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On dimension and shape of inverse limits with set-valued functions

by

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Abstract. Recently, several topological properties of inverse limits of compacta with upper semicontinuous set-valued functions have been studied by many authors. The study of such inverse limits has developed into one of rich topics of geometric topology. There are many differences between the theory of inverse limits with mappings and the theory with set-valued functions. In this paper, we investigate the dimension and the shape of inverse limits with set-valued functions. To evaluate the dimension of the inverse limit $\varprojlim\{X_i, f_{i,i+1}\}$ of a given inverse sequence $\{X_i, f_{i,i+1}\}_{i=1}^{\infty}$ of compacta with set-valued functions satisfying

$$\dim\{x \in X_{i+1} \mid \dim f_{i,i+1}(x) \geq 1\} \leq 0 \quad (i \in \mathbb{N}),$$

we define expand-contract sequences in $\{X_i, f_{i,i+1}\}_{i=1}^{\infty}$ and an index $\tilde{J}(\{X_i, f_{i,i+1}\})$. By use of the index, we prove that

$$\dim \varprojlim\{X_i, f_{i,i+1}\} \leq \tilde{J}(\{X_i, f_{i,i+1}\}) + \sup\{\dim X_i \mid i \in \mathbb{N}\}.$$

Moreover, we evaluate lower bounds of dimensions of some inverse limits of 1-dimensional compacta with set-valued functions. We study the shape of inverse limits with cell-like set-valued functions.

1. Introduction. Inverse limits with bonding maps have played important roles in the development of geometric topology and topological dynamical systems. In fact, every complicated compactum can be represented by an inverse limit of finite polyhedra and simple bonding maps. Conversely, inverse limits with simple bonding maps are useful to produce complicated spaces. These facts imply strong relations between inverse limits and chaotic dynamical systems.

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In 2004, inverse limits with set-valued functions were introduced by Mahavier as inverse limits with closed subsets of the unit square $[0,1] \times [0,1]$ (see e.g. [7, 9]). Since then, the subject has rapidly developed into a rich topic of geometric topology, particularly among continuum theorists (see e.g. [1, 3, 6, 7, 8, 9, 10]). In [9], Ingram and Mahavier discussed several results concerning connectedness, indecomposability and dimension of such inverse limits (see also [7]). Also, they investigated many interesting examples of such inverse limits of set-valued functions from the unit interval $I = [0,1]$ to I. Banič, Nall and Ingram studied the topological dimension of such inverse limits (see also [1, 7, 8, 9, 10]). It is well-known that inverse limits of sequences of single-valued continuous functions (= mappings) have dimension bounded by the dimensions of the factor spaces. In [10], Nall proved that inverse limits of sequences of upper semicontinuous set-valued functions with 0-dimensional values have dimension bounded by the dimensions of the factor spaces. In [3], Charatonik and Roe investigated trivial shape properties of such inverse limits.

In this paper, to evaluate the dimension of the inverse limit $\varprojlim\{X_i, f_{i,i+1}\}$ of a given inverse sequence $\{X_i, f_{i,i+1}\}_{i=1}^{\infty}$ of compacta with set-valued functions satisfying

$$\dim\{x \in X_{i+1} \mid \dim f_{i,i+1}(x) \geq 1\} \leq 0 \qquad (i \in \mathbb{N}),$$

we define expand-contract sequences in $\{X_i, f_{i,i+1}\}_{i=1}^{\infty}$ and an index $\tilde{J}(\{X_i, f_{i,i+1}\})$. In Theorem 4.1, we show that

$$\dim \varprojlim\{X_i, f_{i,i+1}\} \leq \tilde{J}(\{X_i, f_{i,i+1}\}) + \sup\{\dim X_i \mid i \in \mathbb{N}\}.$$

This is a generalization of the results of Banič, Nall and Ingram [1, 8, 9, 10]. Moreover, we evaluate lower bounds of dimensions of some inverse limits of 1-dimensional compacta with set-valued functions. Also, we study the shape of inverse limits with cell-like set-valued functions. Our Theorem 5.2 is a generalization of the results of Charatonik and Roe [3] and Ingram [8].

2. Definitions and notation. In this paper, we assume that all spaces considered are separable metric spaces, and maps (= mappings) are continuous functions. A *compactum* is a compact metric space. A *continuum* is a connected compactum. Let $f, g : X \to Y$ be maps from a compactum X to a metric space (Y, d). Then we define $d(f, g) = \sup\{d(f(x), g(x)) \mid x \in X\}$. Let $\epsilon > 0$. A map $g : X \to Y$ is an *ϵ-map* if the diameter of $g^{-1}(y)$ is less than ϵ for each $y \in g(X)$. We use $\dim X$ for the topological dimension of X.

Let X and Y be compacta. Let 2^X be the collection of all nonempty closed subsets of X and $C(X)$ the collection of all nonempty subcontinua of X. A function $f : X \to 2^Y$ is said to be *upper semicontinuous* if for any $x \in X$ and any open neighborhood V of $f(x)$, there is an open neighborhood

U of x such that if $x' \in U$, then $f(x') \subset V$. A function $f : X \to 2^Y$ is said to be *surjective* if $f(X) = Y$, where $f(A) = \bigcup\{f(x) \mid x \in A\}$ for a subset A of X. If $f : X \to 2^Y$ and $g : Y \to 2^Z$ are set-valued functions, then we define the composition $gf : X \to 2^Z$ by

$$gf(x) = g(f(x)) \ (= \bigcup\{g(y) \mid y \in f(x)\})$$

for $x \in X$. For a function $f : X \to 2^Y$, we set

$$D_1(f) = \{x \in X \mid \dim f(x) \geq 1\}, \qquad D_1(f^{-1}) = \{y \in Y \mid \dim f^{-1}(y) \geq 1\},$$

where $f^{-1}(B) = \{x \in X \mid f(x) \cap B \neq \emptyset\}$ for a subset B of Y.

Let $\mathbb{N} = \{1, 2, \ldots\}$. Let X_i $(i \in \mathbb{N})$ be a sequence of compacta and let $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ be an upper semicontinuous function for each $i \in \mathbb{N}$. The *inverse limit* of the inverse sequence $\{X_i, f_{i,i+1}\}_{i=1}^{\infty}$ is the space

$$\varprojlim\{X_i, f_{i,i+1}\} = \{(x_i)_{i=1}^{\infty} \mid x_i \in f_{i,i+1}(x_{i+1}) \text{ for each } i \in \mathbb{N}\} \subset \prod_{i=1}^{\infty} X_i$$

which has the topology of a subspace of the product space $\prod_{i=1}^{\infty} X_i$.

For $i \leq j$, we define $f_{i,j} : X_j \to 2^{X_i}$ by $f_{i,i} = \mathrm{id}$ and

$$f_{i,j} = f_{i,i+1} f_{i+1,i+2} \cdots f_{j-1,j} : X_j \to 2^{X_i} \qquad (i < j).$$

In particular, if $f : X \to 2^X$ is upper semicontinuous, we consider the inverse sequence $\{X, f\} = \{X_i, f_{i,i+1}\}$, where $X_i = X$, $f_{i,i+1} = f$ $(i \in \mathbb{N})$. We set

$$\varprojlim\{X, f\} = \{(x_i)_{i=1}^{\infty} \mid x_i \in f(x_{i+1}) \text{ for each } i \in \mathbb{N}\}.$$

Let $\{X_i, f_{i,i+1}\}_{i=1}^{\infty}$ be an inverse sequence with set-valued functions. For each $m \leq n$, we set

$$\begin{aligned} &G(f; m, \ldots, n) \\ &\quad = \Bigl\{(x_i)_{i=m}^{n} \in \prod_{i=m}^{n} X_i \Bigm| x_i \in f_{i,i+1}(x_{i+1}) \text{ for each } m \leq i \leq n-1\Bigr\}. \end{aligned}$$

In particular,

$$G(f_{1,2}) = G(f; 1, 2) = \{(x_1, x_2) \in X_1 \times X_2 \mid x_1 \in f_{1,2}(x_2)\}$$

is the graph of $f_{1,2} : X_2 \to 2^{X_1}$. Let $p_j : G(f; m, \ldots, n) \to X_j$ $(m \leq j \leq n)$ be the natural jth projection.

