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ON THE QUADRIC CMC SPACELIKE HYPERSURFACES IN LORENTZIAN SPACE FORMS

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Abstract. We deal with complete spacelike hypersurfaces immersed with constant mean curvature in a Lorentzian space form. Under the assumption that the support functions with respect to a fixed nonzero vector are linearly related, we prove that such a hypersurface must be either totally umbilical or isometric to a hyperbolic cylinder of the ambient space.

1. Introduction. In 2008, Alías, Brasil and Perdomo [3] studied complete hypersurfaces immersed in the unit Euclidean sphere $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$, whose support functions with respect to a fixed nonzero vector of the Euclidean space $\mathbb{R}^{n+2}$ are linearly related. They showed that such a hypersurface having constant mean curvature must be either totally umbilical or isometric to a Clifford torus.

Later on, using a different approach, the first and second authors characterized the totally umbilical and the hyperbolic cylinders of the hyperbolic space $\mathbb{H}^{n+1}$ as the only complete hypersurfaces with constant mean curvature and whose support functions with respect to a fixed nonzero vector a of the Lorentz–Minkowski space are linearly related (see [4, Theorem 4.1] for the case that a is either spacelike or timelike, and [5, Theorem 4.2] for a being a nonzero null vector).

Let L_1^{n+1} be an $(n+1)$-dimensional Lorentz space, that is, a semi-Riemannian manifold of index 1. When L_1^{n+1} has constant sectional curvature c, it is called a *Lorentz space form* and denoted by $L_1^{n+1}(c)$. The Lorentz–Minkowski space $\mathbb{L}^{n+1}$, the de Sitter space $\mathbb{S}_1^{n+1}$ and the anti-de Sitter space $\mathbb{H}_1^{n+1}$ are the standard Lorentz space forms of constant sectional curvature 0, 1 and -1, respectively. We also recall that a hypersurface Σ^n

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immersed in a Lorentz space L_1^{n+1} is said to be *spacelike* if the metric on Σ^n induced from that of the ambient space L_1^{n+1} is positive definite.

Now, let $x : \Sigma^n \to L_1^{n+1}(c)$ be a complete spacelike hypersurface isometrically immersed in $L_1^{n+1}(c)$, with future-pointing Gauss map N. For a fixed nonzero vector a, we define the *support functions* on Σ^n *with respect to* a by

$$l_a(p) = \langle x(p), a\rangle \quad \text{and} \quad f_a(p) = \langle N(p), a\rangle, \qquad p \in \Sigma^n.$$

We say that the support functions l_a and f_a are *linearly related* when

$$l_a = \lambda f_a \tag{1.1}$$

for some $\lambda \in \mathbb{R}$.

Our purpose is to extend the techniques developed in [4] and [5] in order to characterize constant mean curvature spacelike hypersurfaces in $L_1^{n+1}(c)$ whose support functions are linearly related. For this, in Section 2 we recall some standard facts concerning hypersurfaces immersed in $L_1^{n+1}(c)$ and, in particular, we recall a suitable Simons-type formula for such a hypersurface. Then in Section 3 we present some examples of quadric spacelike hypersurfaces satisfying condition (1.1). Finally, in Section 4 we prove our characterization result, stated below:

Theorem 1.1. *Let $x : \Sigma^n \to L_1^{n+1}(c)$ be a complete spacelike hypersurface with constant mean curvature H. If Σ^n satisfies condition* (1.1) *for some fixed nonzero vector a, then Σ^n is either totally umbilical or isometric to*

(a) $\mathbb{R}^k \times \mathbb{H}^{n-k}(c_2)$, *where $c_2 < 0$, when $c = 0$;*
(b) $\mathbb{S}^k(c_1) \times \mathbb{H}^{n-k}(c_2)$, *where $c_1 > 0$, $c_2 < 0$ and $1/c_1 + 1/c_2 = 1$, when $c = 1$;*
(c) $\mathbb{H}^k(c_1) \times \mathbb{H}^{n-k}(c_2)$, *where $c_1 < 0$, $c_2 < 0$ and $1/c_1 + 1/c_2 = -1$, when $c = -1$,*

where $k \in \{1, \dots, n-1\}$.

