 1$. Choose $f \in \mathcal{B}\backslash H^\infty$ such that $\|f\|_{\mathcal{B}} = 1$. Since f is unbounded there is a sequence $\{\omega_n\}_{n=1}^\infty \subset \mathbb{D}$ with $|\omega_n| \to 1^-$ such that $|f(\omega_n)| > n$ for every $n \in \mathbb{N}$. Since $\|\varphi\|_\infty = 1$, there is a sequence $\{z_n\}_{n=1}^\infty \subset \mathbb{D}$ such that $|\varphi(z_n)| = |\omega_n|$ for n large enough, say $n \ge n_0$. Now choose a sequence $\{\theta_n\}_{n=n_0}^\infty$ so that $\omega_n = e^{i\theta_n}\varphi(z_n)$ and define $f_n(z) := f(e^{i\theta_n}z)$. Then $\|f_n\|_{\mathcal{B}} = \|f\|_{\mathcal{B}} = 1$ for every n, but $|C_\varphi f_n(z_n)| = |f_n(\varphi(z_n))| = |f(\omega_n)| > n$ so that $\|C_\varphi f_n\|_\infty > n$. This contradicts the boundedness of $C_\varphi : \mathcal{B} \to H^\infty$, so we must have $\|\varphi\|_\infty < 1$.

It remains to show that $\varphi \in \mathcal{M}(\mathcal{B})$. The boundedness of $uC_\varphi : \mathcal{B} \to \mathcal{B}$ for every $u \in \mathcal{B}$ and (3.2) imply that
$$\sup_{z\in\mathbb{D}}(1-|z|^2)|u(z)\varphi'(z)| < \infty$$
for every $u \in \mathcal{B}$. The operator $M_{\varphi'} : \mathcal{B} \to H_{v_1}^\infty$ is hence well defined and bounded. According to Lemma 3.1 this is equivalent to
$$\sup_{z\in\mathbb{D}}(1-|z|^2)|\varphi'(z)|\log\frac{e}{1-|z|^2} < \infty,$$
so $\varphi \in \mathcal{M}(\mathcal{B})$ and the proof is complete. ■

THEOREM 3.3. *Assume that φ is a finite Blaschke product. Then $\mathcal{M}(\mathcal{B},\varphi) = \mathcal{M}(\mathcal{B})$.*

Proof. If $u \in \mathcal{M}(\mathcal{B})$ then $M_u : \mathcal{B} \to \mathcal{B}$ is bounded and hence $uC_\varphi = M_uC_\varphi : \mathcal{B} \to \mathcal{B}$ is bounded, so that $u \in \mathcal{M}(\mathcal{B},\varphi)$ and $\mathcal{M}(\mathcal{B}) \subset \mathcal{M}(\mathcal{B},\varphi)$.

Conversely, assume that $u \in \mathcal{M}(\mathcal{B}, \varphi)$, so that $uC_\varphi : \mathcal{B} \to \mathcal{B}$ is bounded and $u \in \mathcal{B}$. Since φ is a finite Blaschke product, from [M, Lemma 1] we deduce that

$$\frac{1-|\varphi(z)|^2}{1-|z|^2} \leq \sum_{j=1}^{N} \frac{1+|\omega_j|}{1-|\omega_j|} =: K$$

for every $z \in \mathbb{D}$, where N is the degree of the Blaschke product and $\{\omega_j\}_{j=1}^N$ are the zeros of φ. Hence

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|u'(z)| \log \frac{e}{1-|z|^2} \\ \leq \sup_{z\in\mathbb{D}}(1-|z|^2)|u'(z)| \log \frac{e}{1-|\varphi(z)|^2} + \log K \cdot \|u\|_{\mathcal{B}} < \infty,$$

which is (3.3). It remains to show that $u \in H^\infty$, so let S be the finite supremum in (3.2). For every $z \in \mathbb{D}$ for which $\varphi'(z) \neq 0$ we have

$$|u(z)| \leq S \frac{1-|\varphi(z)|^2}{1-|z|^2} \frac{1}{|\varphi'(z)|} \leq SK \frac{1}{|\varphi'(z)|}.$$

Since φ is a finite Blaschke product, φ' is analytic in an open neighborhood of the closed unit disc, is nonzero on $|z| = 1$ and has only a finite number of zeros elsewhere. This implies that we can choose $\delta > 0$ so that $m_\delta := \min_{\delta \leq |z| \leq 1} |\varphi'(z)| > 0$, which shows that u is bounded by SK/m_δ on $\delta \leq |z| < 1$ and hence bounded on $\mathbb{D}$. Thus $u \in \mathcal{M}(\mathcal{B})$, so that $\mathcal{M}(\mathcal{B}, \varphi) \subset \mathcal{M}(\mathcal{B})$, and hence $\mathcal{M}(\mathcal{B}, \varphi) = \mathcal{M}(\mathcal{B})$. ∎

In the next sections we will study the spectrum of *invertible* weighted composition operators on the Bloch and Dirichlet spaces. The invertibility of weighted composition operators has been characterized on various spaces of functions by many authors: see for example [B, G, HLNS]. The following result is a consequence of [B, Corollary 2.3].

THEOREM 3.4. *Assume that $\mathcal{X}$ is either the Bloch space $\mathcal{B}$ or the Dirichlet space $\mathcal{D}$, and let $uC_\varphi : \mathcal{X} \to \mathcal{X}$ be a bounded weighted composition operator on $\mathcal{X}$. Then uC_φ is invertible on $\mathcal{X}$ if and only if $u \in \mathcal{M}(\mathcal{X})$, u is bounded away from zero on $\mathbb{D}$ and φ is an automorphism of $\mathbb{D}$. In that case the inverse operator of $uC_\varphi : \mathcal{X} \to \mathcal{X}$ is also a weighted composition operator, given by*

$$(uC_\varphi)^{-1} = \frac{1}{u \circ \varphi^{-1}} C_{\varphi^{-1}}.$$

4. Spectra on the Bloch space. In this section we study the spectra of invertible weighted composition operators on the Bloch space. The investigation is divided into three cases, to cover parabolic, hyperbolic and elliptic automorphisms. Our approach is based on the papers [CGP] and [HLNS], but some new ideas are needed. The following lemma will be useful.

