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Tameness in Fréchet spaces of analytic functions

by

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Dedicated to the memory of Tosun Terzioğlu

Abstract. A Fréchet space $\mathcal{X}$ with a sequence $\{\|\cdot\|_k\}_{k=1}^{\infty}$ of generating seminorms is called *tame* if there exists an increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ such that for every continuous linear operator T from $\mathcal{X}$ into itself, there exist N_0 and $C > 0$ such that

$$\|T(x)\|_n \le C\|x\|_{\sigma(n)} \quad \forall x \in \mathcal{X},\ n \ge N_0.$$

This property does not depend upon the choice of the fundamental system of seminorms for $\mathcal{X}$ and is a property of the Fréchet space $\mathcal{X}$. In this paper we investigate tameness in the Fréchet spaces $\mathcal{O}(M)$ of analytic functions on Stein manifolds M equipped with the compact-open topology. Actually we will look into tameness in the more general class of nuclear Fréchet spaces with properties $\underline{\mathrm{DN}}$ and Ω of Vogt and then specialize to analytic function spaces. We show that for a Stein manifold M, tameness of $\mathcal{O}(M)$ is equivalent to hyperconvexity of M.

1. Introduction. Tameness of Fréchet spaces is an important property frequently used in functional analysis since it brings a kind of control on the otherwise chaotic behaviour of continuous linear endomorphisms. This concept was used effectively in the structure theory of nuclear Fréchet spaces, especially in finding bases in complemented subspaces of certain infinite type power series spaces [9]. Frequently the Fréchet spaces that arise in practical applications (e.g. in non-linear analysis) enjoy (versions of) the tameness property [11, 28]. In fact, an inquiry into the tameness of analytic function spaces was undertaken, some time ago, in this context by D. Zarnadze [private communication]. In this paper we answer this question in a kind of negative way. We show that there are no tame analytic function spaces other than the natural ones (Theorem 4.5).

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The organization of the paper is as follows. After establishing the notation and terminology, in Section 2 we recall the definition of the linear topological invariants $\underline{\mathrm{DN}}$ and Ω and introduce a technical tool that we will use in the later sections: local imbeddings of power series spaces of finite type into Fréchet spaces. After some general results on local imbeddings we establish a link between the approximate diametral dimension of the space and the existence of effective local imbeddings from certain finite type power series spaces (Theorem 2.9).

Section 3 is devoted to proving Theorem 3.3, which characterizes tame nuclear Fréchet spaces having a finitely nuclear stable exponent sequence and enjoying properties $\underline{\mathrm{DN}}$ and Ω.

In Subsection 4.1 we show how local imbeddings can be used to construct Green's functions on complex manifolds. Subsection 4.2 is devoted to the diametral dimension of analytic function spaces and the proof of the main theorem of this paper, Theorem 4.5.

Notation and terminology. We will use the terminology of [15] and refer the reader to that book for all undefined concepts and the standard results of functional analysis that we will use. For the notions from complex potential theory used (especially in Section 3) we refer the reader to [12]. By a *grading* on a Fréchet space we mean any fixed sequence of Hilbertian seminorms defining the topology of the space. The pair consisting of a Fréchet space and a fixed grading is called a *graded Fréchet space.*

Power series sequence spaces play an important role in this paper. Recall that these are the Fréchet spaces

$$\Lambda_R(\alpha) \doteq \Big\{(\xi_n)_{n=0}^{\infty} : |(\xi_n)|_r \doteq \Big(\sum_n |\xi_n|^2 e^{2r\alpha_n}\Big)^{1/2} < \infty, \forall\ -\infty < r < R\Big\}$$

where R is either 1 or ∞, and $\alpha = (\alpha_n)_n$ is a sequence of real numbers with $\sup \ln(n)/\alpha_n < \infty$, called the *exponent sequence* of the space. The grading on these spaces will be the Hilbertian grading $\{|\cdot|_r\}_{r<R}$.

These spaces are referred to as *finite type power series spaces* if $R = 1$, and *infinite type power series spaces* if $R = \infty$. We will write $\Lambda_r[\alpha]$ for the Hilbert space $\{(\xi_n)_{n=0}^{\infty} : |(\xi_n)|_r \doteq (\sum_n |\xi_n|^2 e^{2r\alpha_n})^{1/2} < \infty\}$ with the norm $|\cdot|_r$, $0 < r < 1$.

For a pair of Fréchet spaces $(\mathcal{X}, \{\|\cdot\|_k\}_k)$ and $(\mathcal{Y}, \{|\cdot|_k\}_k)$, the set of all continuous linear operators will be denoted by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$. We will use the symbol

$$\|T\|_m^n \doteq \sup_{\|x\|_n \leq 1} |Tx|_m, \qquad n, m \in \mathbb{N},$$

for $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Note that $\|\cdot\|_m^n$ can take the value ∞.

For a graded Fréchet space $(\mathcal{X}, \{\|\cdot\|_k\}_k)$, the local Hilbert spaces corresponding to the norm $\|\cdot\|_k$ will be denoted by $\mathcal{X}_k$, $k = 1, 2, \ldots$. The closed unit ball in $\mathcal{X}$ corresponding to $\|\ \|_k$ will be denoted by U_k, $k = 1, 2, \ldots$.

Throughout the paper we will reserve the symbol ϵ_n for the sequence $(0, \ldots, 0, 1, 0, \ldots)$ where 1 is in the nth place.

2. Local imbeddings of power series spaces of finite type into Fréchet spaces. In this section we will examine local imbeddings of power series spaces of finite type into nuclear Fréchet spaces with properties $\underline{\mathrm{DN}}$ and Ω.

In Subsection 2.1 we recall the definition of Vogt's linear topological invariants $\underline{\mathrm{DN}}$ and Ω, and list some properties of Fréchet spaces with properties $\underline{\mathrm{DN}}$ and Ω that will be used in what follows. We emphasize again that all the Fréchet spaces that we will deal with will be assumed to be nuclear and satisfy properties $\underline{\mathrm{DN}}$ and Ω unless otherwise specified. In Subsection 2.2 we will investigate the existence of local imbeddings. In Subsection 2.3 we consider Fréchet spaces whose diametral approximate dimension is equal to that of a finite type power series space, and show that this property yields local imbeddings.

Definition 2.1. A continuous linear operator T from $\Lambda_1(\alpha)$ into a graded Fréchet space $(\mathcal{X}, \{\|\cdot\|_n\})$ is called an (r,k)-*local imbedding* if

$$\exists C > 0 : \quad \|T(x)\|_k \geq C|x|_r \quad \forall x \in \Lambda_1(\alpha).$$

We will say that a Fréchet space $\mathcal{X}$ admits an r-*local imbedding* of $\Lambda_1(\alpha)$ if there exists a continuous linear operator T from $\Lambda_1(\alpha)$ to $\mathcal{X}$ with the property that there is a continuous seminorm $\|\cdot\|$ on $\mathcal{X}$ and $C > 0$ such that

$$\|T(x)\| \geq C|x|_r \quad \forall x \in \Lambda_1(\alpha).$$

2.1. Linear topological invariants $\underline{\mathrm{DN}}$, Ω and associated exponent sequences

Definition 2.2. A nuclear Fréchet space $\mathcal{X}$ is said to have properties $\underline{\mathrm{DN}}$ and Ω ([21], [24]) if there exists a fundamental system $\{\|\cdot\|_k\}_k$ of Hilbertian norms generating the topology of $\mathcal{X}$ which satisfy

$$(\underline{\mathrm{DN}}) : \quad \forall k\ \exists 0 < \lambda < 1,\ C > 0 :$$
$$\|x\|_{k+1} \leq C\|x\|_k^{\lambda}\ \|x\|_{k+2}^{1-\lambda} \quad \forall x \in \mathcal{X};$$
$$(\Omega) : \quad \forall k\ \forall p\ \exists C > 0,\ j \in \mathbb{N} :$$
$$U_{k+1} \subset Cr^j U_p + \frac{1}{r} U_k \quad \forall r > 0.$$

For more information about these invariants and examples of Fréchet spaces possessing these properties we refer the reader to [15].

Let $V \subseteq U$ be subsets of a Fréchet space $\mathcal{X}$. We denote

$$\mathcal{E}_n(U,V) \doteq -\ln d_n(V,U), \qquad n = 0,1,\dots,$$

where

$$d_n(V,U) \doteq \inf_{L_n} \inf\{\lambda > 0 : V \subseteq \lambda U + L_n\}$$

(the outer infimum is taken over all n-dimensional subspaces L_n of $\mathcal{X}$) is the nth Kolmogorov diameter of V in U.

In the case of a graded Fréchet space $(\mathcal{X}, \{\|\cdot\|_k\})$, we simplify this notation by setting

$$\mathcal{E}_n(p,k) \doteq \mathcal{E}_n(U_p, U_k), \qquad \forall p < k.$$

Let us now fix a nuclear Fréchet space $\mathcal{X}$ with properties $\underline{\mathrm{DN}}$ and Ω [15, p. 367], and choose a strictly increasing grading $(\{\|\cdot\|\}_k)_k$ starting with the "dominating norm" of condition $\underline{\mathrm{DN}}$ and such that the linking maps are nuclear. By passing to a subsequence of $\{\|\cdot\|_k\}_k$ we can choose, by induction, a new grading that satisfies the requirements of Definition 2.2. Observe that the linking maps of this grading are also nuclear. In [5] (cf. [4]) we have shown that all the sequences $\{\mathcal{E}_n(k,k+1)\}_n$, $k = 0,1,2,\dots$, are equivalent, and called this equivalence class the *exponent sequence associated* to $\mathcal{X}$. In this paper, unless otherwise stated, we will use a concrete representation,

$$\mathcal{E}_n \doteq \mathcal{E}_n(0,1), \qquad n = 0,1,\dots,$$

of the associated exponent sequence of $\mathcal{X}$.

We collect [20] some properties of these sequences that we will use.