2. Preliminaries. For $q \in \{1, 2\}$, let $\mathbb{R}_q^{n+2}$ denote the $(n+2)$-dimensional semi-Euclidean space endowed with the following metric of index q:

$$\langle u, v\rangle = -\sum_{i=1}^{q} u_i v_i + \sum_{i=q+1}^{n+2} u_i v_i$$

for $u, v \in \mathbb{R}_q^{n+2}$. In particular, when $q = 1$, $\mathbb{R}_1^{n+2} = \mathbb{L}^{n+2}$ is the *Lorentz–Minkowski space.*

The *de Sitter space* $\mathbb{S}_1^{n+1}$ is the hyperquadric in $\mathbb{L}^{n+2}$ defined by

$$\mathbb{S}_1^{n+1} = \{x \in \mathbb{L}^{n+2};\ \langle x, x\rangle = 1\}.$$

Endowed with the induced metric from $\mathbb{L}^{n+2}$, $n \geq 2$, the de Sitter space is a complete simply connected $(n+1)$-dimensional Lorentzian manifold with constant sectional curvature one.

When $q = 2$, we define the *anti-de Sitter space* $\mathbb{H}_1^{n+1}$ as the hyperquadric in $\mathbb{R}_2^{n+2}$ given by

$$\mathbb{H}_1^{n+1} = \{x \in \mathbb{R}_2^{n+2};\ \langle x, x\rangle = -1\}.$$

Topologically, $\mathbb{H}_1^{n+1}$ corresponds to the product $\mathbb{S}^1 \times \mathbb{R}^n$, and the semi-Euclidean metric on $\mathbb{R}_2^{n+2}$ induces a Lorentzian metric of constant sectional curvature -1 on $\mathbb{H}_1^{n+1}$. Moreover, the universal covering manifold $\widetilde{\mathbb{H}}_1^{n+1}$ of $\mathbb{H}_1^{n+1}$ is topologically the Euclidean space $\mathbb{R}^{n+1}$ (that is, $\widetilde{\mathbb{H}}_1^{n+1}$ is simply connected) and thus is a Lorentzian analogue of the usual Riemannian hyperbolic space $\mathbb{H}^{n+1}$ of negative curvature -1, which is called the *universal anti-de Sitter spacetime* (see, for instance, [6, Section 5.3] or [15, Section 8.6]).

From now on, we consider $L_1^{n+1}(c)$, where $c \in \{-1, 0, 1\}$, denoting the Lorentz–Minkowski space if $c = 0$, the de Sitter space if $c = 1$, and the anti-de Sitter space if $c = -1$. A smooth immersion $x : \Sigma^n \to L_1^{n+1}(c)$ of an n-dimensional connected manifold Σ^n is called a *spacelike hypersurface* if the metric induced via x is a Riemannian metric on Σ^n, which is also denoted by $\langle\ ,\ \rangle$. Since $L_1^{n+1}(c)$ is time-oriented, we can choose a unique unit normal vector field N on Σ^n which is a *future-pointing* timelike vector field in $L_1^{n+1}(c)$, that is, $\langle N, e_1\rangle < 0$ where $e_1 = (1, 0, \ldots, 0) \in L_1^{n+1}(c)$. In this setting, we will assume that Σ^n is oriented by N, and we will denote by $\overline{\nabla}$ and ∇ the Levi-Civita connections of $L_1^{n+1}(c)$ and Σ^n, respectively.

