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VANISHING THEOREMS FOR KILLING VECTOR FIELDS ON COMPLETE HYPERSURFACES IN THE HYPERBOLIC SPACE

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Abstract. We study vanishing theorems for Killing vector fields on complete stable hypersurfaces in a hyperbolic space $\mathbb{H}^{n+1}(-1)$. We derive vanishing theorems for Killing vector fields with bounded L^2-norm in terms of the bottom of the spectrum of the Laplace operator.

1. Introduction. Let M^n be an isometrically immersed submanifold in a complete Riemannian manifold N^m. M^n is called *stable* if for any f in $C_0^\infty(M^n)$,

$$\int_{M^n} \{|\nabla f|^2 - (|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu))f^2\}\,dv \ge 0, \tag{1.1}$$

where A is the second fundamental form, $\overline{\mathrm{Ric}}$ denotes the Ricci curvature of N^m, ν is the unit normal vector of M^n, and dv stands for the volume form on M^n. Recently, several results have appear on nonexistence of L^2 harmonic forms on complete noncompact stable submanifolds of a Riemannian manifold with nonnegative sectional curvature. Kim and Yun [8] proved that for $2 \le n \le 4$, if M^n satisfies the stability inequality (1.1), then there is no nontrivial L^2 harmonic one-form on M^n, which is a generalization of a well-known fact in the case when M^n is a complete stable minimal hypersurface. Extending Kim and Yun's result to the case when $n = 5, 6$, Dung and Seo [4] studied complete hypersurfaces M^n satisfying the δ-stability inequality for a number $0 < \delta < 1$ in a complete manifold of nonnegative sectional curvature. They proved that there is no nontrivial $L^{2\beta}$ harmonic one-form on M^n for some constant β. For related research, see [1, 5, 6, 9, 10, 3, 11] and the references therein.

A vector field V on a Riemannian manifold (M, g) is *Killing* if the Lie derivative of the metric with respect to V vanishes, that is,

$$L_V g = 0. \tag{1.2}$$

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This is equivalent to the fact that the one-parameter group of diffeomorphisms generated by V consists of isometries. Therefore, it is very interesting to study the nonexistence of Killing vector fields since the space of nontrivial Killing vector fields measures the size of the isometry group, in a sense.

It is well known that there is no nontrivial Killing vector field on a compact Riemannian manifold with negative Ricci curvature. A natural question is whether Killing vector fields vanish on a given complete Riemannian manifold under a suitable assumption on the bottom of the spectrum of the Laplace operator. In this paper, we study vanishing theorems for Killing vector fields on complete hypersurfaces in an $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}(-1)$ with constant sectional curvature -1. More precisely, we shall prove the following.

THEOREM 1.1. *Let M^n be an n-dimensional complete noncompact stable hypersurface in a hyperbolic space $\mathbb{H}^{n+1}(-1)$.*

(i) *If $2 \le n \le 4$ and the first eigenvalue $\lambda_1(M)$ of M^n satisfies*

$$\lambda_1(M) > 1,$$

then there is no nontrivial Killing vector field with bounded L^2-norm on M^n.

(ii) *If $4 \le n \le 7$ and*

$$\lambda_1(M) > \frac{(n-2)^2}{8-n},$$

then there is no nontrivial Killing vector field with bounded L^2-norm on M^n.

REMARK 1.2. It is still an interesting problem to find some examples satisfying the assumption of Theorem 1.1.

2. Proofs. It has been shown by Chen [2, Theorem 4] that for an n-dimensional isometrically immersed submanifold M^n in a hyperbolic space $\mathbb{H}^{n+1}(-1)$, the Ricci curvature of M^n satisfies

$$\mathrm{Ric}_M \le H^2/4 - (n-1), \tag{2.1}$$

where H denotes the mean curvature defined by $H = \mathrm{trace}(A)$. Using the Cauchy inequality, we have

$$H^2 \le n|A|^2.$$

Thus, (2.1) becomes

$$\mathrm{Ric}_M \le \frac{n}{4}|A|^2 - (n-1). \tag{2.2}$$

For a Killing vector field V, it has been proved by Hu and Li [7, Lemma 3.2] that

$$\tfrac{1}{2}\Delta|V|^2 \geq 2\big|\nabla|V|\big|^2 - \mathrm{Ric}_M(V,V), \tag{2.3}$$

which gives

$$|V|\Delta|V| \geq \big|\nabla|V|\big|^2 - \mathrm{Ric}_M(V,V), \tag{2.4}$$

where Δ, ∇ denote the Laplace operator and the gradient operator on M^n, respectively. Inserting (2.2) into (2.4), one has

$$|V|\Delta|V| \geq \big|\nabla|V|\big|^2 - \Big(\frac{n}{4}|A|^2 - (n-1)\Big)|V|^2. \tag{2.5}$$