Proposition 2.3. *Let $(\mathcal{X}, \{\|\cdot\|_k\})$ be as above. The Ω-type condition*

$$\exists 0 < \lambda < 1, D > 0 : \quad U_q \subset r^{(1-\lambda)/\lambda} U_p + \frac{D}{r} U_k, \qquad \forall r > C \qquad (p < q < k)$$

implies:

(1) $\exists C > 0 : \mathcal{E}_n(q,k) \le \lambda \mathcal{E}_n(p;k) + C$, $\forall n$,
(2) $\exists C > 0 : \mathcal{E}_n(p,k) \le \frac{1}{1-\lambda} \mathcal{E}_n(p,q) + C$, $\forall n$,
(3) $\exists C > 0 : \mathcal{E}_n(q,k) \le \frac{\lambda}{1-\lambda} \mathcal{E}_n(p,q) + C$, $\forall n$,

The $\underline{\mathrm{DN}}$-type condition

$$\exists 0 < \lambda < 1,\ D > 0 : \quad \|\cdot\|_q \le D \|\cdot\|_k^{1-\lambda} \|\cdot\|_p^{\lambda} \qquad (p < q < k)$$

implies:

(1′) $\exists C > 0 : \mathcal{E}_n(p,k) \le \frac{1}{\lambda} \mathcal{E}_n(q;k) + C$, $n = 0,1,\dots$,
(2′) $\exists C > 0 : \mathcal{E}_n(p,q) \le (1-\lambda) \mathcal{E}_n(p,k) + C$, $n = 0,1,\dots$,
(3′) $\exists C > 0 : \mathcal{E}_n(p,q) \le \frac{\lambda}{1-\lambda} \mathcal{E}_n(q,k) + C$, $n = 0,1,\dots$.

In the presence of both Ω and $\underline{\mathrm{DN}}$ conditions we have

(4) $\exists C > 0,\ 0 < \lambda_1, \lambda_2 < 1 : \mathcal{E}_n \le \frac{\lambda_1\lambda_2}{(1-\lambda_1)(1-\lambda_2)}\mathcal{E}_n(p,q) + C,\ n = 0, 1, \ldots,$

(4′) $\exists C > 0,\ 0 < \lambda_1, \lambda_2 < 1 : \mathcal{E}_n(p,q) \le \frac{\lambda_1\lambda_2}{(1-\lambda_1)(1-\lambda_2)}\mathcal{E}_n + C,\ n = 0, 1, \ldots.$

Proof. See [20] (cf. [5]). ∎

2.2. Local imbeddings of power series spaces of finite type into nuclear Fréchet spaces with properties $\underline{\mathrm{DN}}$ and Ω. We now turn to the question of existence of local imbeddings into the graded Fréchet space $(\mathcal{X}, \{\|\cdot\|_k\})$ fixed in Subsection 2.1 above. But we first present some general considerations concerning local imbeddings.

PROPOSITION 2.4. *Let $\Lambda_1(\alpha)$ be a nuclear power series space of finite type. The following assertions are equivalent:*

(i) *There exists an (r,k)-local imbedding from $\Lambda_1(\alpha)$ into $\mathcal{X}$.*

(ii) *There exists a sequence $\{g_n\}_{n=0}^{\infty}$ in $\mathcal{X}$ that is orthonormal in $\mathcal{X}_k$ and satisfies*

$$\overline{\lim_n} \frac{\ln \|g_n\|_s}{\alpha_n} < 1 - r \qquad \forall s.$$

(iii) *There exists an isometry from the local Hilbert space $\Lambda_r[\alpha]$ into $\mathcal{X}_k$ that induces a continuous linear operator from $\Lambda_1(\alpha)$ into $\mathcal{X}$.*

(iv) *There exists a closed, Hilbertian bounded disc B in $\mathcal{X}$ which satisfies*

$$\exists C > 0: \quad d_n(B, U_k) \ge Ce^{-(1-r)\alpha_n}.$$

Proof. (i)⇒(ii): Fix an (r,k)-local imbedding T with $\|T(x)\|_k \ge C_1|x|_r$ for all $x \in \Lambda_1(\alpha)$. Consider

$$f_n \doteq \frac{\epsilon_n}{e^{r\alpha_n}}, \qquad n = 0, 1, \ldots,$$

and apply the Gram–Schmidt orthogonalization procedure to the linearly independent sequence $\{T(f_n)\}_n$ in $\mathcal{X}_k$ to get a sequence

$$g_n \doteq \sum_{i=0}^{n} c_i^n T(f_i), \qquad n = 1, 2, \ldots,$$

in $\mathcal{X}$ that is orthonormal in $\mathcal{X}_k$. We estimate

$$1 = \|g_n\|_k = \Big\|T\Big(\sum_{i=0}^{n} c_i^n f_i\Big)\Big\|_k \ge C_1\Big|\sum_{i=0}^{n} c_i^n \frac{\epsilon_i}{e^{r\alpha_i}}\Big|_r = C_1\Big(\sum_{i=0}^{n} |c_i^n|^2\Big)^{1/2}.$$

From continuity of T, for every s we get $r < \sigma(s) < 1$ and $C > 0$ such that

$$\|g_n\|_s = \Big\|T\Big(\sum_{i=0}^{n} c_i^n f_i\Big)\Big\|_s \le C\Big|\sum_{i=0}^{n} c_i^n f_i\Big|_{\sigma(s)}$$
$$\le Ce^{(\sigma(s)-r)\alpha_n}\Big(\sum_{i=0}^{n} |c_i|^2\Big)^{1/2} \le \frac{C}{C_1}e^{(\sigma(s)-r)\alpha_n}.$$

Hence
$$\overline{\lim_n} \frac{\ln \|g_n\|_s}{\alpha_n} \le \sigma(s) - r < 1 - r.$$

(ii)⇒(iii): Choose $\{g_n\}_n$ as in (ii) and set
$$T\Big(\sum_{i=0}^{\infty} c_i \epsilon_i\Big) \doteq \sum_{i=0}^{\infty} e^{r\alpha_i} c_i g_i.$$
This operator is plainly a continuous linear operator from $\Lambda_1(\alpha)$ into $\mathcal{X}$ that extends to an isometry from $\Lambda_r[\alpha]$ into $\mathcal{X}_k$.

(iii)⇒(iv): Choose an operator T as in (iii) and set
$$g_n \doteq T\bigg(\frac{\epsilon_n}{e^{r\alpha_n}}\bigg), \quad n = 0, 1, \ldots.$$
Define
$$B \doteq \Big\{x \in \mathcal{X} : \sum_{n=0}^{\infty} |\langle x, g_n\rangle_k|^2 e^{2(1-r)\alpha_n} \le 1\Big\},$$
where $\langle\,,\rangle_k$ is the inner product corresponding to $\|\cdot\|_k$.

A direct computation shows that B is the image under T of the compact set $B_0 = \{(\xi_n) : \sum |\xi_n|^2 e^{2\alpha_n} \le 1\}$ in $\Lambda_1(\alpha)$. So B is a compact disc that is Hilbertian, and hence (see [8, Lemma 6.2.2])
$$d_n(B, U_k) \ge d_n(B \cap T(\mathcal{X}), U_k \cap T(\mathcal{X})) = e^{(r-1)\alpha_n}.$$

(iv)⇒(i): Let H_B denote the Hilbert space generated by B in $\mathcal{X}$. Since the inclusion $H_B \hookrightarrow \mathcal{X}_k$ is a compact operator, we can choose a sequence $\{f_n\}_{n=0}^{\infty}$ of orthogonal vectors in H_B that are orthonormal in the Hilbert space $\mathcal{X}_k$ and have $\|f_n\|_B = (d_n(B, U_k))^{-1}$, $n = 0, 1, \ldots$. Set
$$T((\xi_n)_n) \doteq \sum_{n=0}^{\infty} \xi_n e^{r\alpha_n} f_n \quad \forall (\xi_n) \in \Lambda_1(\alpha).$$
We fix an $s > k$ and, in view of condition $\underline{\mathrm{DN}}$, choose $0 < \lambda < 1$ and $C > 0$ such that $\|x\|_s \le C\|x\|_k^{1-\lambda}\|x\|_B^{\lambda}$ for all $x \in H_B$.

Choose λ^+ so that $0 < r + \lambda(1-r) < \lambda^+ < 1$. There exist positive constants C_1, C_2 such that
$$\begin{aligned}\|T((\xi_n))\|_s &\le \sum_n |\xi_n| e^{r\alpha_n}\|f_n\|_s \le C_1 \sum_n |\xi_n| e^{r\alpha_n} e^{\lambda(1-r)\alpha_n}\\ &\le C_2\Big(\sum_n |\xi_n|^2 e^{2\lambda^+\alpha_n}\Big)^{1/2} = C_2 |(\xi_n)|_{\lambda^+},\end{aligned}$$
so T defines a continuous linear operator from $\Lambda_1(\alpha)$ into $\mathcal{X}$. Moreover
$$\|T((\xi_n))\|_k = \Big(\sum_n |\xi_n|^2 e^{2r\alpha_n}\Big)^{1/2} = |(\xi_n)|_r.$$
Hence T is an (r,k)-local imbedding. ■

REMARK. In contrast to the infinite case [5], the existence of an (r,k)-local imbedding from a finite type power series space into a graded Fréchet space need not imply the existence of an (r^+,k)-local imbedding for some $r^+ > r$. In fact, it is not difficult to see that for a nuclear finite type power series space $\mathcal{X} = \Lambda_1(\alpha)$, there exists an (r_1, r_2)-local imbedding from $\Lambda_1(\alpha)$ into $\mathcal{X}$ if and only if $r_1 \le r_2$. Indeed, if there is a bounded set B in $\mathcal{X}$ such that $d_n(B, U_{r_2}) \ge Ce^{(r_1-1)\alpha_n}$ for some $C > 0$, then for any $r_2 < s < 1$, there exists a $C_1 > 0$ such that $B \subseteq C_1 U_s$, which implies that

$$d_n(B, U_r) \le C_1 d_n(U_s, U_r) = Ce^{(r_2-s)\alpha_n}$$

for every n. Hence $r_1 - 1 \le r_2 - 1$, that is, $r_1 \le r_2$.

The condition in (iv) of the above proposition can also be expressed by using the unit balls of the grading:

PROPOSITION 2.5. *There exists an (r,p)-local imbedding from $\Lambda_1(\alpha)$ into $\mathcal{X}$ if and only if*

$$\sup_{k>p} \overline{\lim_n} \frac{\mathcal{E}_n(p,k)}{\alpha_n} \le 1 - r.$$

Proof. ($\Leftarrow$): Let p and r satisfy the stated inequality. For a given $k > p$, in view of Proposition 2.3(2) we have

$$\exists 0 < \rho < 1 : \quad \overline{\lim_n} \frac{\mathcal{E}_n(p,k)}{\alpha_n} \le \rho \overline{\lim_n} \frac{\mathcal{E}_n(p,k+1)}{\alpha_n} < 1 - r.$$

So for each $k > p$ we can find an $N(k) \in \mathbb{N}$, strictly increasing with respect to k, such that

$$-\ln d_n(U_k, U_p) < (1-r)\alpha_n \quad \text{ for } n > N(k).$$

Let $\delta_1 = \frac{1}{2}e^{-(1-r)\alpha_{N(p+1)}}$. Since U_{p+1} is precompact in $\mathcal{X}_p$, there exists a finite set $Z_1^1 \subseteq U_{p+1}$ such that

$$d_n(U_{p+1}, U_p) \le d_n(Z_1^1, U_p) + \delta_1 \quad \forall n.$$

Hence

$$d_{N(p+1)}(Z_1^1, U_p) \ge e^{-(1-r)\alpha_{N(p+1)}} - \delta_1 = \tfrac{1}{2}e^{-(1-r)\alpha_{N(p+1)}}.$$

For each n with $N(p+1) \le n < N(p+2)$, we obtain as above a finite set $Z_n^1 \subseteq U_{p+1}$ satisfying

$$d_n(Z_n^1, U_p) \ge e^{-(1-r)\alpha_n}/2,$$

and define $Z^1 \doteq \bigcup_{N(p+1)\le n<N(p+2)} Z_n^1$.