Let $y \in X_n$ and $x \in X_{n'}$ $(n \leq n')$. We consider the following conditions:

$$\begin{aligned} \boxed{y \leftarrow x} &: y \in f_{n,n'}(x), \\ \boxed{x \lhd} &: x \in D_1(f_{n',n'+1}^{-1}), \\ \boxed{\rhd y} &: n \geq 2 \text{ and } y \in D_1(f_{n-1,n}). \end{aligned}$$

Also, let $x \in X_m$ and $y \in X_{m'}$ $(m + 2 \leq m')$. We consider the condition

$$\boxed{x \leftarrowtail \rhd y} : \ y \in D_1(f_{m'-1,m'}) \text{ and } \dim[f_{m,m'-1}^{-1}(x) \cap f_{m'-1,m'}(y)] \geq 1.$$

For each $x_n \in X_n$ with $x_n \in D_1(f^{-1}_{n,n+1})$, we consider the sequence

$$\rhd y_{m_1} \leftarrowtail \rhd y_{m_2} \leftarrowtail \cdots \leftarrowtail \rhd y_{m_{k-1}} \leftarrowtail \rhd y_{m_k} \leftarrow x_n \lhd$$

where $2 \leq m_1, m_k \leq n$, $m_i + 2 \leq m_{i+1}$ $(i = 1, \ldots, k-1)$ and $y_{m_i} \in X_{m_i}$ $(i = 1, \ldots, k)$. In this case, we say that the sequence $\{y_{m_i}, x_n \mid 1 \leq i \leq k\}$ is an *expand-contract sequence* in $\{X_i, f_{i,i+1}\}_{i=1}^{\infty}$ of length k. By use of such sequences, we define

$$J(\{X_i, f_{i,i+1}\}) = \sup\{k \in \mathbb{N} \mid \text{there is an expand-contract sequence in } \{X_i, f_{i,i+1}\}_{i=1}^{\infty} \text{ of length } k\}.$$

If there is no such sequence, we set $J(\{X_i, f_{i,i+1}\}) = 0$.

3. Dimensions of inverse limits of 1-dimensional compacta with set-valued functions.

Dimensions of inverse limits with set-valued functions have been studied by Banič, Nall and Ingram (see [1, 8, 10]). In this section, we consider the 1-dimensional case.

Theorem 3.1 (Banič [1]). *Suppose that X is a continuum and A a closed subset of X. Let $g : X \to X$ be a (continuous) map. If $f : X \to 2^X$ is the upper semicontinuous function defined by $G(f) = G(g) \cup (A \times X)$, then* $\dim \varprojlim \{X, f\} \in \{\dim X, \infty\}$.

Theorem 3.2 (Nall [10]). *Let X_i $(i \in \mathbb{N})$ be a sequence of compacta and let $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ be an upper semicontinuous function for each $i \in \mathbb{N}$ such that one of the following conditions is satisfied:*

(1) $\dim f_{i,i+1}(x) = 0$ *for each $i \in \mathbb{N}$ and $x \in X_{i+1}$, i.e., $D_1(f_{i,i+1}) = \emptyset$.*
(2) $\dim f^{-1}_{i,i+1}(x) = 0$ *for each $i \in \mathbb{N}$ and $x \in X_i$, i.e., $D_1(f^{-1}_{i,i+1}) = \emptyset$.*

Then $\dim \varprojlim \{X_i, f_{i,i+1}\} \leq \sup\{\dim X_i \mid i \in \mathbb{N}\}$.

Theorem 3.3 (Ingram [8]). *Let X_i $(i \in \mathbb{N})$ be a sequence of compacta and let $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ be an upper semicontinuous function for each $i \in \mathbb{N}$. If for each $i > 0$, Z_i is a closed 0-dimensional subset of X_i such that $g_{i,i+1} = f_{i,i+1} \mid (X_{i+1} - Z_{i+1})$ is a mapping and $f^{-1}_{i,j}(Z_i)$ is 0-dimensional for each $i \geq 2$ and $j > i$, then* $\dim \varprojlim \{X_i, f_{i,i+1}\} \leq \sup\{\dim X_i \mid i \in \mathbb{N}\}$.

In this section, under the assumption $\dim D_1(f_{i,i+1}) \leq 0$ $(i \in \mathbb{N})$, we will prove the following theorem concerning the 1-dimensional case.

Theorem 3.4. *Let X_i $(i \in \mathbb{N})$ be a sequence of 1-dimensional compacta and let $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ be an upper semicontinuous function for each $i \in \mathbb{N}$. Suppose that $\dim D_1(f_{i,i+1}) \leq 0$ $(i \in \mathbb{N})$. Then*

$$\dim \varprojlim \{X_i, f_{i,i+1}\} \leq J(\{X_i, f_{i,i+1}\}) + 1.$$

To prove this theorem, we need the following well-known theorems.

THEOREM 3.5 (Hurewicz's theorem [5, p. 242]). *If* $f : X \to Y$ *is a closed mapping between separable metric spaces and there is* $k \geq 0$ *such that* $\dim f^{-1}(y) \leq k$ *for each* $y \in Y$*, then* $\dim X \leq \dim Y + k$.

THEOREM 3.6 (Vaĭnšteĭn's theorem [5, p. 244]). *Suppose that* $f : X \to Y$ *is a closed mapping between separable metric spaces. If* $\dim Y \leq n$ *and* $\dim D_i(f^{-1}) \leq n - i$ *for each* $i = 1, \ldots, n+1$*, then* $\dim X \leq n$*, where* $D_i(f^{-1}) = \{y \in Y \mid \dim f^{-1}(y) \geq i\}$. *In particular, if* $f : X \to Y$ *is a closed mapping between separable metric spaces such that* $\dim Y \leq 1$ *and* $\dim D_1(f^{-1}) + \dim f^{-1}(y) \leq 1$ *for each* $y \in Y$*, then* $\dim X \leq 1$.

Proof of Theorem 3.4. We may assume that $J(\{X_i, f_{i,i+1}\}) = k < \infty$. First we will prove the following claim:

(∗) If X_i $(i \in \mathbb{N})$ is a sequence of 1-dimensional compacta and $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ is an upper semicontinuous function for each $i \in \mathbb{N}$ satisfying $\dim D_1(f_{i,i+1}) \leq 0$ $(i \in \mathbb{N})$, then for any $r \geq 2$,
$$\dim G(f; 1, \ldots, r) \leq J(\{X_i, f_{i,i+1}\}) + 1.$$

We proceed by induction on $J(\{X_i, f_{i,i+1}\}) = k$.