Proof of Theorem 1.1(i). Now we fix a point $p \in M^n$ and consider a geodesic ball $B_p(R)$ of radius R centered at p. Choose a test function ϕ such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $B_p(R)$ and $\phi \equiv 0$ on $M^n \backslash B_p(2R)$, and $|\nabla\phi| \leq 1/R$. Replacing f in (1.1) by $\phi|V|$, we obtain

$$\begin{aligned} 0 &\leq \int_{M^n} \{|\nabla(\phi|V|)|^2 - (|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu))\phi^2|V|^2\}\,dv \\ &= \int_{M^n} \{|\nabla(\phi|V|)|^2 - (|A|^2 - n)\phi^2|V|^2\}\,dv, \end{aligned} \tag{2.6}$$

where we have used $\overline{\mathrm{Ric}}(\nu,\nu) = -n$. The divergence theorem gives

$$\begin{aligned} \int_{M^n} |\nabla(\phi|V|)|^2\,dv &= -\int_{M^n} \phi|V|\Delta(\phi|V|)\,dv \\ &= -\int_{M^n} \phi|V|\big(\phi\Delta|V| + |V|\Delta\phi + 2\langle\nabla\phi, \nabla|V|\rangle\big)\,dv \\ &= \int_{M^n} |V|^2|\nabla\phi|^2\,dv - \int_{M^n} \phi^2|V|\Delta|V|\,dv. \end{aligned}$$

Therefore, from (2.6) we derive

$$0 \leq \int_{M^n} \{|V|^2|\nabla\phi|^2 - \phi^2(|V|\Delta|V| + |A|^2|V|^2) + n\phi^2|V|^2\}\,dv. \tag{2.7}$$

Applying (2.5) into (2.7), we obtain

$$0 \leq \int_{M^n} \Big\{|V|^2|\nabla\phi|^2 + \phi^2|V|^2 - \phi^2\big|\nabla|V|\big|^2 + \frac{n-4}{4}\phi^2|A|^2|V|^2\Big\}\,dv. \tag{2.8}$$

From the definition of the bottom of the spectrum, it follows that

$$\begin{aligned}
(2.9)\qquad \lambda_1(M)\int_{M^n}\phi^2|V|^2\,dv &\le \int_{M^n}|\nabla(\phi|V|)|^2\,dv\\
&= \int_{M^n}\{\phi^2\big|\nabla|V|\big|^2+|V|^2|\nabla\phi|^2+2\phi|V|\langle\nabla\phi,\nabla|V|\rangle\}\,dv\\
&\le \left(1+\frac{1}{\varepsilon}\right)\int_{M^n}|V|^2|\nabla\phi|^2\,dv+(1+\varepsilon)\int_{M^n}\phi^2\big|\nabla|V|\big|^2\,dv,
\end{aligned}$$

where the first eigenvalue $\lambda_1(M)$ of a complete noncompact manifold M^n is defined by

$$\lambda_1(M)=\inf_{\Omega}\lambda_1(\Omega).$$

Here the infimum is taken over all compact domains in M^n. Combining (2.8) and (2.9), we obtain

$$\begin{aligned}
(2.10)\qquad 0 &\le \left[\left(1+\frac{1}{\varepsilon}\right)\frac{1}{\lambda_1(M)}+1\right]\int_{M^n}|V|^2|\nabla\phi|^2\,dv\\
&\quad+\left[(1+\varepsilon)\frac{1}{\lambda_1(M)}-1\right]\int_{M^n}\phi^2\big|\nabla|V|\big|^2\,dv\\
&\quad+\frac{n-4}{4}\int_{M^n}\phi^2|A|^2|V|^2\,dv.
\end{aligned}$$

Since $\lambda_1(M)>1$, we can choose a sufficiently small $\varepsilon>0$ such that

$$(1+\varepsilon)\frac{1}{\lambda_1(M)}-1<0.$$

Notice $2\le n\le 4$. Letting $R\to\infty$ and using the fact that the L^2-norm of V is bounded, we obtain

$$(2.11)\qquad \int_{M^n}\big|\nabla|V|\big|^2\,dv=0,$$

which implies that $|V|\equiv$ const. Furthermore, we have $\mathrm{Vol}(M^n)=\infty$ according to [11, Proposition 3.4] which states that if a complete noncompact Riemannian manifold satisfies $\lambda_1(M)>0$, then $\mathrm{Vol}(M^n)=\infty$. Thus, we have $V\equiv 0$ since the L^2-norm of V is bounded. This completes the proof of Theorem 1.1(i). ∎