Continuing in this fashion we get finite sets Z^s, $s = 1, 2, \ldots$, with $Z^s \subseteq U_{p+s}$ and $d_n(Z^s, U_p) \ge e^{-(1-r)\alpha_n}/2$ for $N(p+s) \le n < N(p+s+1)$. Note that $\bigcup_{s=1}^\infty Z^s$ is a bounded set since $Z^{\bar{s}} \subseteq U_k$ for every $k = p + s$ and $\bar{s} > s$. We can find a closed Hilbertian disc B in $\mathcal{X}$, containing $\bigcup_{s=1}^\infty Z^s$ (see for example

[23, Lemma 1.2]). Then for $n > N(p+1)$ (say $N(p+s) \le n < N(p+s+1)$) we have

$$d_n(B, U_p) \ge d_n(Z^s, U_p) \ge e^{-(1-r)\alpha_n}/2.$$

So in view of Proposition 2.4(iv) there is an (r,p)-local imbedding from $\Lambda_1(\alpha)$ into $\mathcal{X}$.

($\Rightarrow$): In view of Proposition 2.4(iv) there exists a bounded set B and $C > 0$ with $d_n(B, U_p) \ge Ce^{-(1-r)\alpha_n}$ for $n = 1, 2, \ldots$. Hence for each k there are $C_i > 0$, $i = 1, 2$, such that

$$\mathcal{E}_n(p,k) \le -\ln d_n(B, U_p) + C_1 \le (1-r)\alpha_n + C_2.$$

So

$$\overline{\lim_n} \frac{\mathcal{E}_n(p,k)}{\alpha_n} \le 1 - r \quad \forall k = p+1, \ldots . \ \blacksquare$$

Remark. Although the structural assumption $\underline{\mathrm{DN}}$ on $\mathcal{X}$ is put into use in the proposition above, the implication ($\Rightarrow$) can be proved without any structural assumptions on $\mathcal{X}$. One can use the argument given in [4, Proposition 2.3] to obtain a proof by just using the definition of local imbeddings.

2.3. Approximate diametral dimension and local imbeddings. In this subsection we investigate the relationship between approximate diametral dimension of a nuclear Fréchet space $\mathcal{X}$ with properties $\underline{\mathrm{DN}}$ and Ω and the existence of local imbeddings from $\Lambda_1(\mathcal{E}_n)$, the finite type power series space corresponding to the associated exponent sequence of $\mathcal{X}$. Since we wish to use the information obtained in the previous subsections, we are forced to make an *additional assumption* on $\{\mathcal{E}_n\}_n$, namely that $\lim_n \ln(n)/\mathcal{E}_n = 0$. This guarantees the nuclearity of $\Lambda_1(\mathcal{E}_n)$ (see [8]), which is needed in what follows and also for the validity of the results of previous subsections.

Definition 2.6 ([7]). *The approximate diametral dimension* of a Fréchet space Y is defined as

$$\delta(Y) \doteq \bigcup_{U \text{ a zero neighborhood of } Y} \ \bigcup_{B \text{ a bounded subset of } Y} \left\{ (t_n) : \lim_n \frac{t_n}{d_n(B,U)} = 0 \right\}$$
$$= \Big\{ (t_n)_n : \exists \text{ a neighbourhood } U \text{ of zero and a bounded subset } B \text{ of } Y \text{ such that } \lim_n t_n/d_n(B,U) = 0 \Big\}.$$

Approximate diametral dimension also admits a representation solely in terms of a zero neighbourhood basis $\{U_n\}_n$ of the Fréchet space Y:

$$\delta(Y) = \{(t_n) : \exists s \ \forall k \ge s : t_n/d_n(U_k, U_s) \to 0\}$$

(see [7]).

Now let us fix, for the rest of this subsection, a graded nuclear Fréchet space $(\mathcal{X}, \{\|\cdot\|_k\}_k)$ with properties $\underline{\mathrm{DN}}$ and Ω, whose associated exponent sequence $(\mathcal{E}_n)$ satisfies $\ln(n)/\mathcal{E}_n \to_n 0$ and the grading $\{\|\cdot\|_k\}_k$ comes from Definition 2.2. Given $p+1 < q$, in view of Proposition 2.3(1), we have $d_n(U_q, U_p)^\lambda \le C d_n(U_q, U_{p+1})$ for some $C > 0$ and $0 < \lambda < 1$. So if a sequence $(t_n)_n$ satisfies for some C,

$$\frac{|t_n|}{d_n(U_q, U_p)} \le C \quad \forall n,$$

then

$$\frac{|t_n|}{d_n(U_q, U_{p+1})} \to 0.$$

Hence under our assumptions, we have

$$\delta(\mathcal{X}) = \Big\{(t_n)_n : \exists p\ \forall q > p : \sup_n |t_n| e^{\mathcal{E}_n(p,q)} < \infty\Big\}.$$

It is not difficult to show, by direct computation, that ([7])

$$\delta(\Lambda_1(\alpha)) = \Big\{(t_n)_n : \overline{\lim_n} \frac{\ln |t_n|}{\alpha_n} < 0\Big\}.$$

Proposition 2.7.

$$\delta(\mathcal{X}) \supseteq \delta(\Lambda_1(\alpha)) \;\Leftrightarrow\; \inf_p \sup_{q>p} \overline{\lim_n} \frac{\mathcal{E}_n(p,q)}{\alpha_n} = 0.$$

Proof. ($\Rightarrow$): For a fixed pair $p < q$ let

$$B(p,q) \doteq \Big\{(t_n) : \sup_n |t_n| e^{\mathcal{E}_n(p,q)} < \infty\Big\}.$$

This linear space becomes a Banach space under the norm

$$|(t_n)|_{pq} \doteq \sup_n |t_n| e^{\mathcal{E}_n(p,q)}.$$

Since $B(p,q^+) \subseteq B(p,q)$ for $p < q < q^+$, the space $B(p) \doteq \bigcap_{q>p} B(p,q)$ with the fundamental generating norms $\{|\cdot|_{pq}\}_{q>p}$ becomes a Fréchet space. Moreover by means of continuous inclusions $B(\bar{p}) \hookrightarrow B(p)$, $\bar{p} < p$, we can endow $\delta(\mathcal{X}) = \bigcup_n B(p)$ with the inductive limit topology and thus view $\delta(\mathcal{X})$ as a locally convex space which is an inductive limit of Fréchet spaces. Let us fix $0 < r < 1$ and define

$$S_r \doteq \Big\{(t_n) : \|(t_n)\| \doteq \sup_n \frac{|t_n|}{r^{\alpha_n}} < \infty\Big\}.$$

Clearly S_r is a Banach space with respect to the norm $\|\cdot\|$, and $S_r \subset \delta(\Lambda_1(\alpha))$. By our assumption, $S_r \subset \delta(\mathcal{X})$ and since projections onto coordinates are continuous in the inductive limit topology on $\delta(\mathcal{X})$, the inclusion $S_r \hookrightarrow \delta(\mathcal{X})$ is a sequentially closed linear operator. In view of Grothendieck's factorization theorem [13, p. 225], there is a $p(r)$ such that for every $q > p(r)$,

$$\exists C > 0 : \quad \sup_n |t_n| e^{\mathcal{E}_n(p(r),q)} \le C \sup_n \frac{|t_n|}{r^{\alpha_n}}$$

for any $(t_n) \in S_r$. In particular for every $q > p(r)$ there exists a $C > 0$ such that $\mathcal{E}_n(p(r), q) \le \ln C - \alpha_n \ln r$, which in turn implies that

$$\inf_p \sup_{q>p} \overline{\lim_n} \frac{\mathcal{E}_n(p,q)}{\alpha_n} \le -\ln r.$$

Since this holds for every $0 < r < 1$, we have

$$\inf_p \sup_{q>p} \overline{\lim_n} \frac{\mathcal{E}_n(p,q)}{\alpha_n} = 0.$$

($\Leftarrow$): Fix an $r < 1$ and choose a $(t_n)_n$ which satisfies

$$|t_n| \le C r^{\alpha_n} \quad \text{for some } C > 0 \text{ and all } n.$$

In view of our assumption, we choose a p so that $\sup_{q\ge p} \overline{\lim}_n \mathcal{E}_n(p,q)/\alpha_n < -\ln r$. It follows that there exists an n_0 such that for $n \ge n_0$,

$$e^{\mathcal{E}_n(p,q)} < \frac{1}{r^{\alpha_n}} \quad \text{for every } q \ge p.$$

Hence, for every $q > p$ there exists $C_q > 0$ such that $e^{\mathcal{E}_n(p,q)} \le C_q/r^{\alpha_n}$, which in turn implies that for every $q > p$, $\sup_n |t_n| e^{\mathcal{E}_n(p,q)} < \infty$. It follows that

$$\delta(\Lambda_1(\alpha)) \subseteq \bigcup_p \bigcap_{q>p} B(p,q) = \delta(\mathcal{X}). \ \blacksquare$$

If we focus our attention on the associated exponent sequence $(\mathcal{E}_n)_n$, we have:

Corollary 2.8.

$$\delta(\mathcal{X}) = \delta(\Lambda_1(\mathcal{E})) \ \Leftrightarrow\ \inf_p \sup_{q>p} \overline{\lim_n} \frac{\mathcal{E}_n(p,q)}{\mathcal{E}_n} = 0.$$

Proof. Fix a sequence $(t_n) \in \delta(\mathcal{X})$, and choose a p such that

$$\forall q > p: \quad \sup_n |t_n| e^{\mathcal{E}_n(p,q)} < \infty.$$

In view of Proposition 2.3(4), there exist constants $C_1, C_2 > 0$ such that

$$\mathcal{E}_n \le C_1 \mathcal{E}_n(p, p+1) + C_2 \quad \forall n.$$

It follows that

$$\sup_n |t_n| e^{\mathcal{E}_n/C_1} < \infty.$$

Hence there is a $D > 0$ such that

$$\ln |t_n| + \frac{\mathcal{E}_n}{C_1} < D,$$

so

$$\overline{\lim} \frac{\ln |t_n|}{\mathcal{E}_n} \le -\frac{1}{C_1} < 0.$$

Therefore we always have $\delta(\mathcal{X}) \subseteq \delta(\Lambda_1(\mathcal{E}))$. Now the corollary follows from Proposition 2.7. $\blacksquare$

We conclude this section with a summary theorem:

THEOREM 2.9. *Let $\mathcal{X}$ be a nuclear Fréchet space with properties* $\underline{\mathrm{DN}}$ *and* Ω. *Assume that the associated exponent sequence* $(\mathcal{E}_n)_n$ *of* $\mathcal{X}$ *is a nuclear exponent sequence of finite type (i.e.* $\Lambda_1(\mathcal{E})$ *is nuclear). Then the following assertions are equivalent:*

(1) $\delta(\mathcal{X}) = \delta(\Lambda_1(\mathcal{E}))$
(2) $\inf_{1\le p<\infty} \sup_{q>p} \overline{\lim}_n \mathcal{E}_n(q,p)/\mathcal{E}_n = 0$.
(3) *For every* $0 < r < 1$ *there exists an* r*-local imbedding from* $\Lambda_1(\mathcal{E})$ *into* $\mathcal{X}$.