CASE (0): $J(\{X_i, f_{i,i+1}\}) = 0$. We will prove by induction on $r \geq 2$ that
$$\dim G(f; 1, \ldots, r) \leq 1.$$

Let $r = 2$. Consider the projection $p_2 : G(f; 1, 2) = G(f_{1,2}) \to X_2$. Since the graph $G(f_{1,2})$ of the upper semicontinuous function $f_{1,2}$ is closed in $X_1 \times X_2$, p_2 is a closed mapping. Note that $\dim D_1(p_2^{-1}) = \dim D_1(f_{1,2}) \leq 0$. Applying Theorem 3.6 to $f = p_2$ and $n = \max\{\dim X_1, \dim X_2\}$, we see that
$$\dim G(f; 1, 2) \leq \max\{\dim X_1, \dim X_2\} \ (= 1).$$

Next we assume that for r (≥ 2)
$$\dim G(f; 1, \ldots, r) \leq 1.$$

We must prove that
$$\dim G(f; 1, \ldots, r+1) \leq 1.$$

By the assumption, we may assume that
$$\dim G(f; 2, \ldots, r+1) \leq 1.$$

Consider the projection $q : G(f; 1, \ldots, r+1) \to G(f; 2, \ldots, r+1)$ defined by $q(x_1, \ldots, x_{r+1}) = (x_2, \ldots, x_{r+1})$. It is a closed mapping and
$$D_1(q^{-1}) = \{(x_2, \ldots, x_{r+1}) \in G(f; 2, \ldots, r+1) \mid x_2 \in D_1(f_{1,2})\}.$$

Note that $\dim D_1(f_{1,2}) \leq 0$. Since $J(\{X_i, f_{i,i+1}\}) = 0$, we see that for any $x_2 \in D_1(f_{1,2})$ and $x_j \in X_j$ $(j \geq 2)$ with $x_2 \leftarrow x_j$, we have $x_j \notin D_1(f_{j,j+1}^{-1})$.

Hence $\dim f_{j,j+1}^{-1}(z) \leq 0$ for $2 \leq j \leq r$ and $z \in p_j(D_1(q^{-1}))$. For each $2 \leq j \leq r$, consider the space

$$\tilde{G}(f;1,\ldots,j) = \{(x_2,\ldots,x_j) \in G(f;2,\ldots,j) \mid x_2 \in D_1(f_{1,2})\}$$

and the projection $q' : \tilde{G}(f;1,\ldots,j+1) \to \tilde{G}(f;1,\ldots,j)$, $q'(x_2,\ldots,x_{j+1}) = (x_2,\ldots,x_j)$. Then q' is 0-dimensional, i.e., $\dim(q')^{-1}(x_2,\ldots,x_j) \leq 0$ for each $(x_2,\ldots,x_j) \in \tilde{G}(f;1,\ldots,j)$. Applying Theorem 3.5 to $f = q'$, we see that

$$\begin{aligned} 0 \geq \dim D(f_{1,2}) &= \dim \tilde{G}(f;1,2) \\ &= \dim \tilde{G}(f;1,2,3) = \cdots = \dim \tilde{G}(f;1,\ldots,r+1) = \dim D_1(q^{-1}). \end{aligned}$$

By the assumption, $\dim G(f;2,\ldots,r+1) \leq 1$. Now Theorem 3.6 for $f = q$ and $n = 1$ yields

$$\dim G(f;1,\ldots,r+1) \leq 1.$$

Case (k): $J(\{X_i, f_{i,i+1}\}) = k$ $(k \geq 1)$. We assume that $(*)$ is true for $J(\{Y_i, g_{i,i+1}\}) \leq k-1$, i.e., if Y_i $(i \in \mathbb{N})$ is a sequence of 1-dimensional compacta and $g_{i,i+1} : Y_{i+1} \to 2^{Y_i}$ is an upper semicontinuous function for each $i \in \mathbb{N}$ satisfying $\dim D_1(g_{i,i+1}) \leq 0$ $(i \in \mathbb{N})$ and $J(\{Y_i, g_{i,i+1}\}) \leq k-1$, then for any $r \in \mathbb{N}$,

$$\dim G(g;1,\ldots,r) \leq J(\{Y_i, g_{i,i+1}\}) + 1.$$

We will show that $(*)$ is true for $J(\{X_i, f_{i,i+1}\}) = k$. Let $r = 2$. As before,

$$\dim G(f;1,2) \leq \max\{\dim X_1, \dim X_2\} \leq J(\{X_i, f_{i,i+1}\}) + 1.$$

Now, we suppose that for some r (≥ 2),

$$\dim G(f;1,\ldots,r) \leq J(\{X_i, f_{i,i+1}\}) + 1.$$

We must show that

$$\dim G(f;1,\ldots,r+1) \leq J(\{X_i, f_{i,i+1}\}) + 1.$$

Consider the projection $q : G(f;1,\ldots,r+1) \to G(f;2,\ldots,r+1)$. Recall that

$$D_1(q^{-1}) = \{(x_2,\ldots,x_{r+1}) \in G(f;2,\ldots,r+1) \mid x_2 \in D_1(f_{1,2})\}$$

and $\dim D_1(f_{1,2}) \leq 0$.

Fix $z \in D_1(f_{1,2}) \subset X_2$ and consider the inverse sequence $\{Y_i, g_{i,i+1}\}_{i=2}^{\infty}$, where $Y_i = f_{2,i}^{-1}(z)$ and $g_{i,i+1} : Y_{i+1} \to 2^{Y_i}$ is defined by $g_{i,i+1}(x) = f_{i,i+1}(x) \cap Y_i$. Note that $\dim Y_i \leq 1$ for each $i \in \mathbb{N}$. Since $J(\{X_i, f_{i,i+1}\}) = k$ and $z \in D_1(f_{1,2})$, we see that $J(\{Y_i, g_{i,i+1}\}_{i=2}^{\infty}) \leq k-1$, and hence by the inductive assumption, for $z \in D_1(f_{1,2})$,

$$\begin{aligned} \dim\{(z,x_3,\ldots,x_{r+1}) \in G(f;2,\ldots,r+1)\} &= \dim G(g;2,\ldots,r+1) \\ &\leq (k-1)+1 = k. \end{aligned}$$

Since $\dim D_1(f_{1,2}) \leq 0$, by Theorem 3.5 we see that $\dim D_1(q^{-1}) \leq k$. Note that

$$\dim G(f; 2, \ldots, r+1) \leq k+1.$$

Applying Theorem 3.6 to $f = q$ and $n = k+1$, we conclude that

$$\dim G(f; 1, \ldots, r+1) \leq k+1.$$

Hence $(*)$ is true for any $r \geq 2$.

For any $\epsilon > 0$ there exists a sufficiently large $r \in \mathbb{N}$ such that the natural projection $p : \varprojlim\{X_i, f_{i,i+1}\} \to G(f; 1, \ldots, r)$ is an ϵ-map. By [5, p. 85, Theorem 1.10.12], we conclude that

$$\dim \varprojlim\{X_i, f_{i,i+1}\} \leq J(\{X_i, f_{i,i+1}\}) + 1.$$

This completes the proof. ■

Let $\{X_i, f_{i,i+1}\}$ be an inverse sequence of compacta with upper semicontinuous functions. Note that $J(\{X_i, f_{i,i+1}\}) = 0$ if and only if

$$f_{i+1,j}(D_1(f_{j,j+1}^{-1})) \cap D_1(f_{i,i+1}) = \emptyset$$

for each $i+1 \leq j$. In this case, we have the following result.

COROLLARY 3.7. *Let X_i $(i \in \mathbb{N})$ be a sequence of* 1*-dimensional compacta and let $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ be an upper semicontinuous function for each $i \in \mathbb{N}$. Suppose that* $\dim D_1(f_{i,i+1}) \leq 0$ $(i \in \mathbb{N})$ *and* $J(\{X_i, f_{i,i+1}\}) = 0$. *Then*

$$\dim \varprojlim\{X_i, f_{i,i+1}\} \leq 1.$$

EXAMPLE 1. In Corollary 3.7, the assumption $\dim D_1(f_{i,i+1}) \leq 0$ $(i \in \mathbb{N})$ is necessary. Let $f : I \to C(I)$ be the surjective upper semicontinuous function defined by $f(x) = 0$ $(x \in [0, 3/4))$, $f(x) = [0, 1/4]$ $(x \in [3/4, 1))$, $f(1) = [0, 1]$. Then $J(\{I, f\}) = 0$ and $1 < 2 = \dim \varprojlim\{I, f\}$. In fact, $\dim D_1(f) = 1$.