For $1 \le r \le n$ and $p \in \Sigma^n$, if $S_r(p)$ denotes the rth elementary symmetric function of the eigenvalues of A_p, we get n smooth functions $S_r : \Sigma^n \to \mathbb{R}$ such that

$$\det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by definition. Taking a local orthonormal frame $\{e_1, \ldots, e_n\}$ on Σ^n such that $Ae_i = \lambda_i e_i$, $i = 1, \ldots, n$, it is easy to verify that

$$S_r = \sigma_r(\lambda_1, \ldots, \lambda_n),$$

where $\sigma_r \in \mathbb{R}[X_1, \ldots, X_n]$ is the rth elementary symmetric polynomial in the indeterminates $X_1, \ldots, X_n$. In particular, when $r = 1$ we know that $H = -(1/n)S_1$ is the mean curvature of Σ^n with respect to its future-pointing Gauss mapping N. Moreover, if $|A|$ stands for the Hilbert–Schmidt norm of A, then it is immediate to check that

$$S_1^2 = |A|^2 + 2S_2. \tag{2.1}$$

Now, for $0 \le r \le n$, let $P_r : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ be the rth *Newton transformation* of Σ^n, defined inductively by setting $P_0 = I$ (the identity operator) and, for $1 \le r \le n$,

$$P_r = (-1)^r S_r I + A P_{r-1}. \tag{2.2}$$

Associated to each Newton transformation P_r, one has the second order linear differential operator $L_r : \mathcal{C}^\infty(\Sigma) \to \mathcal{C}^\infty(\Sigma)$ given by

$$L_r(f) = \operatorname{tr}(P_r \circ \operatorname{Hess} f), \tag{2.3}$$

where $\mathcal{C}^\infty(\Sigma)$ stands for the ring of smooth real functions on Σ^n. In particular, when $r = 0$, the operator L_0 is just the Laplacian Δ.

In [7], Caminha extended a technique due to Alencar, do Carmo and Colares [2] obtaining a suitable formula for $L_q(S_r)$, with $0 \le q < n$ and $0 < r < n$ (cf. [2, Lemma 3.7] and [7, Proposition 3]). In particular, from [7, Corollary 2] we deduce

LEMMA 2.1. *Let Σ^n be a spacelike hypersurface in $L_1^{n+1}(c)$. Then*

$$\begin{aligned} L_1(S_1) = &-\Delta S_2 - |\nabla A|^2 + |\nabla S_1|^2 \\ &- 2S_2(|A|^2 + cn) + S_1\big(S_1 S_2 - 3S_3 + c(n-1)S_1\big). \end{aligned}$$

Now, for a fixed nonzero vector $a \in \mathbb{R}_q^{n+2}$, let us consider the support functions $l_a : \Sigma^n \to \mathbb{R}$ and $f_a : \Sigma^n \to \mathbb{R}$ given, respectively, by $l_a(p) = \langle x(p), a\rangle$ and $f_a(p) = \langle N(p), a\rangle$. Then we can write

$$a = a^\top - f_a N + c l_a x, \tag{2.4}$$

where $a^\top$ denotes the projection of the vector a on the tangent bundle of Σ^n. A direct computation allows us to conclude that

$$\nabla l_a = a^\top \quad \text{and} \quad \nabla f_a = -A(a^\top).$$

So, we get the useful relation

$$\langle a, a\rangle = |\nabla l_a|^2 - f_a^2 + c l_a^2. \tag{2.5}$$

We close this section by quoting convenient formulas for the operator L_r acting on the support functions of spacelike hypersurfaces in $L_1^{n+1}(c)$. Their proofs can be found, for instance, in [11].

LEMMA 2.2. *Let Σ^n be a spacelike hypersurface in a Lorentzian space form $L_1^{n+1}(c)$ of constant sectional curvature c. Then:*

(i) $L_r(l_a) = -(r+1)S_{r+1} f_a - c(n-r)S_r l_a$,

(ii) $L_r(f_a) = -\langle \nabla S_{r+1}, a^\top\rangle + (S_1 S_{r+1} - (r+2)S_{r+2}) f_a + c(r+1)S_{r+1} l_a$.

