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GLOBAL REGULARITY FOR THE 3D MHD SYSTEM WITH DAMPING

BY

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Abstract. We study the Cauchy problem for the 3D MHD system with damping terms $\varepsilon|\boldsymbol{u}|^{\alpha-1}\boldsymbol{u}$ and $\delta|\boldsymbol{b}|^{\beta-1}\boldsymbol{b}$ ($\varepsilon,\delta>0$ and $\alpha,\beta\geq 1$), and show that the strong solution exists globally for any $\alpha,\beta>3$. This improves the previous results significantly.

1. Introduction. The paper deals with the following Cauchy problem for the three-dimensional (3D) incompressible magneto-hydrodynamic (MHD) equations with nonlinear damping terms:

$$
\begin{cases}
\partial_t\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}-\Delta\boldsymbol{u}+\varepsilon|\boldsymbol{u}|^{\alpha-1}\boldsymbol{u}+\nabla\pi=\mathbf{0},\\
\partial_t\boldsymbol{b}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}-(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}-\Delta\boldsymbol{b}+\delta|\boldsymbol{b}|^{\beta-1}\boldsymbol{b}=\mathbf{0},\\
\nabla\cdot\boldsymbol{u}=\nabla\cdot\boldsymbol{b}=0,\\
\boldsymbol{u}(0)=\boldsymbol{u}_0,\quad \boldsymbol{b}(0)=\boldsymbol{b}_0,
\end{cases}
\tag{1.1}
$$

where $\boldsymbol{u}$ is the fluid velocity field, $\boldsymbol{b}$ is the magnetic field, π is a scalar pressure, and $\boldsymbol{u}_0,\boldsymbol{b}_0$ are the prescribed initial data satisfying $\nabla\cdot\boldsymbol{u}_0=\nabla\cdot\boldsymbol{b}_0=0$. In the damping terms, we assume $\varepsilon,\delta>0$ and $\alpha,\beta\geq 1$.

The damping is from the resistance to the motion of the flow. It describes various physical situations such as porous media flow, drag or friction effects, and some dissipative mechanisms (see [1] and references therein). When $\boldsymbol{b}=\mathbf{0}$, system (1.1) reduces to the Navier–Stokes system with damping:

$$
\begin{cases}
\partial_t\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}-\Delta\boldsymbol{u}+\varepsilon|\boldsymbol{u}|^{\alpha-1}\boldsymbol{u}+\nabla\pi=\mathbf{0},\\
\boldsymbol{u}(0)=\boldsymbol{u}_0.
\end{cases}
\tag{1.2}
$$

Cai–Jiu [1] first established the global existence of strong solutions when $\alpha\geq 7/2$, and the strong solution is unique in case $7/2\leq\alpha\leq 5$. This was technically improved by Zhang–Wu–Lu in [3], where the lower bound $7/2$

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was decreased to be 3, which seems to be critical in some sense. This was verified by Zhou [4], where the following three results were obtained:

(1) existence of global strong solutions when $\varepsilon = 1$, $\alpha \geq 3$;
(2) uniqueness of strong solution for any $\alpha \geq 1$, by observing the following non-negativity property:

$$\int_{\mathbb{R}^3} [|\boldsymbol{u}_1|^{\alpha-1}\boldsymbol{u}_1 - |\boldsymbol{u}_2|^{\alpha-1}\boldsymbol{u}_2] \cdot [\boldsymbol{u}_1 - \boldsymbol{u}_2]\, dx \geq 0; \tag{1.3}$$

(3) fundamental regularity criteria involving $\boldsymbol{u}$ or $\nabla \boldsymbol{u}$ for $1 \leq \alpha < 3$.