For the following special cases, we can evaluate lower bounds of dimensions of some inverse limits of set-valued functions.

THEOREM 3.8. *Let X_i $(i \in \mathbb{N})$ be a sequence of* 1*-dimensional compacta and let $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ be a surjective upper semicontinuous function for each $i \in \mathbb{N}$. Suppose that for each $i \geq 2$, Z_i is a* 0*-dimensional closed subset of X_i such that $f_{i,i+1}|X_{i+1} - Z_{i+1} : X_{i+1} - Z_{i+1} \to X_i$ is a mapping for all $i \in \mathbb{N}$. If $J(\{X_i, f_{i,i+1}\}) = k$, then*

$$k \leq \dim \varprojlim\{X_i, f_{i,i+1}\} \leq k+1.$$

Moreover, if there is an expand-contract sequence

$$\rhd y_{m_1} \leftarrowtail \rhd y_{m_2} \leftarrowtail \rhd \cdots \leftarrowtail \rhd y_{m_{k-1}} \leftarrowtail \rhd y_{m_k} \leftarrow x_n \lhd$$

in $\{X_i, f_{i,i+1}\}$ *of length* $J(\{X_i, f_{i,i+1}\}) = k$ *such that* $\dim \pi_n^{-1}(x_n) > 0$, *then*

$$\dim \varprojlim \{X_i, f_{i,i+1}\} = k + 1,$$

where $\pi_n : \varprojlim \{X_i, f_{i,i+1}\}_{i \geq n} \to X_n$ *is defined by* $\pi_n(x_n, x_{n+1}, \ldots) = x_n$.

Proof. By Theorem 3.4, we have $\dim \varprojlim \{X_i, f_{i,i+1}\} \leq k+1$. Set $g_{i,i+1} = f_{i,i+1} \,|\, X_{i+1} - Z_{i+1}$. For $y_i \in X_i$ and $y_j \in X_j$ $(i+2 \leq j)$, we consider the condition

$\boxed{y_i \lhd \overset{\text{map}}{\Longleftarrow} \rhd y_j}$: There exists a (nondegenerate) subcontinuum C in $f_{j-1,j}(y_j)$ such that

$$C \subset X_{j-1} - \bigcup_{m=i+1}^{j-1} g_{m,j-1}^{-1}(Z_m),$$

$g_{m,j-1}|C : C \to g_{m,j-1}(C) \subset X_m$ is a mapping for $i+1 \leq m \leq j-1$ and $g_{i+1,j-1}(C) \subset f_{i,i+1}^{-1}(y_i)$.

Let $y_i \in X_i$ and $y_j \in X_j$ $(i+2 \leq j)$ be such that $y_i \leftarrowtail \rhd y_j$. We will show that there is $y_{i'} \in X_{i'}$ such that $i \leq i', i'+2 \leq j$ and

$$y_i \leftarrow y_{i'} \lhd \overset{\text{map}}{\Longleftarrow} \rhd y_j.$$

Since $y_i \leftarrowtail \rhd y_j$, there is a nondegenerate continuum D in $f_{j-1,j}(y_j)$ such that $y_i \in f_{i,j-1}(z)$ for any $z \in D$. Since Z_{j-1} is a 0-dimensional closed set, there is a nondegenerate subcontinuum C_1 in $D - Z_{j-1}$. Note that $g_{j-2,j-1}|C_1$ is a mapping. If $g_{j-2,j-1}(C_1)$ is a one-point set, we set $y_{i'} = g_{j-2,j-1}(C_1)$. If $g_{j-2,j-1}(C_1)$ is nondegenerate, we choose a nondegenerate subcontinuum C_2 in $C_1 - g_{j-2,j-1}^{-1}(Z_{j-2})$. Then $g_{j-3,j-1}|C_2$ is a mapping. If we continue this procedure finitely many times, we obtain the desired point $y_{i'}$.

Since $J(\{X_i, f_{i,i+1}\}) = k$, there is an expand-contract sequence

$$\rhd y_{m_1} \leftarrowtail \rhd y_{m_2} \leftarrowtail \rhd \cdots \leftarrowtail \rhd y_{m_{k-1}} \leftarrowtail \rhd y_{m_k} \leftarrow x_n \lhd$$

in $\{X_i, f_{i,i+1}\}$ of length k. Hence we have the following sequence:

$$\begin{aligned} \rhd\, y_{m_1} \leftarrow y_{m'_1} \lhd \overset{\text{map}}{\Longleftarrow} \rhd y_{m_2} \leftarrow y_{m'_2} \lhd \overset{\text{map}}{\Longleftarrow} \rhd y_{m_3} \leftarrow \cdots \overset{\text{map}}{\Longleftarrow} \rhd y_{m_{k-1}} \\ \leftarrow y_{m'_{k-1}} \lhd \overset{\text{map}}{\Longleftarrow} \rhd y_{m_k} \leftarrow x_n \lhd . \end{aligned}$$

Note that $f_{i,i+1}$ is surjective. We can choose and fix points as follows:

$$\begin{aligned} &(x_n, x_{n+1}, \ldots) \in \pi_n^{-1}(x_n), \\ &(y_{m_k}, \ldots, x_n) \in G(f; m_k, \ldots, n), \\ &(y_{m_i}, \ldots, y_{m'_i}) \in G(f; m_i, \ldots, m'_i) \quad (i = 1, \ldots, k-1). \end{aligned}$$

Since $y_{m'_{i-1}} \lhd \overset{\text{map}}{\Longleftarrow} \rhd y_{m_i}$ $(i = 2, \ldots, k)$, there are nondegenerate subcontinua A_{m_i-1} in $f_{m_i-1,m_i}(y_{m_i})$ as in the definition of the condition $y_{m'_{i-1}} \lhd \overset{\text{map}}{\Longleftarrow} \rhd y_{m_i}$.

Also, there is a nondegenerate subcontinuum A_{m_1-1} of $f_{m_1-1,m_1}(y_{m_1})$ such that one of the following properties is satisfied;

(i) $g_{i,m_1-1}|A_{m_1-1}$ is a mapping for each $1 \le i \le m_1 - 2$.
(ii) There is $y_{m''_1} \in X_{m''_1}$ such that $1 < m''_1 \le m_1 - 2$, $g_{i,m_1-1}|A_{m_1-1} : A_{m_1-1} \to X_i$ is a mapping for $m''_1 \le i \le m_1-2$ and $g_{m''_1,m_1-1}(A_{m_1-1}) = \{y_{m''_1}\}$.