3. Quadric spacelike hypersurfaces in $L_1^{n+1}(c)$. This section is devoted to a description of quadric spacelike hypersurfaces in $L_1^{n+1}(c)$. In fact, Theorem 1.1 asserts that these examples are the only ones whose support functions with respect to a nonzero vector satisfy condition (1.1).

EXAMPLE 3.1 (Totally umbilical spacelike hypersurfaces in $L_1^{n+1}(c)$). In case $c = 0$, if Σ^n is a spacelike hyperplane orthogonal to a timelike vector $a \in \mathbb{L}^{n+1}$, it is immediate that the functions l_a and f_a on Σ^n are constant, and hence relation (1.1) holds.

So, let $c \in \{-1, 1\}$ and let $a \in \mathbb{R}_q^{n+2}$ be a nonzero vector with $\langle a, a\rangle \in \{-1, 0, 1\}$. Consider the smooth function $g : L_1^{n+1}(c) \to \mathbb{R}$ defined by $g(x) = \langle x, a\rangle$. It is not difficult verify that, for every $\tau \in \mathbb{R}$ with $\langle a, a\rangle - c\tau^2 \neq 0$, the set

$$L_\tau = g^{-1}(\tau) = \{x \in L_1^{n+1}(c);\ \langle x, a\rangle = \tau\} \tag{3.1}$$

is a totally umbilical spacelike hypersurface in $L_1^{n+1}(c)$, with future-pointing Gauss map

$$N_\tau(p) = \frac{1}{\sqrt{|\langle a, a\rangle - c\tau^2|}}(a - c\tau x).$$

Moreover, the shape operator A and mean curvature H are given, respectively, by

$$Av = \frac{c\tau}{\sqrt{|\langle a, a\rangle - c\tau^2|}}v \quad \text{and} \quad H^2 = \frac{\tau^2}{|\langle a, a\rangle - c\tau^2|}.$$

So, after a straightforward computation, it follows that the support functions l_a and f_a satisfy the linear dependence relation

$$l_a = \frac{|\tau|}{\sqrt{|\langle a, a\rangle - c\tau^2|}} f_a = |H| f_a.$$

EXAMPLE 3.2 (Hyperbolic cylinders in $\mathbb{L}_1^{n+1}$). For $\rho > 0$ and k an integer satisfying $0 < k < n$, a hyperbolic cylinder in $\mathbb{R}_1^{n+1}$ is defined by

$$\Sigma^n = \{x \in \mathbb{R}_1^{n+1};\ -x_1^2 + x_2^2 + \cdots + x_{k+1}^2 = -\rho^2\} = \mathbb{H}^k(-1/\rho^2) \times \mathbb{R}^{n-k}.$$

For the timelike unit normal vector field

$$N(x) = -\frac{1}{\rho}(x - \nu(x)),$$

where $\nu(x) = (0, \ldots, 0, x_{k+2}, \ldots, x_{n+2})$ and $x = (x_1, \ldots, x_{n+2})$, we deduce that the Weingarten operator A of Σ^n with respect to N has principal curvatures

$$\lambda_1 = \cdots = \lambda_k = 1/\rho, \qquad \lambda_{k+1} = \cdots = \lambda_n = 0.$$

Considering $a = (1, 1, 0, \ldots, 0)$ for the case of a null vector, $a = (1, 0, \ldots, 0)$ for a timelike vector, and $a = (1, 2, 0, \ldots, 0)$ for a spacelike vector, it is not difficult to verify that

$$l_a = \rho f_a \quad \text{on } \Sigma^n.$$

EXAMPLE 3.3 (Hyperbolic cylinders in $\mathbb{S}_1^{n+1} \subset \mathbb{R}_1^{n+2}$). Let k be an integer satisfying $0 \leq k < n$. We define a smooth function $f : \mathbb{S}_1^{n+1} \to \mathbb{R}$