3. Tame spaces $\mathcal{X}$ with properties $\underline{\mathrm{DN}}$, Ω and $\delta(\mathcal{X}) = \delta(\Lambda_1(\mathcal{E}))$. To every continuous linear operator T between two graded F-spaces $(\mathcal{X}, \{\|\cdot\|_k\}_k)$ and $(\mathcal{Y}, \{|\cdot|_k\}_k)$ one can associate a sequence of natural numbers, $\{\sigma_T(n)\}_n$, called the *characteristic of continuity* of T, via

$$\sigma_T(n) \doteq \inf\{s : \exists C > 0 : |T(x)|_n \le C\|x\|_s,\ \forall x \in \mathcal{X}\}.$$

In general the characteristics of continuity of operators between graded Fréchet spaces could be very disorderly. However, for certain pairs of Fréchet spaces control over the growth of characteristics of continuity can be obtained. For example, in the space of analytic functions on the unit disc, $\mathcal{O}(\Delta)$, with the grading

$$\|f\|_k \doteq \sum_{n=0}^{\infty} \left|\frac{f^n(0)}{n!}\right|^2 e^{-n/k}, \qquad k = 1, 2, \ldots,$$

it is not difficult to see that

$$\forall T \in \mathcal{L}(\mathcal{O}(\Delta), \mathcal{O}(\Delta))\ \exists a \in \mathbb{N} : \quad \sigma_T(n) \le an,\ \forall n = 1, 2, \ldots.$$

Following [22], [9] (cf. [16]) we specify this property as:

DEFINITION 3.1. A pair of Fréchet spaces $\mathcal{X}$ and $\mathcal{Y}$ will be called a *tame pair* (written $(\mathcal{X}, \mathcal{Y}) \in \mathcal{T}$) if for any given pair of sequences of generating seminorms $\{\|\cdot\|_k\}_k$ of $\mathcal{X}$ and $\{|\cdot|_k\}_k$ of $\mathcal{Y}$ there exists an increasing function $\psi : \mathbb{N} \to \mathbb{N}$ such that

$$\forall T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})\ \exists N \in \mathbb{N} : \quad \sigma_T(n) \le \psi(n),\ \forall n \ge N.$$

In case $\mathcal{X} = \mathcal{Y}$ we say that the Fréchet space $\mathcal{X}$ is *tame*.

REMARK 3.2. (1) The definition does not depend on the choice of the seminorms in $\mathcal{X}$ and $\mathcal{Y}$.

(2) Plainly the definition is equivalent to the existence, for a given pair of sequences of generating seminorms $\{\|\cdot\|_k\}_k$ of $\mathcal{X}$ and $\{|\cdot|_k\}_k$ of $\mathcal{Y}$, of a sequence $\{S_K\}_K$ of increasing functions $S_K : \mathbb{N} \to \mathbb{N}$ such that for every $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, there exists a K with $\sigma_T(n) \le S_K(n)$ *for every* $n = 1, 2, \ldots$.

(3) The space $\mathcal{O}(\Delta)$ is tame. More generally, every power series space of finite type is tame, as was observed by various authors (see [9, 2.1] for details). A proof of this appears in [16, 2.1].

(4) If $(\mathcal{X}, \mathcal{Y}) \in \mathcal{T}$, then the space of all continuous linear operators from $(\mathcal{X}, \{\|\cdot\|\}_k)$ into $(\mathcal{Y}, \{|\cdot|\}_k)$ admits a representation of the form

$$\mathcal{L}(\mathcal{X}, \mathcal{Y}) = \bigcup_{K=1}^{\infty} \bigcap_{n=1}^{\infty} \{T \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) : \exists C > 0 : \|T(x)\|_n \le C|x|_{S_K(n)}, \forall x \in \mathcal{X}\}.$$

Using this representation one can endow $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ with a linear topology making it an LF-space (i.e. inductive limit of Fréchet spaces) by considering the seminorms $\|T\|_K^n \doteq \sup_{|x|_{S_K(n)} \le 1} \|T(x)\|_n$ (see [9, 2.1]) on the space

$$\{T \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) : \exists C > 0 : \|T(x)\|_n \le C|x|_{S_K(n)}, \forall x \in \mathcal{X}\}, \qquad K, n \in \mathbb{N}.$$

This structure allows one to use the results on well studied LF-spaces in the study of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ for $(\mathcal{X}, \mathcal{Y}) \in \mathcal{T}$. These ideas are used in the study of nuclear Fréchet spaces $\mathcal{X}$ which form a tame pair with nuclear stable power series spaces of finite or infinite type in [17] where a complete characterization of such spaces in terms of Vogt's linear topological invariants is obtained.

In this section we will once again consider nuclear Fréchet spaces $\mathcal{X}$ with properties $\underline{\mathrm{DN}}$ and Ω. We will assume that the associated exponent sequence $\mathcal{E} = (\mathcal{E}_n)_n$, in addition to our usual assumption of being finitely nuclear, is also *stable*, i.e. $\sup_n \mathcal{E}_{2n}/\mathcal{E}_n < \infty$.

THEOREM 3.3. *Let $\mathcal{X}$ be a nuclear Fréchet space with stable finitely nuclear associated exponent sequence $\mathcal{E} = (\mathcal{E}_n)$ and which has properties $\underline{\mathrm{DN}}$ and Ω. Then $\mathcal{X}$ is isomorphic to a power series space of finite type if and only if $\mathcal{X}$ is tame and $\delta(\mathcal{X}) = \delta(\Lambda_1(\mathcal{E}))$.*

Proof. ($\Rightarrow$): If $\mathcal{X}$ is isomorphic to a power series space of finite type then $\mathcal{X}$ must be isomorphic to $\Lambda_1(\mathcal{E})$ by [5, Proposition 2.3]. So $\delta(\mathcal{X}) = \delta(\Lambda_1(\mathcal{E}))$. Moreover $\mathcal{X}$, being a finite type power series space, is tame in view of Remark 3.2(4) above.

($\Leftarrow$): Let $\mathcal{X}$ be a tame nuclear Fréchet space with stable finitely nuclear associated exponent sequence $\mathcal{E} = (\mathcal{E}_n)$, and which has properties $\underline{\mathrm{DN}}$ and Ω. In view of [22], we can imbed $\mathcal{X}$ into $\Lambda_1(\mathcal{E})$ as a closed subspace. We will consider the grading on $\mathcal{X}$ induced by $(\Lambda_1(\mathcal{E}), \{|\cdot|_r\}_r)$. Throughout this proof, $\epsilon_n = (0, \dots, 0, 1, 0, \dots)$ with 1 in the nth place, $n = 1, 2, \dots$. In view of Theorem 2.9 there exists an r-local imbedding from $\Lambda_1(\mathcal{E})$ into $\mathcal{X}$ for every $0 \le r < 1$. Fix an $r_0 < 1$ and a corresponding (r_0, r_{k_0})-local imbedding T, with $r_{k_0} < 1$, from $\Lambda_1(\mathcal{E})$ into $\mathcal{X}$. Say $|Tx|_{r_{k_0}} \ge C_0|x|_{r_0}$ for all $x \in \Lambda_1(\mathcal{E})$. Let $e_n \doteq \epsilon_n/e^{r_0\mathcal{E}_n}$, $n = 1, 2, \dots$, be the canonical orthonormal basis of $\Lambda_{r_0}[\mathcal{E}]$, and set $g_n \doteq T(e_n)$, $n = 1, 2, \dots$.

We note that $\{g_n\}_{n=1}^{\infty}$ is a finitely linearly independent sequence since T is a local imbedding. We choose a sequence $\{f_n\}$ in $\mathcal{X} \subseteq \Lambda_1(\mathcal{E})$ satisfying;

(1) $f_n \in \operatorname{span}\{g_1, \dots, g_{2n}\}$ for $n = 1, 2, \dots,$
(2) $\langle f_n, f_s\rangle_{r_{k_0}} = 0$ for $s = 1, \dots, n-1$ and $n = 1, 2, \dots,$
(3) $\langle f_n, \epsilon_k\rangle_{r_{k_0}} = 0$ for $k = 1, \dots, n$ and $n = 1, 2, \dots,$
(4) $\langle f_n, f_n\rangle_{r_{k_0}} = 1$ for $n = 1, 2, \dots.$

where $\langle\, ,\rangle_{r_{k_0}}$ is the inner product in $\Lambda_{r_{k_0}}[\mathcal{E}]$.

Such a sequence exists and can be selected by induction since for each $n \in \mathbb{N}$, the space $\{g_1, \dots, g_{2n}\}$ is $2n$-dimensional and we impose $2n-1$ conditions on f_n.