Assume that case (i) holds. Consider the map

$$h : A_{m_1-1} \times \cdots \times A_{m_k-1} \to \varprojlim\{X_i, f_{i,i+1}\}$$

defined by

$$\begin{aligned}h(z_1, \ldots, z_k) = (&g_{1,m_1-1}(z_1), g_{2,m_1-1}(z_1), \ldots, z_1, y_{m_1}, \ldots, y_{m'_1},\\ &g_{m'_1+1,m_2-1}(z_2), g_{m'_1+2,m_2-1}(z_2), \ldots, z_2, y_{m_2}, \ldots, y_{m'_2}, \ldots,\\ &g_{m'_{k-1}+1,m_k-1}(z_k), g_{m'_{k-1}+2,m_k-1}(z_k), \ldots, z_k, y_{m_k}, \ldots, x_n, \ldots)\end{aligned}$$

for $z_i \in A_{m_i-1}$ $(i = 1, \ldots, k)$. We can easily see that h is continuous and moreover it is an embedding. Note that $\dim A_{m_i-1} = 1$. Hence

$$k = \dim(A_{m_1-1} \times \cdots \times A_{m_k-1}) \le \dim \varprojlim\{X_i, f_{i,i+1}\} \le k + 1.$$

In case (ii), we have a similar embedding $h : A_{m_1-1} \times \cdots \times A_{m_k-1} \to \varprojlim\{X_i, f_{i,i+1}\}$. Finally, if $\dim \pi_n^{-1}(x_n) > 0$, as above we have an embedding

$$h' : A_{m_1-1} \times \cdots \times A_{m_k-1} \times \pi_n^{-1}(x_n) \to \varprojlim\{X_i, f_{i,i+1}\}.$$

Hence $\dim \varprojlim\{X_i, f_{i,i+1}\} = k + 1$. ■

COROLLARY 3.9. *Let X_i $(i \in \mathbb{N})$ be a sequence of 1-dimensional compacta and let $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ be a surjective upper semicontinuous function for each $i \in \mathbb{N}$. Suppose that for each $i \ge 2$, Z_i is a 0-dimensional closed subset of X_i such that $f_{i,i+1}|X_{i+1} - Z_{i+1} : X_{i+1} - Z_{i+1} \to X_i$ is a mapping for all $i \in \mathbb{N}$. If $\dim \varprojlim\{X_i, f_{i,i+1}\} = 0$, then $J(\{X_i, f_{i,i+1}\}) = 0$. And if $\dim \varprojlim\{X_i, f_{i,i+1}\} = k$ $(k \ge 1)$, then $J(\{X_i, f_{i,i+1}\}) = k - 1$ or k.*

We give some examples related to the conditions of Theorem 3.8.

EXAMPLE 2. Let $g : I = [0, 1] \to I$ be the map defined by $g(x) = x/2$ and let $g' : I = [0, 1] \to I$ be the map defined by $g'(x) = x/2 + 1/2$. Let $f : I \to 2^I$ be the surjective upper semicontinuous function defined by $f(x) = \{g(x), g'(x)\}$. Then $\varprojlim\{I, f\}$ is a Cantor set and hence $\dim \varprojlim\{I, f\} = 0$.

EXAMPLE 3 (see [7, Example 5.4]). Let $f : I \to C(I)$ be the surjective upper semicontinuous function defined by $f(x) = 0$ $(x \in [0, 1/3))$, $f(1/3) = [0, 1/3]$, $f(x) = 1/3$ $(x \in (1/3, 2/3))$, $f(2/3) = [1/3, 2/3]$, $f(x) = 2/3$ $(x \in (2/3, 1))$, $f(1) = [2/3, 1]$. Note that $J(\{I, f\}) = 2$,

$$\rhd x_2 = 1/3 \leftarrowtail \rhd x_4 = 2/3 \leftarrow x_4 = 2/3 \lhd$$

and $\dim \pi_4^{-1}(x_4) > 0$. Also $\{I, f\}$ satisfies the condition of Theorem 3.8. Hence $\dim \varprojlim \{I, f\} = 3$. In fact, $\varprojlim \{I, f\}$ is a 3-cell with a fin (see [7, Example 5.4]).

Example 4 (see [7, Example 2.3]). Let $f : I \to C(I)$ be the surjective upper semicontinuous function defined by $f(0) = I$ and $f(x) = 0$ $(x \in (0, 1])$. In this case, for any $k \in \mathbb{N}$ we have the following expand-contract sequence of length k:

$$\rhd x_2 = 0 \leftarrowtail \rhd x_4 = 0 \leftarrowtail \rhd x_6 = 0 \leftarrowtail \cdots \leftarrowtail \rhd x_{2k-2} = 0 \leftarrowtail \rhd x_{2k} = 0 \leftarrow x_{2k} = 0 \lhd .$$

Then $J(\{I, f\}) = \infty$ and by Theorem 3.8, $\dim \varprojlim \{I, f\} = \infty$. In fact, $\varprojlim \{I, f\}$ contains a Hilbert cube.

Example 5. Let I_i $(i \in \mathbb{N})$ be a sequence of copies of the unit interval I and let C be a Cantor set in $[0, 1/2]$. Let $u : C \to [0, 1/2]$ be a surjective map. Consider the following surjective upper semicontinuous functions $f_{i,i+1} : I_{i+1} \to 2^{I_i}$ $(i \in \mathbb{N})$:

(1) $f_{1,2}(x) = u^{-1}(x)$ $(x \in [0, 1/2])$ and $f_{1,2}|[1/2, 1] : [1/2, 1] \to I$ is an onto map.
(2) $f_{2,3}(x) = x$ $(x \in [0, 1/2))$, $f_{2,3}(1/2) = [0, 1/2]$, $f_{2,3}(x) = x$ $(x \in (1/2, 1])$.
(3) $f_{3,4}(x) = x$ $(x \in [0, 1/2))$, $f_{3,4}(x) = \{1/2, x\}$ $(x \in [1/2, 1])$.

Also, we will construct $f_{i,i+1}$ $(i \geq 4)$ as follows. For any $\epsilon > 0$, we can construct a surjective upper semicontinuous function $f_\epsilon : [1/2, 1] \to 2^{[1/2,1]}$ such that for some sequence $1/2 = t_0 < t_1 < \cdots < t_{s-1} < t_s = 1$,

(a) $f_\epsilon(1/2) = 1/2$, $f_\epsilon(1) = 1$,
(b) $f_\epsilon|(t_i, t_{i+1})$ $(i = 1, \ldots, t_{s-1})$ is an injective map and $f_\epsilon([1/2, 1]) = [1/2, 1]$,
(c) $f_\epsilon(t_i)$ is a two-point set for $i = 1, \ldots, t_{s-1}$ and the diameter of each $G(f_\epsilon|(t_i, t_{i+1}))$ $(\subset G(f_\epsilon))$ is less than ϵ.

By use of the maps $f_\epsilon : [1/2, 1] \to 2^{[1/2,1]}$ for sufficiently small $\epsilon > 0$ and by induction on i (≥ 4) we can construct surjective upper semicontinuous functions $f_{i,i+1} : I_{i+1} \to 2^{I_i}$ such that $f_{i,i+1}|[0, 1/2] = \mathrm{id}$ and

$$\dim \varprojlim \{[1/2, 1], f_{i,i+1}|[1/2, 1]\}_{i=4}^{\infty} = 0.$$

Note that

$$\rhd x_3 = 1/2 \leftarrow x_3 = 1/2 \lhd \qquad (x_3 \in I_3).$$

In fact, $J(\{I_i, f_{i,i+1}\}) = 1$. Since $\dim \varprojlim \{[1/2, 1], f_{i,i+1}|[1/2, 1]\}_{i=4}^{\infty} = 0$, we see that $\dim \pi_3^{-1}(x_3) = 0$, and hence $\dim \varprojlim \{I_i, f_{i,i+1}\} = 0$. Note that only $f_{1,2}$ does not satisfy the conditions of Theorem 3.8 and Corollary 3.9, and the others satisfy those conditions.

Example 6. Let $f_{i,i+1} : I_{i+1} \to 2^{I_i}$ $(i \geq 2)$ be the surjective upper semicontinuous functions as in Example 5. The inverse sequence $\{I_i, f_{i,i+1}\}_{i=2}^{\infty}$ satisfies the conditions of Theorem 3.8 and Corollary 3.9. In this case, $\dim \varprojlim \{I_i, f_{i,i+1}\}_{i=2}^{\infty} = 1$ and $J(\{I_i, f_{i,i+1}\}_{i=2}^{\infty}) = 1$.