by

$$f(x) = x_2^2 + \cdots + x_{k+2}^2,$$

where $x = (x_1, \ldots, x_{n+2})$. For $\rho > 0$, set $\Sigma^n = f^{-1}(\rho^2)$. If $x = (x_1, \ldots, x_{n+2}) \in \Sigma^n$, then

$$\begin{aligned}\Sigma^n &= \Big\{x \in \mathbb{S}_1^{n+1};\ \sum_{i=2}^{k+2} x_i^2 = \rho^2 \text{ and } -x_1^2 + \sum_{i=k+3}^{n+2} x_i^2 = 1-\rho^2\Big\}\\ &= \mathbb{S}^k(1/\rho^2) \times \mathbb{H}^{n-k}(-1/(1+\rho^2)).\end{aligned}$$

Now, for $X = (X_1, \ldots, X_{n+2})$, we have

$$\langle \overline{\nabla} f(x), X\rangle = \langle 2\nu(x), X\rangle,$$

where $\nu(x) = (0, x_2, \ldots, x_{k+2}, 0, \ldots, 0)$ and $\nu = \nu^\top + \langle \nu, x\rangle x = \nu^\top + \rho^2 x$. Thus, $\overline{\nabla} f(x) = 2(\nu(x) - \rho^2 x)$, and consequently

$$\langle \overline{\nabla} f(x), \overline{\nabla} f(x)\rangle = 4\rho^2(1-\rho^2).$$

So,

$$|\overline{\nabla} f(x)| = 2\rho\sqrt{1-\rho^2}.$$

Hence, the vector field

$$N(x) = \frac{\overline{\nabla} f(x)}{|\overline{\nabla} f(x)|} = \frac{\nu(x) - \rho^2 x}{\rho\sqrt{1-\rho^2}}$$

defines the future-pointing Gauss map of Σ^n. Moreover, the Weingarten operator A of Σ^n with respect to N has principal curvatures

$$\lambda_1 = \cdots = \lambda_k = -\frac{\sqrt{1-\rho^2}}{\rho}, \qquad \lambda_{k+1} = \cdots = \lambda_n = \frac{\rho}{\sqrt{1-\rho^2}}.$$

Finally, if $a = (1, 1, 0, \ldots, 0)$ for the case of a null vector, $a = (1, 0, \ldots, 0)$ for a timelike vector, and $a = (1, 2, 0, \ldots, 0)$ for a spacelike vector, it is not difficult to verify that

$$l_a = -\frac{\sqrt{1-\rho^2}}{\rho} f_a.$$

Example 3.4 (Hyperbolic cylinders in $\mathbb{H}_1^{n+1} \subset \mathbb{R}_2^{n+2}$). In an analogous way to the de Sitter space, we define a smooth function $g : \mathbb{H}_1^{n+1} \to \mathbb{R}$ by

$$g(x) = -x_1^2 + x_3^2 + \cdots + x_{k+2}^2,$$

where $x = (x_1, \ldots, x_{n+2})$. For $\rho > 0$, set $\Sigma^n = g^{-1}(-\rho^2)$. If $x = (x_1, \ldots, x_{n+2}) \in \Sigma^n$, then

$$\begin{aligned}\Sigma^n &= \Big\{x \in \mathbb{H}_1^{n+1};\ -x_1^2 + \sum_{i=3}^{k+2} x_i^2 = -\rho^2 \text{ and } -x_2^2 + \sum_{i=k+3}^{n+2} x_i^2 = \rho^2 - 1\Big\}\\ &= \mathbb{H}^k(-1/\rho^2) \times \mathbb{H}^{n-k}(1/(\rho^2-1)).\end{aligned}$$

Now, for $X = (X_1, \ldots, X_{n+2})$, we have

$$\langle \overline{\nabla} g(x), X\rangle = \langle 2\nu(x), X\rangle.$$

Thus

$$\overline{\nabla} g(x) = 2\nu^\top,$$

where $\nu(x) = (-x_1, 0, x_3, \ldots, x_{k+2}, 0, \ldots, 0)$ and $\nu = \nu^\top - \langle \nu, x\rangle x = \nu^\top + \rho^2 x$.