Hence $f_n = \sum_{i=1}^{2n} c_i^n g_i$ for some scalars $\{c_i^n\}_i$, $n = 1, 2, \dots.$ We have

$$1 = |f_n|_{r_{k_0}} \geq C_0 \Big|\sum_{i=1}^{2n} c_i^n e_i\Big|_{r_0} = C_0\Big(\sum_{i=1}^{2n} |c_i^n|^2\Big)^{1/2}, \qquad n = 1, 2, \dots.$$

So

$$\sum_{i=1}^{2n} |c_i^n|^2 \leq \frac{1}{C_0} \qquad \forall n \in \mathbb{N}. \tag{3.1}$$

Now fix $0 < s < 1$ and estimate, for all $n \in \mathbb{N}$,

$$\begin{aligned} |f_n|_s &= \Big|\sum_{i=1}^{2n} c_i^n g_i\Big|_s \leq \sum_{i=1}^{2n} |c_i^n|\, |g_i|_s \leq C_1 \sum_{i=1}^{2n} |c_i|\, |e_n|_{\sigma_T(s)} \\ &= C_1 \sum_{i=1}^{2n} |c_i^n| \left|\frac{\epsilon_i}{e^{r_0\mathcal{E}_i}}\right|_{\sigma_T(s)} = C_1 \sum_{i=1}^{2n} |c_i^n| e^{(\sigma_T(s)-r_0)\mathcal{E}_i} \end{aligned} \tag{3.2}$$

for some $C_1 > 0$, where σ_T is the characteristic of continuity of T with respect to the canonical grading of $\Lambda_1(\mathcal{E})$.

Choose a $K(s)$ with $\max\{\sigma_T(s), r_0\} < K(s) < 1$, and proceed with the estimate (3.2):

$$\begin{aligned} &\sum_{i=1}^{2n} |c_i^n| e^{(\sigma_T(s)-r_0)\mathcal{E}_i + K(s)\mathcal{E}_i - K(s)\mathcal{E}_i} \\ &\qquad\qquad \leq e^{(K(s)-r_0)\mathcal{E}_{2n}} \Big(\sum_{i=1}^{2n} |c_i^n|^2\Big)^{1/2} \Big(\sum_{i=1}^{2n} e^{2(\sigma_T(s)-K(s))\mathcal{E}_i}\Big)^{1/2} \\ &\qquad\qquad \leq C e^{(K(s)-r_0)\mathcal{E}_{2n}}, \end{aligned}$$

to deduce that for all $s < 1$ there are $C = C(s, T)$ and $K(s) < 1$ such that

$$|f_n|_s \leq C e^{(K(s)-r_0)\mathcal{E}_{2n}}, \qquad n = 1, 2, \dots. \tag{3.3}$$

We choose an increasing sequence $\{K^+(s)\}_s$ with $K(s) < K^+(s) < 1$ for all $s < 1$.

On the other hand we also have, for each $n \in \mathbb{N}$, a representation

$$f_n = \sum_{k=n+1}^{\infty} \beta_k^n \epsilon_k$$

for some sequence $\{\beta_k^n\}_k$. For $-\infty < r < r_{k_0}$ we estimate

$$(3.4)\qquad |f_n|_r^2 = \sum_{s=n+1}^{\infty} |\beta_s^n|^2 e^{2r\mathcal{E}_s} = \sum_{s=n+1}^{\infty} |\beta_s^n|^2 e^{2r_{k_0}\mathcal{E}_s} e^{2(r-r_{k_0})\mathcal{E}_s}$$
$$\le e^{2(r-r_{k_0})\mathcal{E}_{n+1}} |f_n|_{r_{k_0}}^2 = e^{2(r-r_{k_0})\mathcal{E}_{n+1}}, \quad n = 1, 2, \ldots .$$

In view of stability of $\{\mathcal{E}_n\}$ there exists a $C_0 > 0$ such that $C_0\mathcal{E}_{2n} \le \mathcal{E}_{n+1}$ for all $n = 1, 2, \ldots$. Hence proceeding with the estimate (3.4) we have

$$(3.5)\qquad |f_n|_r^2 \le e^{2C_0(r-r_{k_0})\mathcal{E}_{2n}} \quad \forall n \in \mathbb{N},\ -\infty < r < r_{k_0}.$$

We now fix an s_0 with $-\infty < s_0 < -2/C_0$. In view of (3.5) we have

$$(3.6)\quad |f_n|_{s_0} \le e^{(2C_0 s_0 - 2C_0 r_{k_0} + r_0)\mathcal{E}_{2n}} e^{-r_0\mathcal{E}_{2n}} \le e^{-\mathcal{E}_{2n}} e^{-r_0\mathcal{E}_{2n}}, \quad n = 1, 2, \ldots .$$

We stress that s_0 depends only on the associated exponent sequence $\mathcal{E}$. Now for a sequence $\{\lambda_n\}_{n=1}^{\infty}$ and $s < 1$, (3.3) above gives

$$\sum_{i=1}^{\infty} |\lambda_i|\,|f_i|_s e^{r_0\mathcal{E}_{2i}} \le C\Big(\sum |\lambda_i|^2 e^{2K^+(s)\mathcal{E}_{2i}}\Big)^{1/2}$$

for some $C = C(s)$.

It follows that the assignment that sends ϵ_n to $f_n e^{r_0\mathcal{E}_{2n}}$, $n = 1, 2, \ldots$, defines a continuous linear operator $\widehat{T}$ from $\Lambda_1\big((\mathcal{E}_{2n})\big)$ into $\mathcal{X}$ that satisfies

$$|\widehat{T}(x)|_{r_{k_0}} = \Big|\widehat{T}\Big(\sum_{i=1}^{\infty} x_i\epsilon_i\Big)\Big|_{r_{k_0}} = \Big|\sum_{i=1}^{\infty} x_i f_i e^{r_0\mathcal{E}_{2i}}\Big|_{r_{k_0}}$$
$$= \Big(\sum_{i=1}^{\infty} |x_i|^2 e^{2r_0\mathcal{E}_{2i}}\Big)^{1/2} = |x|_{r_0} \quad \forall x \in \Lambda_1((\mathcal{E}_{2n})).$$

Hence $\widehat{T}$ is an (r_0, r_{k_0})-local isomorphism from $\Lambda_1((\mathcal{E}_{2n}))$ into $\mathcal{X}$.

Moreover $\widehat{T}$ extends to a continuous operator from $\Lambda_0((\mathcal{E}_{2n}))$ into $\Lambda_{s_0}(\mathcal{E})$. Indeed, in view of (3.6), for $x \in \Lambda_0(\mathcal{E})$ we have

$$|\widehat{T}(x)|_{s_0} = \Big|\widehat{T}\Big(\sum_i x_i\epsilon_i\Big)\Big|_{s_0} = \Big|\sum_{i=1}^{\infty} x_i f_i e^{r_0\mathcal{E}_{2i}}\Big|_{s_0}$$
$$\le \sum_{i=1}^{\infty} |x_i|\,|f_i|_{s_0} e^{r_0\mathcal{E}_{2i}} \le \sum_{i=1}^{\infty} |x_i| e^{-\mathcal{E}_{2i}} \le \Big(\sum_{i=1}^{\infty} e^{-2\mathcal{E}_{2i}}\Big)^{1/2} |x|_0.$$

We now vary $r < 1$ and obtain a family $\{\widehat{T}_r\}$ of (r, r_{k_r})-local imbeddings from $\Lambda_1\big((\mathcal{E}_{2n})\big)$ into $\mathcal{X}$ with the additional property mentioned above.

Using the elementary inequality

$$|\cdot|_{t_2} \leq |\cdot|_{t_1}^{t_3-t_2/t_3-t_1} |\cdot|_{t_3}^{t_2-t_1/t_3-t_1}$$

for any $t_1 < t_2 < t_3$, which is valid in every power series space, for any $s_0 < s < 1$ we choose a $\rho(s) < 1$ such that

$$\forall 0 \leq r < 1\ \exists C > 0: \quad |\widehat{T}_r(x)|_s \leq C|x|_{\rho(s)}. \tag{3.7}$$

After these preparations we now proceed to show $(\mathcal{X}, \Lambda_1((\mathcal{E}_{2n}))) \in \mathcal{T}$.

According to our assumption, there exists a sequence $\{S_\alpha\}_{\alpha=1}^{\infty}$ of increasing functions from $\mathbb{N}$ into $\mathbb{N}$ such that for every $T \in \mathcal{L}(\mathcal{X}, \mathcal{X})$ there exists $\alpha \in \mathbb{N}$ such that $\sigma(T) \leq S_\alpha$.

Let now S be a given continuous linear operator from $\mathcal{X}$ into $\Lambda_1((\mathcal{E}_{2n}))$. In view of (3.7) above, the family $\{\widehat{T}_r \circ S\}_{r<1}$ of continuous linear operators from $\mathcal{X}$ into $\mathcal{X}$ satisfies

$$\sigma_{\widehat{T}_r \circ S} \leq \sigma_S \circ \rho.$$

Hence this family is in $F \doteq \{U \in \mathcal{L}(\mathcal{X}, \mathcal{X}) : \sigma_U \leq \sigma_S \circ \rho\}$. On F we consider the topology coming from the seminorms $\{\|\cdot\|_s^{\sigma_S \circ \rho(s)}\}_{s=1}^{\infty}$, and on $\mathcal{L}(\mathcal{X}, \mathcal{X}) = \bigcup_{\alpha=1}^{\infty} \bigcap_{s=1}^{\infty} \{U \in \mathcal{L}(\mathcal{X}, \mathcal{X}) : \|U\|_s^{S_\alpha(s)} < \infty\}$, we consider the LF-space structure as explained in Remark 3.2(4) above. Since evaluation at points of $\mathcal{X}$ is continuous in both F and $\mathcal{L}(\mathcal{X}, \mathcal{X})$, the inclusion $F \subseteq \mathcal{L}(\mathcal{X}, \mathcal{X})$ has a sequentially closed graph. It follows that there exists α such that $F \subseteq \bigcap_{s=1}^{\infty} \{U \in \mathcal{L}(\mathcal{X}, \mathcal{X}) : \|U\|_s^{S_\alpha(s)} < \infty\}$ in view of Grothendieck's factorization theorem [14, p. 68]. It follows that there exists an $\alpha \in \mathbb{N}$ such that

$$\sigma_{\widehat{T}_r \circ S} \leq S_\alpha \quad \forall r < 1.$$

In particular for each $r < 1$ there exists a $\widehat{C} > 0$ such that

$$|Sx|_r \leq C|\widehat{T}_r(Sx)|_{r_{k_r}} \leq \widehat{C}|x|_{S_\alpha(r_{k_r})}. \tag{3.8}$$

Now if we set $\widehat{S}_\alpha(r) \doteq S_\alpha(r_{k_r})$ for $r < 1$ and $\alpha \in \mathbb{N}$, the analysis above shows that for every $S \in \mathcal{L}(\mathcal{X}, \Lambda_1((\mathcal{E}_{2n})))$ there is $n \in \mathbb{N}$ such that $\sigma_S \leq \widehat{S}_\alpha$. Hence $(\mathcal{X}, \Lambda_1((\mathcal{E}_{2n}))) \in \mathcal{T}$.