4. Dimensions of inverse limits of higher-dimensional compacta with set-valued functions. For higher-dimensional cases, to find more precise upper bounds of dimensions of inverse limits $\varprojlim \{X_i, f_{i,i+1}\}$ with upper semicontinuous functions, we need a more precise and complicated index $\tilde{J}(\{X_i, f_{i,i+1}\})$. Let $\{X_i, f_{i,i+1}\}_{i=1}^{\infty}$ be an inverse sequence with set-valued functions. For any expand-contract sequence

$$S : \triangleright y_{m_1} \leftarrowtail \triangleright y_{m_2} \leftarrowtail \triangleright \cdots \leftarrowtail \triangleright y_{m_{k-1}} \leftarrowtail \triangleright y_{m_k} \leftarrow x_n \triangleleft$$

we set $d(S) = \sum_{i=1}^{k} \dim f_{m_i - 1, m_i}(y_{m_i})$. We define

$$\tilde{J}(\{X_i, f_{i,i+1}\}) = \sup\{d(S) \mid S \text{ is an expand-contract sequence in } \{X_i, f_{i,i+1}\}_{i=1}^{\infty}\}.$$

Note that $\tilde{J}(\{X_i, f_{i,i+1}\}) \geq J(\{X_i, f_{i,i+1}\})$. If each X_i is 1-dimensional, then $\tilde{J}(\{X_i, f_{i,i+1}\}) = J(\{X_i, f_{i,i+1}\})$.

The following is a more precise result for higher-dimensional cases.

Theorem 4.1. *Let* X_i $(i \in \mathbb{N})$ *be a sequence of compacta and let* $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ *be an upper semicontinuous function for each* $i \in \mathbb{N}$. *Suppose that* $\dim D_1(f_{i,i+1}) \leq 0$ $(i \in \mathbb{N})$. *Then*

$$\dim \varprojlim \{X_i, f_{i,i+1}\} \leq \tilde{J}(\{X_i, f_{i,i+1}\}) + \sup\{\dim X_i \mid i \in \mathbb{N}\}.$$

Proof. The proof is similar to the proof of Theorem 3.4. We use Vaĭnšteĭn's theorem (Theorem 3.6) more precisely than in the proof of Theorem 3.4. For completeness, we give the precise proof.

We may assume that $\sup\{\dim X_i \mid i \in \mathbb{N}\} = m < \infty$ and $\tilde{J}(\{X_i, f_{i,i+1}\}) = k < \infty$. We prove the following claim:

(∗) If X_i $(i \in \mathbb{N})$ is a sequence of compacta and $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ is an upper semicontinuous function for each $i \in \mathbb{N}$ satisfying $\dim D_1(f_{i,i+1}) \leq 0$ $(i \in \mathbb{N})$, then for any $r \geq 2$,

$$\dim G(f; 1, \ldots, r) \leq \tilde{J}(\{X_i, f_{i,i+1}\}) + \sup\{\dim X_i \mid i \in \mathbb{N}\} = n.$$

We proceed by induction on $\tilde{J}(\{X_i, f_{i,i+1}\}) = k$.

Case (0): $\tilde{J}(\{X_i, f_{i,i+1}\}) = 0$. We will prove by induction on $r \geq 2$ that

$$\dim G(f; 1, \ldots, r) \leq \sup\{\dim X_i \mid i \in \mathbb{N}\}.$$

Let $r = 2$. The projection $p_2 : G(f; 1, 2) = G(f_{1,2}) \to X_2$ is a closed mapping and $\dim D_1(p_2^{-1}) = \dim D_1(f_{1,2}) \leq 0$. We apply Theorem 3.6 to $f = p_2$ and

$n = \max\{\dim X_1, \dim X_2\}$ to get

$$\dim G(f;1,2) \le \max\{\dim X_1, \dim X_2\} \le \sup\{\dim X_i \mid i \in \mathbb{N}\}.$$

Next we assume that for some r (≥ 2),

$$\dim G(f;1,\dots,r) \le \sup\{\dim X_i \mid i \in \mathbb{N}\}.$$

We must prove that

$$\dim G(f;1,\dots,r+1) \le \sup\{\dim X_i \mid i \in \mathbb{N}\}.$$

By the assumption, we may assume that

$$\dim G(f;2,\dots,r+1) \le \sup\{\dim X_i \mid i \in \mathbb{N}\}.$$

Consider the projection $q : G(f;1,\dots,r+1) \to G(f;2,\dots,r+1)$ defined by $q(x_1,\dots,x_{r+1}) = (x_2,\dots,x_{r+1})$. We have

$$D_1(q^{-1}) = \{(x_2,\dots,x_{r+1}) \in G(f;2,\dots,r+1) \mid x_2 \in D_1(f_{1,2})\}.$$

Note that $\dim D_1(f_{1,2}) \le 0$. Since $\tilde{J}(\{X_i, f_{i,i+1}\}) = 0$, we see that for any $x_2 \in D_1(f_{1,2})$ and $x_j \in X_j$ $(j \ge 2)$ with $x_2 \leftarrow x_j$, we have $x_j \notin D_1(f^{-1}_{j,j+1})$. Hence $\dim f^{-1}_{j,j+1}(z) \le 0$ for $2 \le j \le r$ and $z \in p_j(D_1(q^{-1}))$. As in the proof of Theorem 3.4, we find that $\dim D_1(q^{-1}) \le 0$. By the assumption, $\dim G(f;2,\dots,r+1) \le \sup\{\dim X_i \mid i \in \mathbb{N}\}$. Theorem 3.6 applied to $f = q$ and $n = \sup\{\dim X_i \mid i \in \mathbb{N}\}$ yields

$$\dim G(f;1,\dots,r+1) \le \sup\{\dim X_i \mid i \in \mathbb{N}\}.$$

Case (k): $\tilde{J}(\{X_i, f_{i,i+1}\}) = k$ $(k \ge 1)$. We assume that $(*)$ is true for $\tilde{J}(\{Y_i, g_{i,i+1}\}) \le k-1$, i.e., if Y_i $(i \in \mathbb{N})$ is a sequence of compacta and $g_{i,i+1} : Y_{i+1} \to 2^{Y_i}$ is an upper semicontinuous function for each $i \in \mathbb{N}$ satisfying $\dim D_1(g_{i,i+1}) \le 0$ $(i \in \mathbb{N})$ and $\tilde{J}(\{Y_i, g_{i,i+1}\}) \le k-1$, then for any $r \in \mathbb{N}$,

$$\dim G(g;1,\dots,r) \le \tilde{J}(\{Y_i, g_{i,i+1}\}) + \sup\{\dim Y_i \mid i \in \mathbb{N}\}.$$

We will show that $(*)$ is true for $\tilde{J}(\{X_i, f_{i,i+1}\}) = k$. Let $r = 2$. As before,

$$\begin{aligned}\dim G(f;1,2) &\le \max\{\dim X_1, \dim X_2\}\\ &\le \tilde{J}(\{X_i, f_{i,i+1}\}) + \sup\{\dim X_i \mid i \in \mathbb{N}\}.\end{aligned}$$

Now, we suppose that for some r (≥ 2),

$$\dim G(f;1,\dots,r) \le \tilde{J}(\{X_i, f_{i,i+1}\}) + \sup\{\dim X_i \mid i \in \mathbb{N}\}.$$

We must show that

$$\dim G(f;1,\dots,r+1) \le \tilde{J}(\{X_i, g_{i,i+1}\}) + \sup\{\dim X_i \mid i \in \mathbb{N}\}.$$

Consider the projection $q : G(f;1,\dots,r+1) \to G(f;2,\dots,r+1)$. Recall that

$$D_1(q^{-1}) = \{(x_2,\dots,x_{r+1}) \in G(f;2,\dots,r+1) \mid x_2 \in D_1(f_{1,2})\}$$

and $\dim D_1(f_{1,2}) \le 0$.