LEMMA 4.1. *If φ is an automorphism of $\mathbb{D}$, then $r_{\mathcal{B}}(C_\varphi) = 1$.*

Proof. By [X, Corollary 2] and [MV, Lemma 6] we have the following estimate of the norm of the iterated composition operator for any automorphism φ of $\mathbb{D}$ and $n \in \mathbb{N}$:

$$1 \le \|C_{\varphi_n}\|_{\mathcal{B}\to\mathcal{B}} \le 1 + \frac{1}{2}\log\frac{1+|\varphi_n(0)|}{1-|\varphi_n(0)|} \le 1 + \varrho(\varphi(0),0)n, \tag{4.1}$$

where

$$\varrho(z,w) := \frac{1}{2}\log\frac{1+\left|\dfrac{z-w}{1-\bar{z}w}\right|}{1-\left|\dfrac{z-w}{1-\bar{z}w}\right|}$$

is the hyperbolic distance on $\mathbb{D}$. If $\varphi(0) = 0$ then obviously $r_{\mathcal{B}}(C_\varphi) = 1$. If on the other hand $\varrho(\varphi(0),0) \neq 0$, then we obtain the following estimate from (4.1) for every $n \in \mathbb{N}$:

$$1 \le \|C_{\varphi_n}\|^{1/n}_{\mathcal{B}\to\mathcal{B}} \le \left[(1+\varrho(\varphi(0),0)n)^{\frac{1}{\varrho(\varphi(0),0)n}}\right]^{\varrho(\varphi(0),0)},$$

and since $\lim_{x\to\infty}(1+x)^{1/x} = 1$ we conclude that $r_{\mathcal{B}}(C_\varphi) = 1$. ■

4.1. The parabolic case. We begin by describing the spectrum of invertible weighted composition operators uC_φ on the Bloch space induced by parabolic automorphisms φ. For the proof we need the following result.

LEMMA 4.2. *Suppose that φ is a parabolic automorphism of $\mathbb{D}$ with the unique fixed point $a \in \partial\mathbb{D}$, and assume that $u \in A(\mathbb{D})$ is bounded away from zero on $\mathbb{D}$. Then*

$$\lim_{n\to\infty} \|u_{(n)}\|^{1/n}_\infty = |u(a)|.$$

Proof. See the proof of [HLNS, Lemma 4.2]. ■

THEOREM 4.3. *Suppose the weighted composition operator $uC_\varphi\colon \mathcal{B}\to\mathcal{B}$ is invertible on the Bloch space and assume that the automorphism φ is parabolic, with the unique fixed point $a \in \partial\mathbb{D}$. If $u \in A(\mathbb{D})$, then*

$$\sigma_{\mathcal{B}}(uC_\varphi) = \{\lambda \in \mathbb{C} : |\lambda| = |u(a)|\}.$$

Proof. We begin by showing that the spectrum is contained in the given circle of radius $|u(a)|$. According to Theorem 3.4, u belongs to $\mathcal{M}(\mathcal{B})$ and is bounded away from zero on $\mathbb{D}$, which implies that $u(a) \neq 0$. Since $r_{\mathcal{B}}(C_\varphi) = 1$ by Lemma 4.1 and $(uC_\varphi)^n = u_{(n)}C_{\varphi_n} = M_{u_{(n)}}C_{\varphi_n}$ by Theorem 3.3, we only need to focus on the operator norm of $M_{u_{(n)}} : \mathcal{B}\to\mathcal{B}$:

$$\begin{aligned}
\|M_{u_{(n)}}\|_{\mathcal{B}\to\mathcal{B}} &= \sup_{\|f\|_{\mathcal{B}}\le 1} \|u_{(n)}\cdot f\|_{\mathcal{B}} \\
&\le \sup_{\|f\|_{\mathcal{B}}\le 1}\sup_{z\in\mathbb{D}}(1-|z|^2)|u'_{(n)}(z)f(z)| + \sup_{\|f\|_{\mathcal{B}}\le 1}\sup_{z\in\mathbb{D}}(1-|z|^2)|u_{(n)}(z)f'(z)| \\
&\quad + \sup_{\|f\|_{\mathcal{B}}\le 1}|u_{(n)}(0)f(0)| \\
&= \sup_{\|f\|_{\mathcal{B}}\le 1}\sup_{z\in\mathbb{D}}(1-|z|^2)\left|\sum_{j=0}^{n-1}\frac{u_{(n)}(z)}{u\circ\varphi_j(z)}\cdot(u\circ\varphi_j)'(z)\right||f(z)| \\
&\quad + \sup_{\|f\|_{\mathcal{B}}\le 1}\sup_{z\in\mathbb{D}}(1-|z|^2)|u_{(n)}(z)f'(z)| + \sup_{\|f\|_{\mathcal{B}}\le 1}|u_{(n)}(0)f(0)| \\
&\le \sum_{j=0}^{n-1}\left\|\frac{u_{(n)}}{u\circ\varphi_j}\right\|_\infty \sup_{\|f\|_{\mathcal{B}}\le 1}\sup_{z\in\mathbb{D}}(1-|z|^2)|u'(\varphi_j(z))|\,|\varphi_j'(z)|\,|f(z)| + 2\|u_{(n)}\|_\infty \\
&= \sum_{j=0}^{n-1}\left\|\frac{u_{(n)}}{u\circ\varphi_j}\right\|_\infty \sup_{\|f\|_{\mathcal{B}}\le 1}\sup_{z\in\mathbb{D}}(1-|\varphi_j(z)|^2)|u'(\varphi_j(z))f(z)| + 2\|u_{(n)}\|_\infty \\
&= \sum_{j=0}^{n-1}\left\|\frac{u_{(n)}}{u\circ\varphi_j}\right\|_\infty \sup_{\|f\|_{\mathcal{B}}\le 1}\sup_{z\in\mathbb{D}}(1-|z|^2)|u'(z)f(\psi_j(z))| + 2\|u_{(n)}\|_\infty,
\end{aligned}$$