Now [17, Theorem 11] implies that $\mathcal{X}$ satisfies Vogt's strong Ω condition, $\bar{\Omega}$. This together with our assumption that $\mathcal{X}$ has $\underline{\mathrm{DN}}$ allows us to conclude that $\mathcal{X}$ is isomorphic to a finite type power series space [15, Proposition 2.9.18]. Hence $\mathcal{X} \cong \Lambda_1(\mathcal{E})$ [5, Proposition 1.1]. ∎

4. Spaces of analytic functions. In this section we will focus on a particular class of nuclear Fréchet spaces with properties $\underline{\mathrm{DN}}$ and Ω, namely the spaces of analytic functions on Stein manifolds. Stein manifolds, being closed connected submanifolds of complex Euclidean spaces $\mathbb{C}^N$, possess a rich supply of analytic functions. These spaces, with the usual topology of uniform convergence on compact subsets, form an important subclass of Fréchet spaces with properties $\underline{\mathrm{DN}}$ and Ω. The linear topological properties of $\mathcal{O}(M)$, the Fréchet space of analytic functions on a Stein manifold M, and the complex analytic properties of M that are reflected by the type of $\mathcal{O}(M)$, have been studied by several authors (see [3], [25] and the references therein).

In this context, we show in Subsection 4.1 that local imbeddings of finite type power series spaces into $\mathcal{O}(M)$ can be used to construct Green's functions in M. In Subsection 4.2 we classify Stein manifolds M for which $\mathcal{O}(M)$ is tame. Some results and concepts from pluripotential theory will be used. For all undefined terminology and background we refer the reader to [12].

4.1. Local imbeddings of finite type power series spaces into $\mathcal{O}(M)$ and Green's functions. Let M be a complex manifold and fix $z_0 \in M$. We will write $\mathrm{PSH}(M)$ for the set of all plurisubharmonic functions on M. Employing norms in a local chart centred at z_0 we consider

$$\mathcal{L}_{z_0} \doteq \{u \in \mathrm{PSH}(M) : u \le 0 \text{ and } u(z) - \ln \|z - z_0\| \text{ is bounded near } z_0\}$$

and set

$$g(\xi, z_0) \doteq \sup\{u(\xi) : u \in \mathcal{L}_{z_0}\}.$$

This assignment, if not $\equiv -\infty$, defines a plurisubharmonic function on M. We will call $g(\cdot, z_0)$ *Green's function* of M with pole at z_0, and say that Green's function with pole at z_0 *exists* if $g(\cdot, z_0)$ is not identically $-\infty$. If M is parabolic, i.e. has no non-constant bounded plurisubharmonic function, then of course, no Green's function exist. In one variable, non-parabolicity characterizes existence of Green's functions [19], but in several complex variables there is no such general result. The difficulty seems to be in constructing a negative plurisubharmonic function with pole at a given point.

PROPOSITION 4.1. *Let M be a Stein manifold of dimension d and let $z_0 \in M$. If there exists a local imbedding from $\Lambda_1(\alpha)$ into $\mathcal{O}(M)$ for some finitely nuclear exponent sequence $\{\alpha_n\}_n$ with $\underline{\lim}\, n^{1/d}/\alpha_n > 0$, then Green's function with pole at z_0 exists.*

Proof. By [10] there exists a local biholomorphism $\Phi : \Delta_e^d \twoheadrightarrow M$ from the polydisc with centre 0 and radius e of $\mathbb{C}^d$ onto M such that $\Phi(0) = z_0$. We use Φ to imbed $\mathcal{O}(M)$ into $\mathcal{O}(\Delta_e^d)$ via $f \mapsto f \circ \Phi$, $f \in \mathcal{O}(M)$ (see [6]). We choose a bijection $\rho : \mathbb{N} \to \mathbb{N}^d$ such that $|\rho(n)|$ is strictly increasing in n and $(\rho(n))$ is ordered lexicographically, and use it to define an isomorphism between $\Lambda_1(n^{1/d})$ and $\mathcal{O}(\Delta_e^d)$ by the correspondence $\epsilon_i \mapsto z^{p(i)} = z_1^{p_1(i)} \ldots z_d^{p_d(i)}$, $i = 1, 2, \ldots$. We note that there exist constants $\beta_1, \beta_2 > 0$ such that $\beta_1(\rho(n)) \le n^{1/d} \le \beta_2(\rho(n))$, $n = 1, 2, \ldots$ (see e.g. [18, p. 362]).

Fix an (r_0, k_0)-local imbedding T from $\Lambda_1(\alpha_n)$ into $\Lambda_1(n^{1/d})$, with $T(\Lambda_1(\alpha_n)) \subseteq \mathcal{O}(M) \subseteq \Lambda_1(n^{1/d})$, which exists in view of our assumptions. Let

$$f_n \doteq T\left(\frac{\epsilon_n}{e^{r_0\alpha_n}}\right), \qquad n = 1, 2, \ldots.$$

The sequence $\{f_n\}_n$ is linearly independent, so we can choose a sequence $\{g_n\}_{n=1}^{\infty}$ of elements of $\mathcal{O}(M)$ with

(1) $g_n \in \operatorname{span}\{f_1, \ldots, f_n\}$ for $n = 1, 2, \ldots$,
(2) $\langle g_n, \epsilon_i \rangle_{k_0} = 0$ for $i = 1, \ldots, n-1$, where $\langle\, ,\rangle_{k_0}$ is the inner product in $\Lambda_{k_0}[n^{1/d}]$ for $n = 1, 2, \ldots$,
(3) $|g_n|_{k_0} = 1$ for $n = 1, 2, \ldots$.

Note that if $g_n = \sum_{k=1}^{n} c_k^n f_k$ then, as in the above arguments,

$$\begin{aligned}\sum_{k=1}^{n} |c_k^n|^2 &= \sum_{k=1}^{n} \left|\frac{c_k^n}{e^{2r_0\alpha_k}}\right|^2 e^{2r_0\alpha_k} = \left|\sum_{k=1}^{n} c_k^n \frac{\epsilon_k}{e^{2r_0\alpha_k}}\right|_{r_0}^2 \\ &\le C\left|T\left(\sum_{k=1}^{n} c_k^n \frac{\epsilon_k}{e^{2r_0\alpha_k}}\right)\right|_{k_0}^2 = C|g_n|_{k_0}^2 \le C\end{aligned}$$

for some $C > 0$ and for all $n = 1, 2, \ldots$.

Hence, for a given $r < 1$, using the Cauchy–Schwarz inequality we have

$$\begin{aligned}|g_n|_r &= \left|\sum_{i=1}^{n} c_i^n f_i\right|_r \le C_1 \sum_{i=1}^{n} |c_i^n| \left|\frac{\epsilon_i}{e^{r_0\alpha_i}}\right|_{\sigma_T(r)} \\ &= \sum_{i=1}^{n} |c_i^n| e^{(\sigma_T(r)-r_0)\alpha_i} \le C_2 e^{\rho(r)\alpha_n}, \qquad n = 1, 2, \ldots,\end{aligned} \tag{4.1}$$

for some constants $C_1, C_2 > 0$ where $\sigma_T(r) - r_0 < \rho(r) < 1$ is a chosen number that depends on r (and T).

On the other hand, in view of condition (2) above, each g_n, $n = 1, 2, \ldots$, has an expansion as

$$g_n = \sum_{i \ge n} d_i^n \epsilon_i \leftrightarrow \sum_{i \ge n} d_i^n z^{\rho(i)}$$

in $\Lambda_1(n^{1/d})$ and in $\mathcal{O}(\Delta_e^d)$, respectively. By abuse of notation we will think of $g_n \in \mathcal{O}(M) \subseteq \mathcal{O}(\Delta_e^d)$, $n = 1, 2, \ldots$, as an analytic function on Δ_e^d with Taylor series

$$g_n(z) = \sum_{|t| \ge \frac{1}{\beta_2} n^{1/d}} \alpha_t^n z_1^{t_1} \ldots z_d^{t_d}. \tag{4.2}$$

Choose $h_n \in \mathcal{O}(M)$ such that $g_n = h_n \circ \Phi$. For a given compact set $K \subseteq M$ choose $r < 1$ so that $\Phi(\Delta_{e^r}^d) \supseteq K$. Then in view of (4.1),

$$\sup_{z \in K} |h_n(z)| \le \sup_{\xi \in \Delta_{e^r}^d} |g_n(\xi)| \le C|g|_{r^+} \le CC_2 e^{\rho(r^+)\alpha_n}$$

for some constant $C > 0$ and a choice of $r < r^+ < 1$. Hence

$$u(z) \doteq \overline{\lim_{\xi\to z}}\ \overline{\lim_{n}} \frac{\ln|h_n(\xi)|}{\alpha_n}, \qquad z \in M,$$

is a plurisubharmonic function on M that is bounded by 1. This function is not $\equiv -\infty$. Indeed, assume it is. Fix large $k_0 < r < 1$ such that $|x|_{k_0} \le C\sup_{z\in\overline{\Delta}^d_{e^r}} |x(z)|$ for all $x \in \mathcal{O}(\Delta^d_e)$ and choose a compact set $K \subset M$ that contains $\Phi(\overline{\Delta}^d_{e^r})$. Our assumption and the Hartogs theorem [12, p. 70] give, for each $N \in \mathbb{N}$, an $n_0 \in \mathbb{N}$ such that

$$\sup_{z\in K} |h_n(z)| \le e^{-N\alpha_n}, \qquad n \ge n_0.$$

This in turn implies that for some $C > 0$,

$$1 = |g_n|_{k_0} \le Ce^{-N\alpha_n} \qquad \text{for } n \ge n_0.$$

So u is not identically $-\infty$.

Now consider a $z \in M$ near z_0, say with $\|z - z_0\| = e^r$ for some very large negative r. Since Φ is a local biholomorphism, there exists a $C_0 > 0$, independent of r, and $\xi \in \Delta^d_e$ with $\|\xi\| < C_0e^r$ such that $\Phi(\xi) = z$. Using (4.2) we estimate, with $k_0^- < k_0$ and $C_1 > 0$,

$$\begin{aligned}|g_n(\xi)| &\le \sum_{|t|\ge Cn^{1/d}} |\alpha^n_t| C_0^{|t|} e^{r|t|} e^{-k_0^-|t|} e^{k_0^-|t|} \\ &\le C_0 e^{C(r-k_0^-)n^{1/d}} \Big(\sum_{|t|} |\alpha^n_t|^2 e^{2k_0|t|}\Big)^{1/2} \\ &\le C_1 e^{C(r-k_0^-)n^{1/d}}, \qquad n = 1, 2, \ldots.\end{aligned}$$

Hence our assumption on (α_n) implies that there are $C_2, C_3 > 0$ such that

$$\overline{\lim_{n}} \frac{\ln|h_n(z)|}{\alpha_n} \le \overline{\lim_{n}}\, C(r - k_0^-)\frac{n^{1/d}}{\alpha_n} \le C_2 \ln\|z - z_0\| + C_3.$$

So u/C_2 has a logarithmic singularity at z_0, and is a bounded plurisubharmonic function. It follows that $g_m(\cdot, z_0)$ is not identically $-\infty$. ∎

4.2. Diametral dimensions of analytic function spaces. In this subsection we will investigate the diametral dimension of the spaces of analytic functions on Stein manifolds. These invariants for Fréchet spaces are in a sense dual to approximate diametral dimensions, but they have been more extensively studied.