For the damped MHD system (1.1), Ye [2] showed the global regularity in case the pair α, β belongs to

$$\begin{aligned}\{(\alpha,\beta);\ \alpha,\beta \geq 4\} &\cup \left\{(\alpha,\beta);\ 3 < \alpha \leq \frac{7}{2},\ \frac{7}{2\alpha-5} \leq \beta \leq \frac{3\alpha+5}{\alpha-1}\right\}\\ &\cup \left\{(\alpha,\beta);\ \frac{7}{4} < \alpha < 4,\ \frac{5\alpha+7}{2\alpha} \leq \beta \leq \frac{3\alpha+5}{\alpha-1}\right\}\\ &\cup \left\{(\alpha,\beta);\ 4 \leq \alpha \leq \frac{17}{3},\ \frac{5\alpha+7}{2\alpha} \leq \beta < 4\right\}\\ &\cup \left\{(\alpha,\beta);\ \frac{17}{3} < \alpha \leq 7,\ \frac{5\alpha+7}{2\alpha} \leq \beta \leq \frac{\alpha+5}{\alpha-3}\right\}.\end{aligned}$$

This is quite unsatisfactory, and the aim of this paper is to establish the global regularity for any $\alpha, \beta > 3$, which is the exact result we have for the Navier–Stokes system with damping. Moreover, we treat all α, β in a unified approach.

MAIN THEOREM 1.1. *Assume that* $\alpha, \beta > 3$ *and* $\boldsymbol{u}_0, \boldsymbol{b}_0 \in H^1(\mathbb{R}^3)$ *with*

$$\nabla \cdot \boldsymbol{u}_0 = \nabla \cdot \boldsymbol{b}_0 = 0.$$

Then there exists a unique global strong solution to system (1.1). *Moreover,*

$$\begin{aligned}&\boldsymbol{u}, \boldsymbol{b} \in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^2(\mathbb{R}^3)),\\ &\boldsymbol{u} \in L^\infty(0,T;L^{\alpha+1}(\mathbb{R}^3)), \qquad \boldsymbol{b} \in L^\infty(0,T;L^{\beta+1}(\mathbb{R}^3)),\\ &\partial_t \boldsymbol{u}, \partial_t \boldsymbol{b}, |\boldsymbol{u}|^{\frac{\alpha-1}{2}}\nabla \boldsymbol{u}, |\boldsymbol{b}|^{\frac{\beta-1}{2}}\nabla \boldsymbol{b}, \nabla|\boldsymbol{u}|^{\frac{\alpha+1}{2}}, \nabla|\boldsymbol{b}|^{\frac{\beta+1}{2}} \in L^2(0,T;L^2(\mathbb{R}^3)),\end{aligned} \tag{1.4}$$

for any $T > 0$.

2. Proof of Theorem 1.1. We only need to show the a priori bounds (1.4), which ensure the global regularity. The proof of the uniqueness of the solution can be found in [2] (see also [4]).

Taking the inner product of $(1.1)_1$ with $-\Delta \boldsymbol{u}$ and of $(1.1)_2$ with $-\Delta \boldsymbol{b}$ in $L^2(\mathbb{R}^3)$, and summing the results, we obtain