where we have used the fact that

$$|\varphi'(z)| = \frac{1-|\varphi(z)|^2}{1-|z|^2}$$

if φ is an automorphism of $\mathbb{D}$, and introduced the function $\psi := \varphi^{-1}$ to simplify notation. Note that ψ is also a parabolic automorphism with fixed point a, and $\psi_j = (\varphi^{-1})_j = \varphi_j^{-1}$. Furthermore,

$$\begin{aligned}
\sup_{z\in\mathbb{D}}(1-|z|^2)|u'(z)f(\psi_j(z))| &= \|M_{u'}(f\circ\psi_j)\|_{H^\infty_{v_1}} \\
&\le \|M_{u'}\|_{\mathcal{B}\to H^\infty_{v_1}}\|C_{\psi_j}\|_{\mathcal{B}\to\mathcal{B}}\|f\|_{\mathcal{B}},
\end{aligned}$$

where the norm $\|M_{u'}\|_{\mathcal{B}\to H^\infty_{v_1}}$ is finite since $u\in\mathcal{M}(\mathcal{B})$, as discussed in Lemma 3.1. Now using inequality (4.1) and the fact that u is bounded away from zero on $\mathbb{D}$, we obtain

$$\begin{aligned}
\|M_{u_{(n)}}\|_{\mathcal{B}\to\mathcal{B}} &\le \sum_{j=0}^{n-1}\left\|\frac{u_{(n)}}{u\circ\varphi_j}\right\|_\infty \|M_{u'}\|_{\mathcal{B}\to H^\infty_{v_1}}\|C_{\psi_j}\|_{\mathcal{B}\to\mathcal{B}} + 2\|u_{(n)}\|_\infty \\
&\le \|M_{u'}\|_{\mathcal{B}\to H^\infty_{v_1}}\sum_{j=0}^{n-1}\left\|\frac{u_{(n)}}{u\circ\varphi_j}\right\|_\infty \big(1+\varrho(\psi(0),0)j\big) + 2\|u_{(n)}\|_\infty \\
&\le \left[\|M_{u'}\|_{\mathcal{B}\to H^\infty_{v_1}}\left\|\frac{1}{u}\right\|_\infty + 2\right] n\big(1+\varrho(\psi(0),0)n\big)\|u_{(n)}\|_\infty.
\end{aligned}$$

Applying this to the spectral radius and using Lemmas 4.1 and 4.2 gives

$$\begin{aligned} r_{\mathcal{B}}(uC_\varphi) &= \lim_{n\to\infty} \|(uC_\varphi)^n\|_{\mathcal{B}\to\mathcal{B}}^{1/n} \le \limsup_{n\to\infty} \|M_{u_{(n)}}\|_{\mathcal{B}\to\mathcal{B}}^{1/n} r_{\mathcal{B}}(C_\varphi) \\ &\le \lim_{n\to\infty} \left[\|M_{u'}\|_{\mathcal{B}\to H^\infty_{v_1}} \left\| \frac{1}{u} \right\|_\infty + 2 \right]^{1/n} n^{1/n} \big(1 + \varrho(\psi(0), 0) n\big)^{1/n} \|u_{(n)}\|_\infty^{1/n} \\ &= |u(a)|. \end{aligned}$$

Since

$$(uC_\varphi)^{-1} = \frac{1}{u \circ \varphi^{-1}} C_{\varphi^{-1}},$$

by Theorem 3.4, where φ^{-1} is a parabolic automorphism with unique fixed point a, the above result also shows that

$$r_{\mathcal{B}}((uC_\varphi)^{-1}) \le \left| \frac{1}{u(\varphi^{-1}(a))} \right| = |u(a)|^{-1}.$$

Now if $\lambda \in \sigma_{\mathcal{B}}(uC_\varphi)$, then we also have $\lambda^{-1} \in \sigma_{\mathcal{B}}((uC_\varphi)^{-1})$. It follows that $|\lambda| \le r_{\mathcal{B}}(uC_\varphi) \le |u(a)|$ and $|\lambda|^{-1} \le r_{\mathcal{B}}((uC_\varphi)^{-1}) \le |u(a)|^{-1}$, so $|\lambda| = |u(a)|$. Thus

$$\sigma_{\mathcal{B}}(uC_\varphi) \subseteq \big\{\lambda \in \mathbb{C} : |\lambda| = |u(a)|\big\}, \tag{4.2}$$

and obviously $r_{\mathcal{B}}(uC_\varphi) = |u(a)|$.

In order to prove the reverse inclusion in (4.2), let λ be a complex number of modulus $|\lambda| = |u(a)| = r_{\mathcal{B}}(uC_\varphi)$. As in the proof of [HLNS, Theorem 4.3] it is then enough, by the Spectral Mapping Theorem, to show that

$$r_{\mathcal{B}}(\lambda - uC_\varphi) \ge 2 r_{\mathcal{B}}(uC_\varphi),$$

which is done as follows. The sequence $\{z_n\}_{n=0}^\infty$ defined by $z_n = \varphi_n(0)$ is interpolating for H^∞ since φ is a parabolic automorphism (see the comment preceding [HLNS, Theorem 4.3]), so by the Open Mapping Theorem there is a constant $c > 0$ and a sequence $\{f_n\}_{n=0}^\infty \subset H^\infty$ such that for all $n \in \mathbb{N}$ we have $\|f_n\|_\infty \le c$ and

$$f_n(\varphi_k(z_n)) = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \ne n. \end{cases} \tag{4.3}$$

Note that $\mathcal{B} \subset H^\infty_{v_s}$ for every $s > 0$, since if $f \in \mathcal{B}$ then by (3.5),

$$\|f\|_{H^\infty_{v_s}} \le \alpha \|f\|_{\mathcal{B}} \sup_{z\in\mathbb{D}} (1 - |z|^2)^s \log \frac{e}{1 - |z|^2} < \infty.$$