For a nuclear Fréchet space $\mathcal{X}$ with a neighbourhood basis $\{U_p\}_{p=1}^\infty$ of 0 consisting of discs, we define (in the notation of Section 1)

$$\begin{aligned}\Delta(\mathcal{X}) &= \{(t_n) : \forall p\ \exists q : t_n d_n(U_q, U_p) \to 0\} \\ &= \{(t_n) : \forall p\ \exists q : t_n e^{-\mathcal{E}_n(p,q)} \to 0\}.\end{aligned}$$

As the notation suggests, it is easy to see that this sequence space does not depend upon the neighbourhood basis chosen and is an invariant of the Fréchet space $\mathcal{X}$.

In this subsection we will use a special generating norm system for analytic function spaces, unless stated otherwise. To describe these norms, suppose a Stein manifold M of dimension d is given. We choose a strictly plurisubharmonic C^∞ exhaustion function ρ of M and consider a sequence $r_p \uparrow \infty$ such that the sublevel sets $D_p \doteq \{z \in M : \rho(z) < r_p\}$, $p = 1, 2, \ldots$, are strictly pseudoconvex. We set $K_p \doteq \bar{D}_p$, $p = 1, 2, \ldots$. Following [27] (cf. [25]) we choose for each p a Hilbert space H_p that satisfies

$$A(K_p) \hookrightarrow H_p \hookrightarrow AC(K_p) \hookrightarrow \mathcal{O}(D_p)$$

where $A(K_p)$ denotes the germs of analytic functions on the compact set K_p with the inductive limit topology, $AC(K_p)$ denotes the Banach space that is the closure of $A(K_p)$ in $C(K_p)$, the Banach space of continuous functions on K_p with the sup-norm, and the $\hookrightarrow$'s are imbeddings with dense range, $p = 1, 2, \ldots$. The norms that we will use to generate the topology $\mathcal{O}(M)$ will be the Hilbertian norms $\{\|\cdot\|_p\}_p$ of H_p's. Their corresponding unit balls in $\mathcal{O}(M)$ will be denoted by U_p, $p = 1, 2, \ldots$. The primary reason of our usage of these seminorms is the beautiful formula of Nivoche, Poletsky, and Zaharyuta:

$$\forall p < q : \quad \lim_n \frac{\mathcal{E}_n(p,q)}{\left(\frac{2\pi d! n}{\tilde{C}(K_p, D_q)}\right)^{1/d}} = 1 \tag{4.3}$$

where $\tilde{C}(K_p, D_q) = \sup\{\int_{K_p} (dd^c u)^n : u \in \mathrm{PSH}(D_q),\ -1 \le u \le 0\}$ [27, Propositions 4.6, 4.8, 4.12]. We refer the reader to [27] and [25] for a proof and a discussion of the history of this formula.

An immediate consequence of (4.3) is the following proposition.

Proposition 4.2. *Let M be a Stein manifold of dimension d. Then*

$$\Delta(\mathcal{O}(M)) \equiv \left\{(t_n) : \forall p\ \exists q : \sup_n |t_n| e^{-\alpha_n / c(p,q)} < \infty\right\}$$

where for $p < q$, $c(p,q) \doteq \tilde{C}(K_p, D_q)^{1/d}$, and $\alpha_n \doteq (2\pi d!)^{1/d} n^{1/d}$, $n = 1, 2, \ldots$.

Proof. ($\subseteq$): Choose a $(t_n)_n \in \Delta(\mathcal{O}(M))$. Fix p and choose q such that

$$\sup_n |t_n| e^{-\mathcal{E}_n(p,q)} < \infty.$$

Now in view of Proposition 2.3(2′) there exist q^+, $C > 0$ and $0 < \lambda < 1$ such that

$$\mathcal{E}_n(p,q) \le (1-\lambda)\mathcal{E}_n(p,q^+) + C.$$

For $\varepsilon > 0$ such that $(1+\varepsilon) < 1/(1-\lambda)$, in view of (4.3) there exists an N such that

$$(1-\varepsilon)\frac{\alpha_n}{c(p,q^+)} \le \mathcal{E}_n(p,q^+) \le (1+\varepsilon)\frac{\alpha_n}{c(p,q^+)}, \qquad n \ge N.$$

Hence

$$\begin{aligned}|t_n| \le C_1 e^{\mathcal{E}_n(p,q)} &\le C_2 e^{(1-\lambda)\mathcal{E}_n(p,q^+)} \\ &\le e^{(1-\lambda)(1+\varepsilon)\alpha_n/c(p,q^+)} \le e^{\alpha_n/c(p,p^+)}, \qquad n \ge N.\end{aligned}$$

($\supseteq$): Choose a sequence (t_n) from the right hand side. For a fixed p, choose a q such that $\sup_n |t_n| e^{-\alpha_n/c(p,q)} < \infty$. By Proposition 2.3(2′) choose k, $0 < \rho < 1$ and $C > 0$ such that $\mathcal{E}_n(p,q) \le (1-\rho)\mathcal{E}_n(p,k) + C$. Let $\varepsilon > 0$ be such that $1-\rho < 1-\varepsilon$ and choose an N (from (4.3)) for which

$$\frac{-(1+\varepsilon)\alpha_n}{c(p,q)} \le -\mathcal{E}_n(p,q) \le -\frac{(1-\varepsilon)\alpha_n}{c(p,q)}, \qquad n \ge N.$$

Hence

$$|t_n|^{(1-\varepsilon)} e^{-\mathcal{E}_n(p,q)} \le (|t_n| e^{-\alpha_n/c(p,q)})^{1-\varepsilon} \le Q < \infty$$

for some $Q > 0$. Moreover, since $-\mathcal{E}_n(p,q) \ge -(1-\varepsilon)\mathcal{E}_n(p,k) - C$, we have

$$\sup_n (|t_n| e^{-\mathcal{E}_n(p,k)})^{1-\varepsilon} \le Q e^C, \quad \text{so} \quad \sup_n |t_n| d_n(U_k, U_p) < \infty.$$

It follows that $(t_n) \in \Delta(\mathcal{O}(M))$. ∎

Corollary 4.3. *Let M be a Stein manifold of dimension d. Then*

$$\begin{aligned}\Delta(\mathcal{O}(M)) = \Delta(\Lambda_1(n^{1/d})) \;&\Leftrightarrow\; \inf_p \sup_{q\ge p} \frac{1}{c(p,q)} = 0 \\ &\Leftrightarrow\; \delta(\mathcal{O}(M)) = \delta(\Lambda_1(n^{1/d})).\end{aligned}$$

Proof. Suppose that $\Delta(\mathcal{O}(M)) = \Delta(\Lambda_1(n^{1/d}))$. Suppose that there is an $\varepsilon > 0$ such that $\inf_p \sup_{q\ge p} 1/c(p,q) > \varepsilon > 0$. In view of (4.3) this implies

$$\inf_p \sup_{q\ge p} \lim_n \frac{\mathcal{E}_n(p,q)}{n^{1/d}} > \varepsilon (d!2\pi)^{1/d}.$$

Setting $\kappa \doteq \varepsilon (d!2\pi)^{1/d}$, we find that

$$\forall p\ \exists q, N: \quad \frac{1}{d_n(U_q,U_p)} \ge e^{\kappa n^{1/d}},\ n \ge N.$$

Hence,

$$e^{(\kappa/2)n^{1/d}} \in \Delta(\mathcal{O}(M)) = \Delta(\Lambda_1(n^{1/d})) = \Big\{(\xi_n) : \forall r < 1 : \lim_n |\xi_n| r^{n^{1/d}} = 0\Big\}.$$

This contradiction shows that $\inf_p \sup_{q\ge p} 1/c(p,q) = 0$.

On the other hand, if $\inf_p \sup_{q\geq p} 1/c(p,q) = 0$, and $r < 1$ is given, choose a p such that, in the notation of Proposition 4.2,

$$\sup_{q\geq p} \frac{\alpha_n}{c(p,q)n^{1/d}} \leq -\ln r \qquad \forall n.$$

In view of Proposition 4.2, for p there is a $q \gg p$ such that $\sup_n |t_n| e^{-\alpha_n/c(p,q)} < \infty$. Hence $\sup_n |t_n| r^{n^{1/d}} < \infty$. This shows that

$$(t_n) \in \Big\{(\xi_n) : \forall r < 1 : \lim_n |\xi_n| r^{n^{1/d}} = 0\Big\} = \Delta(\Lambda_1(n^{1/d})).$$

So $\Delta(\mathcal{O}(M)) \subseteq \Delta(\Lambda_1(n^{1/d}))$.

On the other hand, $(n^{1/d})_n$, being the associated exponent sequence of $\mathcal{O}(M)$, always satisfies $\Delta(\Lambda_1(n^{1/d})) \subseteq \Delta(\mathcal{O}(M))$ [5, Proposition 1.1]. So $\Delta(\Lambda_1(n^{1/d})) = \Delta(\mathcal{O}(M))$. Since $\mathcal{O}(M)$ is isomorphic to a closed subspace of $\Lambda_1(n^{1/d})$, the other equivalence follows directly from Proposition 2.7 and the fact that the approximate diametral dimension of a nuclear Fréchet space is greater than the approximate diametral dimension of its subspaces [7]. ■

THEOREM 4.4. *Let M be a Stein manifold of dimension d. Then either $\Delta(\mathcal{O}(M)) = \Delta(\mathcal{O}(\Delta^d))$ or $\Delta(\mathcal{O}(M)) = \Delta(\mathcal{O}(\mathbb{C}^d))$.*

Proof. We will use the grading on $\mathcal{O}(M)$ described at the beginning of this section. Using the notation above, for a given $p \geq 1$, we set $c(p) \doteq \lim_{q>p} c(p,q) = \inf_{q>p} c(p,q)$. Since $c(\cdot,\cdot)$ is increasing in the first variable, the sequence $\{c(p)\}$ is increasing. We have two cases:

CASE 1: *$c(p)$ is zero for all $p \geq 1$.* Choose $(t_n)_n \in \Delta(\mathcal{O}(\mathbb{C}^d)) = \Delta(\Lambda_\infty(n^{1/d}))$, $R > 1$ and $C > 0$ so that $|t_n| \leq CR^{n^{1/d}}$ for all n. For a given p choose a q such that $c(p,q) \leq 1/\ln(\tilde{R})$ for some $\tilde{R}$ with $R^{1/(2\pi d!)^{1/d}} \ll \tilde{R}$. Then for $\alpha_n = n^{1/d}$,

$$|t_n| e^{-\alpha_n/c(p,q)} \leq |t_n| e^{-\alpha_n \ln \tilde{R} + n^{1/d}\ln R - n^{1/d}\ln R} \leq C.$$

So $(t_n)_n \in \Delta(\mathcal{O}(M))$ in view of Proposition 4.1. Hence

$$\Delta(\Lambda_\infty(n^{1/d})) \subseteq \Delta(\mathcal{O}(M)).$$

However, $(\alpha_n)_n = (n^{1/d})_n$ being the associated exponent sequence of $\mathcal{O}(M)$, the inclusion $\Delta(\mathcal{O}(M)) \subseteq \Delta(\Lambda_\infty(n^{1/d}))$ is always true. It follows that in this case $\Delta(\mathcal{O}(M)) = \Delta(\mathcal{O}(\mathbb{C}^d))$.