Fix $z \in D_1(f_{1,2}) \subset X_2$. Consider the inverse sequence $\{Y_i, g_{i,i+1}\}_{i=2}^{\infty}$, where $Y_i = f_{2,i}^{-1}(z)$ and $g_{i,i+1} : Y_{i+1} \to 2^{Y_i}$ is defined by $g_{i,i+1}(x) = f_{i,i+1}(x) \cap Y_i$. Since $\tilde{J}(\{X_i, f_{i,i+1}\}) = k$ and $z \in D_1(f_{1,2})$, we see that $\tilde{J}(\{Y_i, g_{i,i+1}\}_{i=2}^{\infty}) \le k - \dim f_{1,2}(z)$, and hence by the inductive assumption, for the fixed $z \in D_1(f_{1,2})$,

$$\begin{aligned}\dim\{(z,x_3,\dots,x_{r+1}) \in G(f;2,\dots,r+1)\} &= \dim G(g;2,\dots,r+1)\\ &\le (k - \dim f_{1,2}(z)) + \sup\{\dim Y_i \mid i \in \mathbb{N}\}\\ &\le (k - \dim f_{1,2}(z)) + \sup\{\dim X_i \mid i \in \mathbb{N}\}.\end{aligned}$$

Recall $D_i(q^{-1}) = \{(z,x_3,\dots,x_{r+1}) \in G(f;2,\dots,r+1) \mid \dim f_{1,2}(z) \ge i\}$. By Theorem 3.5, for each $i \ge 1$,

$$\dim D_i(q^{-1}) \le (k-i) + \sup\{\dim X_i \mid i \in \mathbb{N}\}.$$

Vaĭnšteĭn's theorem (Theorem 3.6) applied to $f = q$ and $n = \tilde{J}(\{X_i, f_{i,i+1}\}) + \sup\{\dim X_i \mid i \in \mathbb{N}\}$ yields

$$\dim G(f;1,\dots,r+1) \le \tilde{J}(\{X_i, f_{i,i+1}\}) + \sup\{\dim X_i \mid i \in \mathbb{N}\}.$$

Hence $(*)$ is true for any $r \ge 2$.

For any $\epsilon > 0$ there exists a sufficiently large $r \in \mathbb{N}$ such that the natural projection $p : \varprojlim\{X_i, f_{i,i+1}\} \to G(f;1,\dots,r)$ is an ϵ-map. By [5] again, we conclude that

$$\dim \varprojlim\{X_i, f_{i,i+1}\} \le \tilde{J}(\{X_i, f_{i,i+1}\}) + \sup\{\dim X_i \mid i \in \mathbb{N}\}.$$

This completes the proof. ■

Corollary 4.2. *Let X_i $(i \in \mathbb{N})$ be a sequence of compacta and let $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ be a surjective upper semicontinuous function for each $i \in \mathbb{N}$. Suppose that for each $i \ge 2$, Z_i is a 0-dimensional closed subset of X_i such that $f_{i,i+1} \,|\, X_{i+1} - Z_{i+1} : X_{i+1} - Z_{i+1} \to X_i$ is a mapping for all $i \in \mathbb{N}$. Then*

$$J(\{X_i, f_{i,i+1}\}) \le \dim \varprojlim\{X_i, f_{i,i+1}\} \le \tilde{J}(\{X_i, g_{i,i+1}\}) + \sup\{\dim X_i \mid i \in \mathbb{N}\}.$$

Proof. The proof is similar to the proof of Theorem 3.8. ■

Example 7. Under the assumption of Corollary 4.2, we cannot conclude

$$\tilde{J}(\{X_i, f_{i,i+1}\}) \le \dim \varprojlim\{X_i, f_{i,i+1}\}.$$

In fact, it is well-known that there is a 2-dimensional continuum P such that $\dim(P \times P) = 3$ (see [11]). Let $X_i = P$ $(i = 1,3)$, $X_i = \{*\}$ $(i = 2,4)$ and $X_i = [1/2, 1]$ $(i = 5,6,\dots)$. Let $f_{i,i+1} : X_{i+1} \to 2^{X_i}$ be defined as follows: $f_{1,2}(*) = P$, $f_{2,3}(P) = \{*\}$, $f_{3,4}(*) = P$, $f_{4,5}([1/2,1]) = \{*\}$ and $f_{i,i+1} : [1/2,1] \to [1/2,1]$ $(i \geq 5)$ are set-valued functions as in Example 4 satisfying $\dim \varprojlim \{[1/2,1], f_{i,i+1}\}_{i\geq 5} = 0$. Then $\{X_i, f_{i,i+1}\}$ satisfies the conditions of Corollary 4.2 and $J(\{X_i, f_{i,i+1}\}) = 2$. Note that $\dim \varprojlim \{X_i, f_{i,i+1}\} = \dim(P\times P) = 3$ and $\tilde{J}(\{X_i, f_{i,i+1}\}) = \dim P + \dim P = 4$.

5. Shape of inverse limits with cell-like set-valued functions. Let (X, d_X) and (Y, d_Y) be metric spaces. We assume that the product space $X \times Y$ has the fixed metric $d_{X\times Y}$ defined by $d_{X\times Y}((x,y),(x',y')) = d_X(x,x') + d_Y(y,y')$. Let $f : X \to 2^Y$ be a set-valued function and $\epsilon > 0$. A map $g : X \to Y$ is an *ϵ-selection* of f if $G(g) \subset U(G(f);\epsilon)$, where $U(G(f);\epsilon)$ is the ϵ-neighborhood of the graph $G(f)$ in $Y \times X$. If two compacta X and Y have the same shape (i.e., X and Y are shape equivalent), we write $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$. If X has trivial shape (i.e., X and the one-point set $*$ are shape equivalent), we write $\mathrm{Sh}(X) = \mathrm{Sh}(*)$. Recall that a continuum X in the Hilbert cube Q has trivial shape if and only if for any open neighborhood U of X in Q, X is contractible in U. Trivial shape of some inverse limits with set-valued functions was studied by Charatonik and Roe [3].

A set-valued function $f : X \to C(Y)$ is *cell-like* if $\mathrm{Sh}(f(x)) = \mathrm{Sh}(*)$ for each $x \in X$. In this section, we study the shape of inverse limits with cell-like set-valued functions (for shape theory, see e.g. [2] and [4]).

THEOREM 5.1 (Charatonik and Roe [3]). *Let X_i $(i \in \mathbb{N})$ be a sequence of finite-dimensional continua and let $f_{i,i+1} : X_{i+1} \to C(X_i)$ be an upper semicontinuous function for each $i \in \mathbb{N}$. Suppose that $f_{i,i+1}$ is cell-like and $\mathrm{Sh}(X_i) = \mathrm{Sh}(*)$ for each $i \in \mathbb{N}$. Then $\mathrm{Sh}(\varprojlim \{X_i, f_{i,i+1}\}) = \mathrm{Sh}(*)$.*

Now, we prove the following main theorem of this section.