Hence, $\overline{\nabla} g(x) = 2(\nu(x) - \rho^2 x)$, and consequently

$$\langle \overline{\nabla} g(x), \overline{\nabla} g(x)\rangle = 4\rho^2(\rho^2 - 1).$$

Then

$$|\overline{\nabla} g(x)| = 2\rho\sqrt{\rho^2 - 1}.$$

So,

$$N(x) = \frac{\overline{\nabla} g(x)}{|\overline{\nabla} g(x)|} = \frac{\nu(x) - \rho^2 x}{\rho\sqrt{\rho^2 - 1}}$$

defines the future-pointing Gauss map of Σ^n. Moreover, the Weingarten operator A of Σ^n with respect the N has principal curvatures

$$\lambda_1 = \cdots = \lambda_k = -\frac{\sqrt{\rho^2 - 1}}{\rho}, \qquad \lambda_{k+1} = \cdots = \lambda_n = \frac{\rho}{\sqrt{\rho^2 - 1}}.$$

Furthermore, if we take $a = (0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$ for the case of a null vector, $a = (0, 1, 0, \ldots, 0)$ for a timelike vector, and $a = (0, 1, 0, \ldots, 0, 2, 0, \ldots, 0)$ for a spacelike vector, it is not difficult to verify that

$$l_a = -\frac{\sqrt{\rho^2 - 1}}{\rho} f_a.$$

4. Proof of Theorem 1.1. First, we observe that if $\lambda = 0$, from (1.1) and (2.5), and taking into account [15, Lemma 2.6], it is not difficult to verify that a must be a timelike vector. So, using (2.5) once more, we obtain $f_a^2 = 1$, and consequently, assuming without loss of generality that $\langle a, a\rangle = -1$, we have $N = \pm a$. Thus,

$$\Sigma^n \subset a^\perp = \{v \in T_p\Sigma^n;\ \langle a, v\rangle = 0\}, \quad p \in \Sigma^n, \tag{4.1}$$

and, by completeness, Σ^n must be a leaf of the foliation of $L_1^{n+1}(c)$ orthogonal to the vector a. From [12, Proposition 1], such leaves are totally umbilical spacelike hypersurfaces in $L_1^{n+1}(c)$. But, since $l_a = 0$, we see that Σ^n is totally geodesic. Therefore, Σ^n is either a spacelike hyperplane in $\mathbb{L}^{n+1}$, when $c = 0$, or a totally geodesic round sphere in $\mathbb{S}_1^{n+1}$, when $c = 1$, or is isometric to a totally geodesic hyperbolic space in $\mathbb{H}_1^{n+1}$, when $c = -1$.

Now, let $\lambda \neq 0$. Since λ is constant, (1.1) guarantees that $\Delta l_a = \lambda \Delta f_a$. Taking $r = 0$ in Lemma 2.2, we obtain

$$(S_1 + nc\lambda + \lambda S_1^2 - 2\lambda S_2 + c\lambda^2 S_1) l_a = 0 \tag{4.2}$$

on Σ^n. Let $h : \Sigma^n \to \mathbb{R}$ be defined by

$$h = S_1 + nc\lambda + \lambda S_1^2 - 2\lambda S_2 + c\lambda^2 S_1.$$

If there exists $p_0 \in \Sigma^n$ such that $h(p_0) \neq 0$, then there exists a neighborhood $\mathcal{U}$ of p_0 in Σ^n in which $h(p) \neq 0$ for all $p \in \mathcal{U}$. Then, from (4.2) it follows that $l_a = 0$ in $\mathcal{U}$. This implies that f_a and l_a are simultaneously zero in $\mathcal{U}$ and, taking into account (2.5), a must be a null vector. On the other hand, $f_a = 0$ implies that a is a spacelike vector. Hence, we reach a contradiction.