Proof of Theorem 1.1(ii). By replacing f by $\phi|V|$ in (1.1), we obtain the following inequality (see (2.6)):

$$0\le\int_{M^n}\{|\nabla(\phi|V|)|^2-(|A|^2-n)\phi^2|V|^2\}\,dv,$$

which gives

$$0 \leq \int_{M^n} \{\phi^2 \big|\nabla|V|\big|^2 + |V|^2|\nabla\phi|^2 + 2\phi|V|\langle\nabla\phi, \nabla|V|\rangle - (|A|^2 - n)\phi^2|V|^2\}\, dv. \tag{2.12}$$

This yields

$$\int_{M^n} \phi^2|A|^2|V|^2\, dv \leq \int_{M^n} \{\phi^2\big|\nabla|V|\big|^2 + |V|^2|\nabla\phi|^2 + 2\phi|V|\langle\nabla\phi, \nabla|V|\rangle + n\phi^2|V|^2\}\, dv. \tag{2.13}$$

On the other hand, from (2.5) we derive

$$\frac{n}{4}\int_{M^n} \phi^2|A|^2|V|^2\, dv \geq \int_{M^n} \{-\phi^2|V|\Delta|V| + \phi^2\big|\nabla|V|\big|^2 + (n-1)\phi^2|V|^2\}\, dv. \tag{2.14}$$

Using the divergence theorem again gives

$$-\int_{M^n} \phi^2|V|\Delta|V|\, dv = \int_{M^n} \{2\phi|V|\langle\nabla\phi, \nabla|V|\rangle + \phi^2\big|\nabla|V|\big|^2\}\, dv. \tag{2.15}$$

Therefore, the inequality (2.14) can be written as

$$\frac{n}{4}\int_{M^n} \phi^2|A|^2|V|^2\, dv \geq \int_{M^n} \{2\phi|V|\langle\nabla\phi, \nabla|V|\rangle + 2\phi^2\big|\nabla|V|\big|^2 + (n-1)\phi^2|V|^2\}\, dv. \tag{2.16}$$

Combining (2.13) with (2.16), we obtain

$$\begin{aligned} 0 &\leq \int_{M^n} \left\{\frac{n-8}{4}\phi^2\big|\nabla|V|\big|^2 + \frac{n}{4}|V|^2|\nabla\phi|^2 + \frac{n-4}{2}\phi|V|\langle\nabla\phi, \nabla|V|\rangle + \frac{(n-2)^2}{4}\phi^2|V|^2\right\} dv \\ &\leq \int_{M^n} \left\{\left(\frac{n-8}{4} + \frac{(n-4)\varepsilon}{4}\right)\phi^2\big|\nabla|V|\big|^2 + \left(\frac{n}{4} + \frac{n-4}{4\varepsilon}\right)|V|^2|\nabla\phi|^2 + \frac{(n-2)^2}{4}\phi^2|V|^2\right\} dv \end{aligned} \tag{2.17}$$

for any positive constant ε. Inserting (2.9) into (2.17), we obtain

$$0 \le \left[\frac{n-8}{4} + \frac{(n-4)\varepsilon}{4} + \frac{(n-2)^2}{4}\frac{1}{\lambda_1(M)}(1+\varepsilon)\right] \int_{M^n} \phi^2 \big|\nabla|V|\big|^2 \, dv + \left[\frac{n}{4} + \frac{n-4}{4\varepsilon} + \frac{(n-2)^2}{4}\frac{1}{\lambda_1(M)}\left(1+\frac{1}{\varepsilon}\right)\right] \int_{M^n} |V|^2 |\nabla\phi|^2 \, dv. \tag{2.18}$$

Clearly,

$$\frac{n}{4} + \frac{n-4}{4\varepsilon} + \frac{(n-2)^2}{4}\frac{1}{\lambda_1(M)}\left(1+\frac{1}{\varepsilon}\right) > 0$$

for any ε since $4 \le n \le 7$. Moreover, since

$$\lambda_1(M) > \frac{(n-2)^2}{8-n},$$

we can choose a sufficiently small $\varepsilon > 0$ such that

$$\frac{n-8}{4} + \frac{(n-4)\varepsilon}{4} + \frac{(n-2)^2}{4}\frac{1}{\lambda_1(M)}(1+\varepsilon) < 0.$$

Letting $R \to \infty$ and using the fact that the L^2-norm of V is bounded, we obtain

$$\int_{M^n} \big|\nabla|V|\big|^2 \, dv = 0, \tag{2.19}$$

which implies that $|V| \equiv$ const. Furthermore, we have $\mathrm{Vol}(M^n) = \infty$ from $\lambda_1(M) > (n-2)^2/(8-n)$, and hence $V \equiv 0$ as the L^2-norm of V is bounded. The proof of Theorem 1.1(ii) is complete. ∎

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