THEOREM 5.2. *Let X_i $(i \in \mathbb{N})$ be a sequence of finite-dimensional compact ANRs and let $f_{i,i+1} : X_{i+1} \to C(X_i)$ be an upper semicontinuous function for each $i \in \mathbb{N}$. If $f_{i,i+1}$ is cell-like for each $i \in \mathbb{N}$, then $\mathrm{Sh}(\varprojlim \{X_i, f_{i,i+1}\}) = \mathrm{Sh}(\varprojlim \{X_i, s(f_{i,i+1})\})$, where $s(f_{i,i+1}) : X_{i+1} \to X_i$ is an ϵ_i-selection of $f_{i,i+1}$ such that $\epsilon_i > 0$ is so small that any two ϵ_i-selections of $f_{i,i+1}$ are homotopic.*

Proof. For each $i < j$, consider the space $G(f; i, \ldots, j)$ $(i < j)$ and the natural projection

$$p_B[i,j] : G(f; i, \ldots, j) \to G(f; i, \ldots, j-1)$$

defined by $p_B[i,j](x_i, \ldots, x_j) = (x_i, \ldots, x_{j-1})$. Note that

$$\varprojlim \{X_i, f_{i,i+1}\} = \varprojlim \{G(f; 1, \ldots, i,), p_B[1, i+1]\}.$$

Also let $p_F[i,j] : G(f; i, \ldots, j) \to G(f; i+1, \ldots, j)$ be the projection defined by

$$p_F[i,j](x_i,\dots,x_j) = (x_{i+1},\dots,x_j).$$

Note that $(p_F[i,j])^{-1}(x_{i+1},\dots,x_j)$ is homeomorphic to $f_{i,i+1}(x_{i+1})$, and hence $p_F[i,j]$ is a cell-like map (i.e. the set-valued function $(p_F[i,j])^{-1}$ is cell-like). Since $G(f;i,\dots,j)$ and $G(f;i+1,\dots,j)$ are finite-dimensional, $p_F[i,j]$ is a hereditary shape equivalence (see e.g. [4, Section 8]). In particular, $p_F[i,i+1] : G(f;i,i+1) = G(f_{i,i+1}) \to X_{i+1}$ is a hereditary shape equivalence.

Fix $i \in \mathbb{N}$. Note that $X_i \times X_{i+1}$ is an ANR and $\dim X_i + \dim X_{i+1} = n < \infty$. Since $p_F[i,i+1]$ is cell-like, there is a sufficiently small $\epsilon_i > 0$ such that any maps g, g' from any n-dimensional compactum to $U(G(f;i,i+1);\epsilon_i)$ with $d(\tilde{p}_F[i,i+1]\cdot g, \tilde{p}_F[i,i+1]\cdot g') < \epsilon_i$ are homotopic in $X_i \times X_{i+1}$ (see e.g. [4, Section 8]), where $\tilde{p}_F[i,i+1] : X_i \times X_{i+1} \to X_{i+1}$ denotes the projection. Also, we have a map $s_{i,i+1} : X_{i+1} \to U(G(f;i,i+1);\epsilon_i)$ such that $\tilde{p}_F[i,i+1]\cdot s_{i,i+1} = \mathrm{id}_{X_{i+1}}$. Set $s(f_{i,i+1}) = \tilde{p}_B[i,i+1]\cdot s_{i,i+1}$, where $\tilde{p}_B[i,i+1] : X_i \times X_{i+1} \to X_i$ is the projection. Note that $G(s(f_{i,i+1})) \subset U(G(f_{i,i+1});\epsilon_i)$ and $\epsilon_i > 0$ satisfy the property of Theorem 5.2. Then we have the following diagram commutative up to homotopy:

$$\begin{tikzcd}
X_i \arrow[d, "\mathrm{id}"] & G(f;i,i+1) \arrow[l, "p_B{[i,i+1]}"'] \arrow[d, "p_F{[i,i+1]}"] \\
X_i & X_{i+1} \arrow[l, "s(f_{i,i+1})"']
\end{tikzcd}$$

For each $k = 1,\dots,i-1$, we have the following commutative diagram:

$$\begin{tikzcd}
G(f;k,\dots,i) \arrow[d, "p_F{[k,i]}"] & G(f;k,\dots,i+1) \arrow[l, "p_B{[k,i+1]}"'] \arrow[d, "p_F{[k,i+1]}"] \\
G(f;k+1,\dots,i) & G(f;k+1,\dots,i+1) \arrow[l, "p_B{[k+1,i+1]}"']
\end{tikzcd}$$

The above diagrams yield the following diagram commutative up to homotopy:

$$\begin{tikzcd}
G(f;1,\dots,i) \arrow[d, "p_F"] & G(f;1,\dots,i+1) \arrow[l, "p_B{[1,i+1]}"'] \arrow[d, "p_F"'] \\
X_i & X_{i+1} \arrow[l, "s(f_{i,i+1})"']
\end{tikzcd}$$

where $p_F : G(f;1,\dots,i) \to X_i$ denotes the composition of the projections $p_F[k,i]$ $(k = 1,\dots,i)$. Since p_F is a shape equivalence, we see that

$$\mathrm{Sh}(\varprojlim\{X_i, f_{i,i+1}\}) = \mathrm{Sh}(\varprojlim\{X_i, s(f_{i,i+1})\}). \ \blacksquare$$

Let $\mathcal{P}$ be a collection of compacta. A compactum X is *$\mathcal{P}$-like* if for any $\epsilon > 0$ there exist $P \in \mathcal{P}$ and an ϵ-map from X onto P. A compactuum X is *circle-like* if $\mathcal{P} = \{\text{circle}\}$ and X is $\mathcal{P}$-like. A compactum X is *tree-like* if $\mathcal{P} = \{T \mid T \text{ is a tree}\}$ and X is $\mathcal{P}$-like.

COROLLARY 5.3. *Let X_i $(i \in \mathbb{N})$ be a sequence of graphs (i.e., 1-dimensional finite connected polyhedra) and for each $i \in \mathbb{N}$ let $f_{i,i+1} : X_{i+1} \to C(X_i)$ be an upper semicontinuous function such that $f_{i,i+1}(x)$ is a tree in X_i and $x \in X_{i+1}$. Then $\varprojlim\{X_i, f_{i,i+1}\}$ is shape equivalent to an $\{X_i \mid i \in \mathbb{N}\}$-like continuum Y.*

Proof. Theorem 5.2 implies that $\varprojlim\{X_i, f_{i,i+1}\}$ is shape equivalent to $\varprojlim\{X_i, s(f_{i,i+1})\}$. For each i, we can easily construct a surjective map $g_{i,i+1} : X_{i+1} \to X_i$ such that $s(f_{i,i+1})$ and $g_{i,i+1}$ are homotopic. Hence

$$\mathrm{Sh}(\varprojlim\{X_i, f_{i,i+1}\}) = \mathrm{Sh}(\varprojlim\{X_i, s(f_{i,i+1})\}) = \mathrm{Sh}(\varprojlim\{X_i, g_{i,i+1})\}.$$

Thus $Y = \varprojlim\{X_i, g_{i,i+1}\}$ is $\{X_i \mid i \in \mathbb{N}\}$-like. ∎

COROLLARY 5.4. *Let X_i $(i \in \mathbb{N})$ be a sequence of simple closed curves and for each $i \in \mathbb{N}$ let $f_{i,i+1} : X_{i+1} \to C(X_i)$ be an upper semicontinuous function such that $f_{i,i+1}(x)$ is an arc in X_i and $x \in X_{i+1}$. Then $\varprojlim\{X_i, f_{i,i+1}\}$ is shape equivalent to a circle-like continuum.*

COROLLARY 5.5. *Let X_i $(i \in \mathbb{N})$ be a sequence of graphs and for each $i \in \mathbb{N}$ let $f_{i,i+1} : X_{i+1} \to C(X_i)$ be a surjective upper semicontinuous function such that $f_{i,i+1}(x)$ is a tree in X_i and $x \in X_{i+1}$. If $\dim D_1(f_{i,i+1}) \leq 0$ for each i and $J(\{X_i, f_{i,i+1}\}) = 0$, then $\varprojlim\{X_i, f_{i,i+1}\}$ is a 1-dimensional continuum which is shape equivalent to an $\{X_i \mid i \in \mathbb{N}\}$-like continuum Y. Moreover, if X_i is a tree for each i, then $\varprojlim\{X_i, f_{i,i+1}\}$ is tree-like.*

Proof. Since $f_{i,i+1}(x)$ is connected for each $x \in X_{i+1}$, we see that $\varprojlim\{X_i, f_{i,i+1}\}$ is a nondegenerate continuum (see [9]) and hence it is 1-dimensional. ∎

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