Consequently, $h = 0$ on Σ^n. Thus, from (4.2) we get

$$S_1 + cn\lambda + \lambda S_1^2 - 2\lambda S_2 + c\lambda^2 S_1 = 0, \tag{4.3}$$

and since $\lambda \neq 0$, it follows from (4.3) that S_2 is also constant.

Now, suppose that, for some $1 \leq r < n$, S_j is constant for $j \in \{1, \dots, r\}$. Since $L_{r-1} l_a = \lambda L_{r-1} f_a$, from Lemma 2.2 we have

$$\begin{aligned} L_{r-1} l_a &= -rS_r f_a - c(n - r + 1)S_{r-1} l_a, \\ L_{r-1} f_a &= (S_1 S_r - (r+1)S_{r+1}) f_a + crS_r l_a. \end{aligned}$$

So, we can reason in a similar way to the case $r = 0$ to obtain

$$S_{r+1} = \frac{r}{\lambda(r+1)} S_r + \frac{c(n-r+1)}{r+1} S_{r-1} + \frac{1}{r+1} S_1 S_r + \frac{cr\lambda}{r+1} S_r. \tag{4.4}$$

Since by induction we are supposing that S_r and S_{r-1} are constant, it follows that S_{r+1} must also be constant. Hence, S_r is constant for all $r \in \{1, \dots, n\}$.

Consequently, from (2.1) and Lemma 2.1 we have

$$|\nabla A|^2 - S_1^2 S_2 + 4S_2^2 - 2cnS_2 - 3S_1 S_3 + c(n-1)S_1^2 = 0. \tag{4.5}$$

Suppose that $S_1 = 0$, that is, Σ^n is maximal. When $c = 0$, from the classical theorem of Cheng–Yau [8] we see that Σ^n must be a spacelike hyperplane in $\mathbb{L}^{n+1}$. When $c = 1$, from [13, Theorem A] we conclude that Σ^n is isometric to a totally geodesic round sphere in $\mathbb{S}_1^{n+1}$. Finally, when $c = -1$, from (2.1) we have $|A|^2 = -2S_2$. On the other hand, from (4.3) we get $2S_2 = -n$, and consequently $|A|^2 = n$. Hence, [9, Theorem 1.3] shows that Σ^n is isometric to a maximal hyperbolic cylinder

$$\mathbb{H}^m\left(-\frac{n}{m}\right) \times \mathbb{H}^{n-m}\left(-\frac{n}{n-m}\right), \qquad 1 \leq m \leq n-1.$$

Now, suppose that $S_1 \neq 0$. Taking $r = 2$ in (4.4) and multiplying by S_1, we get

$$-3\lambda S_1 S_3 + c\lambda(n-1)S_1^2 = -2S_1 S_2 - 2c\lambda^2 S_1 S_2 - \lambda S_1^2 S_2. \tag{4.6}$$

Since $\lambda \neq 0$, we can multiply (4.5) by λ to get

$$\lambda|\nabla A|^2 - \lambda S_1^2 S_2 + 4\lambda S_2^2 - 2\lambda cnS_2 - 3\lambda S_1 S_3 + c\lambda(n-1)S_1^2 = 0. \tag{4.7}$$

Applying (4.7) in (4.6), we obtain

$$\lambda|\nabla A|^2 - 2\lambda S_1^2 S_2 + 4\lambda S_2^2 - 2\lambda cnS_2 - 2S_1S_2 - 2c\lambda^2 S_1S_2 = 0. \tag{4.8}$$