Choose some $s > 0$ and let

$$c_s := \alpha \sup_{z\in\mathbb{D}} (1 - |z|^2)^s \log \frac{e}{1 - |z|^2},$$

so that $\|f\|_{H^\infty_{v_s}} \le c_s \|f\|_{\mathcal{B}}$ for every $f \in \mathcal{B}$. The interpolating sequence

$\{f_n\}_{n=0}^{\infty}$ satisfies $\|f_n\|_{\mathcal{B}} \le \|f_n\|_\infty \le c$ for all $n \in \mathbb{N}$, which gives

$$\begin{aligned}\|(\lambda - uC_\varphi)^{2n}\|_{\mathcal{B}\to\mathcal{B}} &\ge c^{-1}\|(\lambda - uC_\varphi)^{2n} f_n\|_{\mathcal{B}}\\ &\ge (cc_s)^{-1}\|(\lambda - uC_\varphi)^{2n} f_n\|_{H^\infty_{v_s}}\\ &\ge (cc_s)^{-1}(1-|z_n|^2)^s|[(\lambda - uC_\varphi)^{2n} f_n](z_n)|\end{aligned}$$

for all $n \in \mathbb{N}$. Furthermore,

$$\begin{aligned}[(\lambda - uC_\varphi)^{2n} f_n](z_n) &= \sum_{k=0}^{2n}\binom{2n}{k}\lambda^{2n-k}[(-uC_\varphi)^k f_n](z_n)\\ &= \sum_{k=0}^{2n}\binom{2n}{k}\lambda^{2n-k}(-1)^k u_{(k)}(z_n) f_n(\varphi_k(z_n))\\ &= \binom{2n}{n}(-1)^n\lambda^n u_{(n)}(z_n)\end{aligned}$$

by (4.3), so

$$\begin{aligned}&\|(\lambda - uC_\varphi)^{2n}\|_{\mathcal{B}\to\mathcal{B}} \ge (cc_s)^{-1}\binom{2n}{n}|\lambda|^n|u_{(n)}(z_n)|(1-|z_n|^2)^s\\ &= (cc_s)^{-1}\binom{2n}{n}|\lambda|^n|u_{(n)}(z_n)|\left(\frac{1-|z_n|^2}{1-|\varphi_n(z_n)|^2}\right)^s(1-|\varphi_n(z_n)|^2)^s\\ &= (cc_s)^{-1}\binom{2n}{n}|\lambda|^n\left|\frac{u_{(n)}(z_n)}{\varphi_n'(z_n)^s}\right|(1-|\varphi_{2n}(0)|^2)^s\\ &\ge (cc_s)^{-1}\binom{2n}{n}|\lambda|^n|\omega_{(n)}(z_n)|(1-|\varphi_{2n}(0)|)^s,\end{aligned}$$

where the function

$$\omega(z) := \frac{u(z)}{\varphi'(z)^s}$$

introduced in the proof of [HLNS, Theorem 4.3] satisfies

$$\omega_{(n)}(z) = \frac{u_{(n)}(z)}{\varphi_n'(z)^s}.$$

In the same proof it was also shown that

$$\lim_{n\to\infty}|\omega_{(n)}(z_n)|^{1/(2n)} = \frac{|u(a)|^{1/2}}{\varphi'(a)^{s/2}} = |u(a)|^{1/2},$$

and mentioned that $\lim_{n\to\infty}\binom{2n}{n}^{1/(2n)} = 2$. Using the parabolic version of the limit (2.1), we see that

$$
\begin{aligned}
r_{\mathcal{B}}(\lambda - uC_\varphi) &= \lim_{n\to\infty} \|(\lambda - uC_\varphi)^{2n}\|_{\mathcal{B}\to\mathcal{B}}^{1/(2n)} \\
&\geq \lim_{n\to\infty} (cc_s)^{-1/(2n)} \binom{2n}{n}^{1/(2n)} |\lambda|^{1/2} |\omega_{(n)}(z_n)|^{1/(2n)} (1 - |\varphi_{2n}(0)|)^{s/(2n)} \\
&= 2|\lambda|^{1/2}|u(a)|^{1/2} = 2|u(a)| = 2r_{\mathcal{B}}(uC_\varphi).
\end{aligned}
$$

Since $r_{\mathcal{B}}(\lambda - uC_\varphi) \geq 2r_{\mathcal{B}}(uC_\varphi)$, from the proof of [HLNS, Theorem 4.3] we get $-\lambda \in \sigma_{\mathcal{B}}(uC_\varphi)$, which shows that

$$
\sigma_{\mathcal{B}}(uC_\varphi) \supseteq \{-\lambda \in \mathbb{C} : |\lambda| = |u(a)|\} = \{\lambda \in \mathbb{C} : |\lambda| = |u(a)|\},
$$

and the proof is complete. ∎

4.2. The hyperbolic case. In this subsection we investigate the spectrum of a weighted composition operator $uC_\varphi\colon \mathcal{B} \to \mathcal{B}$ generated by a hyperbolic symbol φ. The results in this case are not complete. We obtain the spectral radius $r_{\mathcal{B}}(uC_\varphi)$ and thereby an inclusion of the spectrum in an annulus, which turns out to coincide with the spectrum under additional assumptions on u. The main result is given in Theorem 4.5.