CASE 2: *$c(p)$ increases to a non-zero c.* We have two possibilities: either $c \in \mathbb{R}$ or $c = \infty$.

Suppose that $c > 0$ is a real number. Fix natural numbers $\alpha < \gamma < \beta$. For given natural numbers n and m, plainly

$$d_{n+m}(U_\beta, U_\alpha) \leq d_n(U_\beta, U_\gamma) d_m(U_\gamma, U_\alpha),$$

where U_t's, $1 \le t < \infty$, are the unit balls corresponding to our grading. In particular,

$$\mathcal{E}_{n+m}(\alpha,\beta) \ge \mathcal{E}_n(\gamma,\beta) + \mathcal{E}_m(\alpha,\gamma), \qquad n,m \ge 1. \tag{4.4}$$

Fix an $\varepsilon > 0$ and using (4.3), choose $N = N(\alpha,\gamma,\beta)$ such that

$$\begin{aligned}
&(1-\varepsilon)\frac{\alpha_{n+m}}{c(\alpha,\beta)} \le \mathcal{E}_{n+m}(\alpha,\beta) \le (1+\varepsilon)\frac{\alpha_{n+m}}{c(\alpha,\beta)},\\
&(1-\varepsilon)\frac{\alpha_n}{c(\gamma,\beta)} \le \mathcal{E}_n(\gamma,\beta) \le (1+\varepsilon)\frac{\alpha_n}{c(\gamma,\beta)},\\
&(1-\varepsilon)\frac{\alpha_m}{c(\alpha,\gamma)} \le \mathcal{E}_m(\alpha,\gamma) \le (1+\varepsilon)\frac{\alpha_m}{c(\alpha,\gamma)}, \qquad \text{for } n,m \ge N.
\end{aligned}$$

Hence

$$(1-\varepsilon)\frac{m^{1/d}}{c(\alpha,\gamma)} + (1-\varepsilon)\frac{n^{1/d}}{c(\gamma,\beta)} \le (1+\varepsilon)\frac{(n+m)^{1/d}}{c(\alpha,\beta)}$$

for $n,m \ge N$; taking $m = n$ and $n > N$, we get, after cancellation,

$$\frac{1-\varepsilon}{c(\alpha,\gamma)} + \frac{1-\varepsilon}{c(\gamma,\beta)} \le \frac{(1+\varepsilon)2^{1/d}}{c(\alpha,\beta)},$$

which, upon letting first β then γ and α go to infinity, gives $2 \le 2^{1/d}$. Hence for $d > 1$, c must be ∞.

For $d = 1$, we will use the exhaustion given in [1, p. 145] and an associated fundamental Hilbertian norm system as explained at the beginning of this section. In this context we will use the *modulus inequality* of [19, p. 14], which in our notation states that for $1 \le p < q < s < \infty$,

$$\frac{1}{c(p,s)} \ge \frac{1}{c(p,q)} + \frac{1}{c(q,s)}. \tag{4.5}$$

First letting $s \to \infty$ and then $q \to \infty$ we see that $1/c(q) \to 0$. Hence in this case $c = \infty$ as well. So the first possibility does not occur and we conclude that

$$\lim_p \frac{1}{c(p)} = \inf_p \sup_{q \ge p} \frac{1}{c(p,q)} = 0.$$

The theorem now follows from Corollary 4.3. ∎

Now we turn our to tameness in spaces of analytic functions. Recall that a Stein manifold M is called *hyperconvex* if it has a bounded plurisubharmonic exhaustion function. We refer the reader to [12] and the references therein for an account of hyperconvex manifolds. From a functional analysis point of view, hyperconvex Stein manifolds M are precisely those Stein manifolds that satisfy $\mathcal{O}(M) \approx \mathcal{O}(\Delta^d)$, $d = \dim M$ [26, 2]. Hence for a hyperconvex manifold M, $\mathcal{O}(M)$ is a tame Fréchet space. Our next and final result tells us that they are the only ones with this property:

MAIN THEOREM 4.5. *Let M be a Stein manifold. Then $\mathcal{O}(M)$ is tame if and only if M is hyperconvex.*

Proof. Let M have dimension d. In view of the remarks preceding the theorem, it suffices to show that if $\mathcal{O}(M)$ is tame then $\mathcal{O}(M) \approx \mathcal{O}(\Delta^d)$. In view of Theorem 4.4 either $\Delta(\mathcal{O}(M)) = \Delta(\mathcal{O}(\mathbb{C}^d))$ or $\Delta(\mathcal{O}(M)) = \Delta(\mathcal{O}(\Delta^d))$. The first case cannot occur. To see this first observe that the assumption on the diametral dimension implies that $\mathcal{O}(M)$ contains a complemented copy of $\mathcal{O}(\mathbb{C}^d)$ by [4, Theorem 1.3]. Since plainly tameness passes to complemented subspaces and $\mathcal{O}(\mathbb{C}^d) = \Lambda_\infty(n^{1/d})$ is not tame [9], indeed this case cannot occur. Now the theorem follows from Corollary 4.3 and Theorem 3.3. ∎

References

[1] L. Ahlfors and L. Sario, *Riemann Surfaces*, Princeton Univ. Press, Princeton, NJ, 1960.

[2] A. Aytuna, *On Stein manifolds M for which $\mathcal{O}(M)$ is isomorphic to $\mathcal{O}(\Delta^n)$ as Fréchet spaces*, Manuscripta Math. 62 (1988), 297–315.

[3] A. Aytuna, *Stein spaces M for which $\mathcal{O}(M)$ is isomorphic to a power series space*, in: Advances in the Theory of Fréchet Spaces, T. Terzioğlu (ed.), NATO ASI Ser. 287, Kluwer, 1989, 115–154.

[4] A. Aytuna, J. Krone, and T. Terzioğlu, *Complemented infinite type power series subspaces of nuclear Fréchet spaces*, Math. Ann. 283 (1989), 193–202.

[5] A. Aytuna, J. Krone, and T. Terzioğlu, *Imbedding of power series spaces and spaces of analytic functions*, Manuscripta Math. 67 (1990), 125–142.

[6] A. Aytuna and T. Terzioğlu, *On certain subspaces of nuclear power series spaces of finite type*, Studia Math. 69 (1981), 79–86.

[7] Cz. Bessaga, A. Pełczyński, and S. Rolewicz, *On diametral approximative dimension and linear homogeneity of F-spaces*, Bull. Acad. Polon. Sci. 9 (1961), 307–318.

[8] E. Dubinsky, *The Structure of Nuclear Fréchet Spaces*, Lecture Notes in Math. 720, Springer 1979, 108–127.

[9] E. Dubinsky and D. Vogt, *Complemented subspaces in tame power series spaces*, Studia Math. 93 (1989), 71–85.

[10] J. E. Fornæss and E. L. Stout, *Spreading polydiscs on complex manifolds*, Amer. J. Math. 99 (1977), 933–960.

[11] R. S. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. 7 (1982), 65–222.

[12] M. Klimek, *Pluripotential Theory*, Clarendon Press, Oxford, 1991.

[13] G. Köthe, *Topological Vector Spaces I*, Grundlehren Math. Wiss. 159, Springer, New York, 1969.

[14] G. Köthe, *Topological Vector Spaces II*, Grundlehren Math. Wiss. 237, Springer, New York, 1979.

[15] R. Meise and D. Vogt, *Introduction to Functional Analysis*, Clarendon Press, Oxford, 1997.

[16] K. Nyberg, *Tameness of pairs of nuclear power series spaces and related topics*, Trans. Amer. Math. Soc. 283 (1984), 645–660.

[17] K. Piszczek, *On tame pairs of Fréchet spaces*, Math. Nachr. 282 (2009), 270–287.

[18] S. Rolewicz, *Metric Linear Spaces*, PWN–Polish Sci. Publ. Warszawa, and Reidel, Dordrecht, 1985.
[19] L. Sario and M. Nakai, *Classification Theory of Riemann Surfaces*, Springer, 1970.
[20] T. Terzioğlu, *On the diametral dimension of some classes of F-spaces*, J. Karadeniz Univ. 8 (1985), 1–13.
[21] D. Vogt, *Charakterisierung der Unterraüme eines nuklearen stabilen Potenzreihenraumes von endlichem Typ*, Studia Math. 71 (1982), 251–270.
[22] D. Vogt, *Eine Charakterisierung der Potenzreihenräume von endlichem Typ und ihre Folgerungen*, Manuscripta Math. 37 (1982), 269–301.
[23] D. Vogt, *Power series space representations of nuclear Fréchet spaces*, Trans. Amer. Math. Soc. 319 (1990), 191–208.
[24] D. Vogt und M. Wagner, *Charakterisierung der Quotientenräume von s und eine Vermutung von Martineau*, Studia Math. 67 (1980), 225–240.
[25] V. Zakhariuta, *Spaces of analytic functions and complex potential theory*, Linear Topol. Spaces Complex Anal. 1 (1994), 74–146.
[26] V. Zaharyuta, *Isomorphism of spaces of analytic functions*, Dokl. Akad. Nauk SSSR 255 (1980), 11–14 (in Russian).
[27] V. Zakharyuta, *Kolmogorov problem on widths asymptotics and pluripotential theory*, in: Contemp. Math. 481, Amer. Math. Soc., 2009, 171–196.
[28] D. N. Zarnadze, *A generalization of the method of least squares for operator equations in some Fréchet spaces*, Izv. Math. 59 (1995), 935–948.

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