$$
\begin{aligned}
(2.1)\qquad & \frac{1}{2}\frac{d}{dt}\|(\nabla \boldsymbol{u}, \nabla \boldsymbol{b})\|_{L^2}^2 + \|(\Delta \boldsymbol{u}, \Delta \boldsymbol{b})\|_{L^2}^2 + \big\|\big(|\boldsymbol{u}|^{\frac{\alpha-1}{2}}\nabla \boldsymbol{u}, |\boldsymbol{b}|^{\frac{\beta-1}{2}}\nabla \boldsymbol{b}\big)\big\|_{L^2}^2 \\
& \quad + \frac{4(\alpha-1)}{(\alpha+1)^2}\big\|\nabla |\boldsymbol{u}|^{\frac{\alpha+1}{2}}\big\|_{L^2}^2 + \frac{4(\beta-1)}{(\beta+1)^2}\big\|\nabla |\boldsymbol{b}|^{\frac{\beta+1}{2}}\big\|_{L^2}^2 \\
& = \int_{\mathbb{R}^3} [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot \Delta \boldsymbol{u}\,dx - \int_{\mathbb{R}^3} [(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}]\cdot \Delta \boldsymbol{u}\,dx \\
& \quad + \int_{\mathbb{R}^3} [(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}]\cdot \Delta \boldsymbol{b}\,dx + \int_{\mathbb{R}^3} [(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}]\cdot \Delta \boldsymbol{b}\,dx \\
& \le \|(\boldsymbol{u},\boldsymbol{b})\|_{L^{12}}\|(\nabla \boldsymbol{u}, \nabla \boldsymbol{b})\|_{L^{12/5}}\|(\Delta \boldsymbol{u}, \Delta \boldsymbol{b})\|_{L^2} \qquad \text{(Hölder inequality)} \\
& \le C\|(\nabla \boldsymbol{u}, \nabla \boldsymbol{b})\|_{L^{12/5}}^2\|(\Delta \boldsymbol{u}, \Delta \boldsymbol{b})\|_{L^2} \qquad \text{(Sobolev inequality)} \\
& \le C\|(\nabla \boldsymbol{u}, \nabla \boldsymbol{b})\|_{L^{12/5}}^4 + \tfrac{1}{4}\|(\Delta \boldsymbol{u}, \Delta \boldsymbol{b})\|_{L^2}^2.
\end{aligned}
$$

Similarly, taking the $L^2(\mathbb{R}^3)$ inner product of $(1.1)_1$ and $(1.1)_2$ with $\partial_t \boldsymbol{u}$ and $\partial_t \boldsymbol{b}$ respectively, and collecting the resulting equations, we find that

$$
\begin{aligned}
(2.2)\qquad & \frac{1}{2}\frac{d}{dt}\|(\nabla \boldsymbol{u}, \nabla \boldsymbol{b})\|_{L^2}^2 \\
& \qquad + \frac{d}{dt}\bigg(\frac{1}{\alpha+1}\|\boldsymbol{u}\|_{L^{\alpha+1}}^{\alpha+1} + \frac{1}{\beta+1}\|\boldsymbol{b}\|_{L^{\beta+1}}^{\beta+1}\bigg) + \|(\partial_t \boldsymbol{u}, \partial_t \boldsymbol{b})\|_{L^2}^2 \\
& \quad = -\int_{\mathbb{R}^3} [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot \partial_t \boldsymbol{u}\,dx + \int_{\mathbb{R}^3} [(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}]\cdot \partial_t \boldsymbol{u}\,dx \\
& \qquad - \int_{\mathbb{R}^3} [(\boldsymbol{u}\cdot\nabla)\boldsymbol{b}]\cdot \partial_t \boldsymbol{b}\,dx + \int_{\mathbb{R}^3} [(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}]\cdot \partial_t \boldsymbol{b}\,dx \\
& \quad \le C\|(\nabla \boldsymbol{u}, \nabla \boldsymbol{b})\|_{L^{12/5}}^4 + \tfrac{1}{2}\|(\partial_t \boldsymbol{u}, \partial_t \boldsymbol{b})\|_{L^2}^2.
\end{aligned}
$$

To proceed further, we establish a technical inequality, which could be of independent interest.

LEMMA 2.1. *For any $\gamma > 3$ and $\varepsilon_1, \varepsilon_2 > 0$, we have*

$$
(2.3)\qquad \|\nabla f\|_{L^{12/5}}^4 \le C\|\nabla f\|_{L^2}^4 + \varepsilon_1\big\|\nabla |f|^{\frac{\gamma+1}{2}}\big\|_{L^2}^2 + \varepsilon_2\|\Delta f\|_{L^2}^2.
$$