LEMMA 4.4. *Suppose that φ is a hyperbolic automorphism of $\mathbb{D}$ with fixed points $a, b \in \partial\mathbb{D}$, and assume that $u \in A(\mathbb{D})$ is bounded away from zero on $\mathbb{D}$. Then*

$$
\lim_{n\to\infty} \|u_{(n)}\|_\infty^{1/n} = \max\{|u(a)|, |u(b)|\}.
$$

Proof. See the proof of [HLNS, Lemma 4.4]. ∎

THEOREM 4.5. *Suppose the weighted composition operator $uC_\varphi\colon \mathcal{B} \to \mathcal{B}$ is invertible on the Bloch space and assume that the automorphism φ is hyperbolic, with attractive fixed point $a \in \partial\mathbb{D}$ and repulsive fixed point $b \in \partial\mathbb{D}$. If $u \in A(\mathbb{D})$, then $r_{\mathcal{B}}(uC_\varphi) = \max\{|u(a)|, |u(b)|\}$ and*

$$
\sigma_{\mathcal{B}}(uC_\varphi) \subseteq \big\{\lambda \in \mathbb{C} : \min\{|u(a)|, |u(b)|\} \leq |\lambda| \leq \max\{|u(a)|, |u(b)|\}\big\}.
$$

Proof. As in the proof of Theorem 4.3, u belongs to $\mathcal{M}(\mathcal{B})$ and is bounded away from zero on $\mathbb{D}$, so that $u(a), u(b) \neq 0$. Since $r_{\mathcal{B}}(C_\varphi) = 1$ by Lemma 4.1 and $(uC_\varphi)^n = M_{u_{(n)}} C_{\varphi_n}$, it is again enough to consider the operator norm of $M_{u_{(n)}} : \mathcal{B} \to \mathcal{B}$. Through identical calculations as in the proof of Theorem 4.3, observing that $\psi := \varphi^{-1}$ is also a hyperbolic automorphism, we obtain

$$
\|M_{u_{(n)}}\|_{\mathcal{B}\to\mathcal{B}} \leq \bigg[\|M_{u'}\|_{\mathcal{B}\to H_{v_1}^\infty} \bigg\|\frac{1}{u}\bigg\|_\infty + 2\bigg] n\big(1 + \varrho(\psi(0), 0)n\big) \|u_{(n)}\|_\infty,
$$

and so by Lemmas 4.1 and 4.4,

$$\begin{aligned} r_{\mathcal{B}}(uC_\varphi) &= \lim_{n\to\infty} \|(uC_\varphi)^n\|_{\mathcal{B}\to\mathcal{B}}^{1/n} \le \limsup_{n\to\infty} \|M_{u_{(n)}}\|_{\mathcal{B}\to\mathcal{B}}^{1/n} r_{\mathcal{B}}(C_\varphi) \\ &\le \lim_{n\to\infty} \left[\|M_{u'}\|_{\mathcal{B}\to H^\infty_{v_1}} \left\| \frac{1}{u} \right\|_\infty + 2 \right]^{1/n} n^{1/n} (1+\varrho(\psi(0),0)n)^{1/n} \|u_{(n)}\|_\infty^{1/n} \\ &= \max\{|u(a)|, |u(b)|\}. \end{aligned}$$

On the other hand, $\|u_{(n)}\|_\infty \le \|M_{u_{(n)}}\|_{\mathcal{B}\to\mathcal{B}}$ by [ADMV, Lemma 1], so

$$\begin{aligned} \|u_{(n)}\|_\infty^{1/n} \le \|M_{u_{(n)}}\|_{\mathcal{B}\to\mathcal{B}}^{1/n} &= \big\|(uC_\varphi)^n (C_{\varphi_n})^{-1}\big\|_{\mathcal{B}\to\mathcal{B}}^{1/n} \\ &\le \big\|(uC_\varphi)^n\big\|_{\mathcal{B}\to\mathcal{B}}^{1/n} \big\|(C_{\varphi^{-1}})^n\big\|_{\mathcal{B}\to\mathcal{B}}^{1/n}. \end{aligned}$$

Letting n tend to infinity and observing that $r_{\mathcal{B}}(C_{\varphi^{-1}}) = 1$ (by Lemma 4.1 since $\varphi^{-1} \in \mathrm{Aut}(\mathbb{D})$) we see that

$$\max\{|u(a)|, |u(b)|\} \le r_{\mathcal{B}}(uC_\varphi),$$

and thus

$$r_{\mathcal{B}}(uC_\varphi) = \max\{|u(a)|, |u(b)|\}.$$

Applying the above result to the inverse operator $(uC_\varphi)^{-1} = \frac{1}{u\circ\varphi^{-1}} C_{\varphi^{-1}}$, where φ^{-1} is a hyperbolic automorphism with attractive fixed point b and repulsive fixed point a, we get

$$\begin{aligned} r_{\mathcal{B}}((uC_\varphi)^{-1}) &= \max\{|u(\varphi^{-1}(a))|^{-1}, |u(\varphi^{-1}(b))|^{-1}\} \\ &= \frac{1}{\min\{|u(a)|, |u(b)|\}}. \end{aligned}$$

Now if $\lambda \in \sigma_{\mathcal{B}}(uC_\varphi)$, then we also have $\lambda^{-1} \in \sigma_{\mathcal{B}}((uC_\varphi)^{-1})$, so that

$$|\lambda| \le r_{\mathcal{B}}(uC_\varphi) = \max\big\{|u(a)|, |u(b)|\big\}$$

and

$$|\lambda|^{-1} \le r_{\mathcal{B}}((uC_\varphi)^{-1}) = \frac{1}{\min\big\{|u(a)|, |u(b)|\big\}},$$

which shows that the spectrum is contained in the indicated annulus. ■

In the case when $|u(a)| = |u(b)|$ we are able to improve the previous theorem and give a complete description of the spectrum. However, we have not been able to compute the spectrum when $|u(a)| \ne |u(b)|$. It seems one has to consider the cases $|u(a)| < |u(b)|$ and $|u(a)| > |u(b)|$ separately (see for example [HLNS, Theorem 4.9]).