If $S_2 = 0$, from (4.8) we have $\lambda|\nabla A|^2 = 0$, and since $\lambda \neq 0$, it follows that $|\nabla A|^2 = 0$, that is, Σ^n has parallel second fundamental form. Therefore, Σ^n is an isoparametric spacelike hypersurface in $\mathbb{S}_1^{n+1}$. Now, if $S_2 \neq 0$, multiplying (4.3) by $2S_2$ we get

$$2S_1S_2 + 2cn\lambda S_2 + 2\lambda S_1^2 S_2 - 4\lambda S_2^2 + 2c\lambda^2 S_1S_2 = 0. \tag{4.9}$$

From $\lambda \neq 0$, and equations (4.8) and (4.9), we obtain $|\nabla A|^2 = 0$. Thus, we can proceed as before to conclude that Σ^n is an isoparametric spacelike hypersurface in $\mathbb{S}_1^{n+1}$.

Hence, a classical result due to Nomizu [14] shows that Σ^n has at most two distinct principal curvatures in the Lorentz–Minkowski and de Sitter spaces. The same holds for the case of the anti-de Sitter space, according to Li and Xie [10]. Therefore, we can apply Theorem 5.1 of Abe et al. [1] to finish the proof of Theorem 1.1.

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REFERENCES

[1] N. Abe, N. Naoyuki and S. Yamaguchi, *Congruence theorems for proper semi-Riemannian hypersurfaces in a real space form*, Yokohama Math. J. 35 (1987), 123–136.

[2] H. Alencar, M. do Carmo and A. G. Colares, *Stable hypersurfaces with constant scalar curvature*, Math. Z. 213 (1993), 117–131.

[3] L. J. Alías, A. Brasil, Jr. and O. Perdomo, *A characterization of quadric constant mean curvature hypersurfaces of spheres*, J. Geom. Anal. 18 (2008), 687–703.

[4] C. P. Aquino and H. F. de Lima, *On the Gauss map of complete CMC hypersurfaces in the hyperbolic space*, J. Math. Anal. Appl. 386 (2012), 862–869.

[5] C. P. Aquino and H. F. de Lima, *On the geometry of horospheres*, Comment. Math. Helv. 89 (2014), 617–629.

[6] J. K. Beem, P. E. Ehrlich and K. L. Easley, *Global Lorentzian Geometry*, 2nd ed., CRC Press, New York, 1996.

[7] A. Caminha, *On spacelike hypersurfaces of constant sectional curvature Lorentzian manifolds*, J. Geom. Phys. 56 (2006), 1144–1174.

[8] S. Y. Cheng and S. T. Yau, *Maximal spacelike hypersurfaces in Lorentz–Minkowski spaces*, Ann. of Math. 104 (1976), 407–419.

[9] T. Ishihara, *Maximal spacelike submanifolds of a pseudoriemannian space of constant curvature*, Michigan Math. J. 35 (1988), 345–352.

[10] Z. Q. Li and X. H. Xie, *Space-like isoparametric hypersurfaces in the Lorentzian space forms*, J. Nanchang Univ. Natur. Sci. Ed. 28 (2004), 113–117 (in Chinese); English transl.: Front. Math. China 1 (2006), 130–137.

[11] P. Lucas and H. F. Ramírez-Ospina, *Hypersurfaces in pseudo-Euclidean spaces satisfying a linear condition on the linearized operator of a higher order mean curvature*, Differential Geom. Appl. 31 (2013), 175–189.

[12] S. Montiel, *Uniqueness of spacelike hypersurfaces of constant mean curvature in foliated spacetimes*, Math. Ann. 314 (1999), 529–553.

[13] S. Nishikawa, *On maximal spacelike hypersurfaces in a Lorentzian manifolds*, Nagoya Math. J. 95 (1984), 117–124.

[14] K. Nomizu, *On isoparametric hypersurfaces in the Lorentzian space forms*, Japan. J. Math. 7 (1981), 217–226.

[15] B. O'Neill, *Semi-Riemannian Geometry, with Applications to Relativity*, Academic Press, New York, 1983.

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