Proof. This follows from intricate applications of well-known inequalities as

$$
\begin{aligned}
\|\nabla f\|_{L^{12/5}}^4 & \le C\|f\|_{L^{\frac{15\gamma-33}{2\gamma-5}}}^{\frac{2(5\gamma-11)}{3\gamma-7}}\|\Delta f\|_{L^2}^{\frac{2(\gamma-3)}{3\gamma-7}} \qquad \text{(Gagliardo–Nirenberg inequality)} \\
& \le C\|f\|_{L^{3(\gamma+1)}}^{\frac{2(\gamma+1)}{3\gamma-7}}\|\nabla f\|_{L^2}^{\frac{8(\gamma-3)}{3\gamma-7}}\|\Delta f\|_{L^2}^{\frac{2(\gamma-3)}{3\gamma-7}} \\
& \qquad\qquad \text{(Gagliardo–Nirenberg inequality)} \\
& \le C\big\||f|^{\frac{\gamma+1}{2}}\big\|_{L^6}^{\frac{4}{3\gamma-7}}\|\nabla f\|_{L^2}^{\frac{8(\gamma-3)}{3\gamma-7}}\|\Delta f\|_{L^2}^{\frac{2(\gamma-3)}{3\gamma-7}}
\end{aligned}
$$

$$\leq C\big\|\nabla|f|^{\frac{\gamma+1}{2}}\big\|_{L^2}^{\frac{4}{3\gamma-7}}\|\nabla f\|_{L^2}^{\frac{8(\gamma-3)}{3\gamma-7}}\|\Delta f\|_{L^2}^{\frac{2(\gamma-3)}{3\gamma-7}} \qquad \text{(Sobolev inequality)}$$

$$\leq C\|\nabla f\|_{L^2}^4+\varepsilon_1\big\|\nabla|f|^{\frac{\gamma+1}{2}}\big\|_{L^2}^2+\varepsilon_2\|\Delta f\|_{L^2}^2 \qquad \text{(Young inequality).} \ \blacksquare$$

With Lemma 2.1 in hand, we estimate

$$\|(\nabla \boldsymbol{u},\nabla \boldsymbol{b})\|_{L^{12/5}}^4 \leq C\|(\nabla \boldsymbol{u},\nabla \boldsymbol{b})\|_{L^2}^4 +\varepsilon_1\big\|\big(\nabla|\boldsymbol{u}|^{\frac{\alpha+1}{2}},\nabla|\boldsymbol{b}|^{\frac{\beta+1}{2}}\big)\big\|_{L^2}^2+\varepsilon_2\|(\Delta \boldsymbol{u},\Delta \boldsymbol{b})\|_{L^2}^2. \tag{2.4}$$

Plugging (2.4) into (2.1) + (2.2), and taking ε_1 and ε_2 sufficiently small, we conclude that

$$\begin{aligned}\frac{d}{dt}\|(\nabla \boldsymbol{u},\nabla \boldsymbol{b})\|_{L^2}^2&+\frac{d}{dt}\left(\frac{1}{\alpha+1}\|\boldsymbol{u}\|_{L^{\alpha+1}}^{\alpha+1}+\frac{1}{\beta+1}\|\boldsymbol{b}\|_{L^{\beta+1}}^{\beta+1}\right)\\&+\big\|\big(|\boldsymbol{u}|^{\frac{\alpha-1}{2}}\nabla \boldsymbol{u},|\boldsymbol{b}|^{\frac{\beta-1}{2}}\nabla \boldsymbol{b}\big)\big\|_{L^2}^2\\&+\frac{2(\alpha-1)}{(\alpha+1)^2}\big\|\nabla|\boldsymbol{u}|^{\frac{\alpha+1}{2}}\big\|_{L^2}^2+\frac{2(\beta-1)}{(\beta+1)^2}\big\|\nabla|\boldsymbol{b}|^{\frac{\beta+1}{2}}\big\|_{L^2}^2\\&+\tfrac{1}{2}\|(\Delta \boldsymbol{u},\Delta \boldsymbol{b})\|_{L^2}^2+\tfrac{1}{2}\|(\partial_t \boldsymbol{u},\partial_t \boldsymbol{b})\|_{L^2}^2\\&\leq C\|(\nabla \boldsymbol{u},\nabla \boldsymbol{b})\|_{L^2}^4.\end{aligned}$$

Applying the Gronwall inequality, we deduce the desired a priori bounds (1.4). This concludes the proof of Theorem 1.1.

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