THEOREM 4.6. *Suppose the weighted composition operator $uC_\varphi\colon \mathcal{B} \to \mathcal{B}$ is invertible on the Bloch space and assume that the automorphism φ is hyperbolic, with attractive fixed point $a \in \partial\mathbb{D}$ and repulsive fixed point $b \in \partial\mathbb{D}$. If $u \in A(\mathbb{D})$ and $|u(a)| = |u(b)|$, then*

$$\sigma_{\mathcal{B}}(uC_\varphi) = \big\{\lambda \in \mathbb{C} : |\lambda| = |u(a)|\big\}.$$

Proof. By Theorem 4.5 it suffices to prove that

$$\sigma_{\mathcal{B}}(uC_\varphi) \supseteq \{\lambda \in \mathbb{C} : |\lambda| = |u(a)|\},$$

so let λ be a complex number of modulus $|\lambda| = |u(a)| = r_{\mathcal{B}}(uC_\varphi)$. As in the proof of Theorem 4.3 it is then enough to show that $r_{\mathcal{B}}(\lambda - uC_\varphi) \geq 2r_{\mathcal{B}}(uC_\varphi)$. This can be done exactly as in that proof since the sequence $\{z_n\}_{n=0}^{\infty} \subset \mathbb{D}$ defined by $z_n = \varphi_n(0)$ is interpolating for H^∞ by [CM, Theorem 2.65]. Using the same notation and performing the same calculations as in the proof of Theorem 4.3, we obtain

$$\|(\lambda - uC_\varphi)^{2n}\|_{\mathcal{B}\to\mathcal{B}} \geq (cc_s)^{-1}\binom{2n}{n}|\lambda|^n|\omega_{(n)}(z_n)|(1-|\varphi_{2n}(0)|)^s.$$

Now since

$$\lim_{n\to\infty}|\omega_{(n)}(z_n)|^{1/(2n)} = \frac{|u(a)|^{1/2}}{\varphi'(a)^{s/2}},$$

as in [HLNS, Theorem 4.3], this gives

$$\begin{aligned} &r_{\mathcal{B}}(\lambda - uC_\varphi) \\ &\quad\geq \lim_{n\to\infty}(cc_s)^{-1/(2n)}\binom{2n}{n}^{1/(2n)}|\lambda|^{1/2}|\omega_{(n)}(z_n)|^{1/(2n)}(1-|\varphi_{2n}(0)|)^{s/(2n)} \\ &\quad= 2|\lambda|^{1/2}\frac{|u(a)|^{1/2}}{\varphi'(a)^{s/2}}\varphi'(a)^s = 2|u(a)|\varphi'(a)^{s/2} = 2r_{\mathcal{B}}(uC_\varphi)\varphi'(a)^{s/2}, \end{aligned}$$

where we have used the limit (2.1) with $0 < \varphi'(a) < 1$. The inequality

$$r_{\mathcal{B}}(\lambda - uC_\varphi) \geq 2r_{\mathcal{B}}(uC_\varphi)\varphi'(a)^{s/2}$$

holds for every $s > 0$ (see the last part of the proof of Theorem 4.3) so we may let $s \to 0$ to obtain $r_{\mathcal{B}}(\lambda - uC_\varphi) \geq 2r_{\mathcal{B}}(uC_\varphi)$. ∎

4.3. The elliptic case. We now turn to the spectrum of invertible weighted composition operators uC_φ on the Bloch space when φ is an elliptic automorphism of $\mathbb{D}$. The methods of proof here are standard, but some minor modifications are necessary and we thus present them.

LEMMA 4.7. *If $u \in \mathcal{M}(\mathcal{B})$ and $1/u \in H^\infty$, then $1/u \in \mathcal{M}(\mathcal{B})$.*

Proof. The function $f(z) := 1/u(z)$ is bounded by assumption, so we only need to show that it satisfies condition (3.3):

$$\begin{aligned} \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|\log\frac{e}{1-|z|^2} &= \sup_{z\in\mathbb{D}}(1-|z|^2)\left|\frac{u'(z)}{u(z)^2}\right|\log\frac{e}{1-|z|^2} \\ &\leq \left\|\frac{1}{u}\right\|_\infty^2 \sup_{z\in\mathbb{D}}(1-|z|^2)|u'(z)|\log\frac{e}{1-|z|^2}, \end{aligned}$$

which is finite since $u \in \mathcal{M}(\mathcal{B})$. ∎

THEOREM 4.8. *Suppose that the weighted composition operator $uC_\varphi : \mathcal{B} \to \mathcal{B}$ is bounded on the Bloch space and assume that $u \in A(\mathbb{D})$ and φ is an elliptic automorphism, with the unique fixed point $a \in \mathbb{D}$.*

(1) *If there is a positive integer j such that $\varphi_j(z) = z$ for all $z \in \mathbb{D}$, then letting m be the smallest such integer, we have*
$$\sigma_{\mathcal{B}}(uC_\varphi) = \overline{\{\lambda \in \mathbb{C} : \lambda^m = u_{(m)}(z) \text{ for some } z \in \mathbb{D}\}},$$

(2) *If $\varphi_n \neq \mathrm{Id}$ for every $n \in \mathbb{N}$, and if $uC_\varphi : \mathcal{B} \to \mathcal{B}$ is invertible, then*
$$\sigma_{\mathcal{B}}(uC_\varphi) = \{\lambda \in \mathbb{C} : |\lambda| = |u(a)|\}.$$

Proof. The proof of (1) is identical to the proof of [HLNS, Theorem 4.11], because as noted in [CGP, Section 3.1] we can use the result of Lemma 4.7 to prove that
$$\sigma_{\mathcal{B}}(uC_\varphi) \subseteq \overline{\{\lambda \in \mathbb{C} : \lambda^m = u_{(m)}(z) \text{ for some } z \in \mathbb{D}\}}.$$

The proof of (2) goes as in [HLNS, Theorem 4.14] and it relies on [HLNS, Lemma 4.13], which is also true for the Bloch space after a minor modification in the proof. Namely, one assumes that $\varphi(z) = \mu z$, where $\mu = e^{2\pi\theta i}$ and θ is irrational, $u \in A(\mathbb{D})$ and uC_φ is invertible, and proves that $r_{\mathcal{A}}(uC_\varphi) = |u(0)|$ by methods also valid for the Bloch space, except for the proof of
$$r_{\mathcal{A}}(uC_\varphi) \le \limsup_{n\to\infty} \|u_{(n)}\|_\infty^{1/n}. \tag{4.4}$$
Here $\mathcal{A}$ stands for a space satisfying conditions (C1)–(C3) of [HLNS, Section 2.2]. However, by the same calculations used to prove Theorem 4.3 we get
$$\begin{aligned}\|M_{u_{(n)}}\|_{\mathcal{B}\to\mathcal{B}}^{1/n} \\ \le \left[\|M_{u'}\|_{\mathcal{B}\to H_{v_1}^\infty}\left\|\frac{1}{u}\right\|_\infty + 2\right]^{1/n} n^{1/n}\big(1+\varrho(\psi(0),0)n\big)^{1/n}\|u_{(n)}\|_\infty^{1/n},\end{aligned}$$
and hence
$$r_{\mathcal{B}}(uC_\varphi) \le r_{\mathcal{B}}(C_\varphi)\limsup_{n\to\infty}\|M_{u_{(n)}}\|_{\mathcal{B}\to\mathcal{B}}^{1/n} \le \limsup_{n\to\infty}\|u_{(n)}\|_\infty^{1/n},$$
where we have used the fact that $r_{\mathcal{B}}(C_\varphi) = 1$ by Lemma 4.1. Thus (4.4) also holds for the Bloch space, and we can use the proof of [HLNS, Theorem 4.14] to obtain (2). ■

5. Spectra on the Dirichlet space. The spectra of invertible weighted composition operators induced by parabolic and elliptic automorphisms on the Dirichlet space were completely described in [CGP]. In the hyperbolic case and under essentially the same assumptions as in Theorem 5.2 below, it was also shown in [CGP, Theorem 3.3] that $r_{\mathcal{D}}(uC_\varphi) \le \max\{|u(a)|, |u(b)|\}/\mu$

and

$$\sigma_{\mathcal{D}}(uC_\varphi) \subseteq \big\{\lambda \in \mathbb{C} : \min\{|u(a)|, |u(b)|\}\mu \le |\lambda| \le \max\{|u(a)|, |u(b)|\}/\mu\big\},$$

where φ is conjugate to the automorphism

$$\psi(z) = \frac{(1+\mu)z + (1-\mu)}{(1-\mu)z + (1+\mu)}$$

for $0 < \mu < 1$. In Theorem 5.2 we improve this result to obtain an exact expression for the spectral radius.

Lemma 5.1 ([MV, Theorem 7]). *If φ is a univalent selfmap of $\mathbb{D}$, then $r_{\mathcal{D}}(C_\varphi) = 1$.*

Theorem 5.2. *Suppose the weighted composition operator $uC_\varphi \colon \mathcal{D} \to \mathcal{D}$ is invertible on the Dirichlet space and assume that the automorphism φ is hyperbolic, with attractive fixed point $a \in \partial\mathbb{D}$ and repulsive fixed point $b \in \partial\mathbb{D}$. If $u \in A(\mathbb{D})$, then $r_{\mathcal{D}}(uC_\varphi) = \max\{|u(a)|, |u(b)|\}$ and*

$$\sigma_{\mathcal{D}}(uC_\varphi) \subseteq \big\{\lambda \in \mathbb{C} : \min\{|u(a)|, |u(b)|\} \le |\lambda| \le \max\{|u(a)|, |u(b)|\}\big\}.$$

Proof. The weight u belongs to $\mathcal{M}(\mathcal{D})$ and is bounded away from zero on $\mathbb{D}$ by Theorem 3.4, which implies $u(a), u(b) \ne 0$. Since $r_{\mathcal{D}}(C_\varphi) = 1$ by Lemma 5.1, we begin by estimating the operator norm of $M_{u_{(n)}} : \mathcal{D} \to \mathcal{D}$:

$$\begin{aligned}
\|M_{u_{(n)}}\|_{\mathcal{D}\to\mathcal{D}} &= \sup_{\|f\|_{\mathcal{D}}\le 1} \|u_{(n)} \cdot f\|_{\mathcal{D}} \\
&= \sup_{\|f\|_{\mathcal{D}}\le 1} \Big[|u_{(n)}(0)f(0)|^2 + \int_{\mathbb{D}} |(u_{(n)} \cdot f)'(z)|^2 \, dA(z)\Big]^{1/2} \\
&\le \sup_{\|f\|_{\mathcal{D}}\le 1} \Big[|u_{(n)}(0)f(0)| + \Big(\int_{\mathbb{D}} |u'_{(n)}(z)f(z) + u_{(n)}(z)f'(z)|^2 dA(z)\Big)^{1/2}\Big] \\
&\le \|u_{(n)}\|_\infty + \sup_{\|f\|_{\mathcal{D}}\le 1} \Big(\int_{\mathbb{D}} |u'_{(n)}(z)f(z)|^2 \, dA(z)\Big)^{1/2} \\
&\qquad + \sup_{\|f\|_{\mathcal{D}}\le 1} \Big(\int_{\mathbb{D}} |u_{(n)}(z)f'(z)|^2 \, dA(z)\Big)^{1/2} \\
&\le \sup_{\|f\|_{\mathcal{D}}\le 1} \Big(\int_{\mathbb{D}} |u'_{(n)}(z)f(z)|^2 \, dA(z)\Big)^{1/2} + 2\|u_{(n)}\|_\infty,
\end{aligned}$$

where we have used the subadditivity of the square root function and the triangle inequality for the L^2-norm. Now since

$$u'_{(n)}(z) = \sum_{j=0}^{n-1} \frac{u_{(n)}(z)}{u \circ \varphi_j(z)} \cdot (u \circ \varphi_j)'(z),$$

we can continue the above estimation as follows:

$$\begin{aligned}\|M_{u_{(n)}}\|_{\mathcal{D}\to\mathcal{D}} &\le \sup_{\|f\|_{\mathcal{D}}\le 1} \sum_{j=0}^{n-1} \Big(\int_{\mathbb{D}} \Big| \frac{u_{(n)}(z)}{u\circ\varphi_j(z)} (u\circ\varphi_j)'(z) f(z)\Big|^2 \, dA(z)\Big)^{1/2} \\ &\quad + 2\|u_{(n)}\|_\infty \\ &\le \sum_{j=0}^{n-1} \Big\| \frac{u_{(n)}}{u\circ\varphi_j}\Big\|_\infty \sup_{\|f\|_{\mathcal{D}}\le 1} \Big(\int_{\mathbb{D}} |u'(\varphi_j(z))|^2 |\varphi_j'(z)|^2 |f(z)|^2 \, dA(z)\Big)^{1/2} \\ &\quad + 2\|u_{(n)}\|_\infty.\end{aligned}$$

After substituting $w = \varphi_j(z)$, the above integral takes the form

$$\begin{aligned}\Big(\int_{\mathbb{D}} |u'(w) f(\psi_j(w))|^2 \, dA(w)\Big)^{1/2} &= \|M_{u'}(f\circ\psi_j)\|_{A^2} \\ &\le \|M_{u'}\|_{\mathcal{D}\to A^2} \|C_{\psi_j}\|_{\mathcal{D}\to\mathcal{D}} \|f\|_{\mathcal{D}},\end{aligned}$$

where $\psi := \varphi^{-1}$ and the norm $\|M_{u'}\|_{\mathcal{D}\to A^2}$ is finite since $u \in \mathcal{M}(\mathcal{D})$ (see the section on multiplier spaces). In [MV, Theorem 7], Martín and Vukotić proved that

$$\|C_{\psi_j}\|_{\mathcal{D}\to\mathcal{D}} \le \sqrt{2}\big(1 + \varrho(\psi(0), 0) j\big)^{1/2}$$

for every $j \in \mathbb{N}$, which combined with the results above gives

$$\begin{aligned}\|M_{u_{(n)}}\|_{\mathcal{D}\to\mathcal{D}} &\le \sum_{j=0}^{n-1} \Big\| \frac{u_{(n)}}{u\circ\varphi_j}\Big\|_\infty \|M_{u'}\|_{\mathcal{D}\to A^2} \|C_{\psi_j}\|_{\mathcal{D}\to\mathcal{D}} + 2\|u_{(n)}\|_\infty \\ &\le \Big[\sqrt{2} \|M_{u'}\|_{\mathcal{D}\to A^2} \Big\| \frac{1}{u}\Big\|_\infty + 2\Big] n(1 + \varrho(\psi(0), 0)n)^{1/2} \|u_{(n)}\|_\infty.\end{aligned}$$

Applying this to the spectral radius and using Lemmas 4.4 and 5.1 gives

$$\begin{aligned} r_{\mathcal{D}}(uC_\varphi) &= \lim_{n\to\infty} \|(uC_\varphi)^n\|_{\mathcal{D}\to\mathcal{D}}^{1/n} \le \limsup_{n\to\infty} \|M_{u_{(n)}}\|_{\mathcal{D}\to\mathcal{D}}^{1/n} r_{\mathcal{D}}(C_\varphi) \\ &\le \lim_{n\to\infty} \Big[\sqrt{2} \|M_{u'}\|_{\mathcal{D}\to A^2} \Big\| \frac{1}{u}\Big\|_\infty + 2\Big]^{1/n} n^{1/n} (1 + \varrho(\psi(0), 0)n)^{1/(2n)} \|u_{(n)}\|_\infty^{1/n} \\ &= \max\{|u(a)|, |u(b)|\}.\end{aligned}$$

On the other hand, by [ADMV, Lemma 1], we have

$$\begin{aligned}\|u_{(n)}\|_\infty^{1/n} \le \|M_{u_{(n)}}\|_{\mathcal{D}\to\mathcal{D}}^{1/n} &= \big\|(uC_\varphi)^n (C_{\varphi_n})^{-1}\big\|_{\mathcal{D}\to\mathcal{D}}^{1/n} \\ &\le \big\|(uC_\varphi)^n\big\|_{\mathcal{D}\to\mathcal{D}}^{1/n} \big\|(C_{\varphi^{-1}})^n\big\|_{\mathcal{D}\to\mathcal{D}}^{1/n},\end{aligned}$$

so letting n tend to infinity and observing that $r_{\mathcal{D}}(C_{\varphi^{-1}}) = 1$ (by Lemma 5.1 since $\varphi^{-1} \in \mathrm{Aut}(\mathbb{D})$) we see that

$$\max\{|u(a)|, |u(b)|\} \le r_{\mathcal{D}}(uC_\varphi),$$

and thus

$$r_{\mathcal{D}}(uC_\varphi) = \max\{|u(a)|, |u(b)|\}.$$

The statement regarding the spectrum $\sigma_{\mathcal{D}}(uC_\varphi)$ can now be justified exactly as in Theorem 4.5. ∎

Remark 5.3. As already noted, we have not been able to compute the spectrum of invertible weighted composition operators with hyperbolic symbols φ, either for the Bloch or the Dirichlet space, except when $|u(a)| = |u(b)|$ in the Bloch case. However, the following conjecture seems plausible and we leave it as an open problem:

Suppose that the weighted composition operator $uC_\varphi\colon \mathcal{X} \to \mathcal{X}$ is invertible on $\mathcal{X}$, where $\mathcal{X}$ is either the Bloch space $\mathcal{B}$ or the Dirichlet space $\mathcal{D}$, and assume that the automorphism φ is hyperbolic, with attractive fixed point $a \in \partial\mathbb{D}$ and repulsive fixed point $b \in \partial\mathbb{D}$. If $u \in A(\mathbb{D})$, then

$$\sigma_{\mathcal{X}}(uC_\varphi) = \big\{\lambda \in \mathbb{C} : \min\{|u(a)|, |u(b)|\} \le |\lambda| \le \max\{|u(a)|, |u(b)|\}\big\}.$$

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