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TOPOLOGICAL CONJUGATION CLASSES OF TIGHTLY TRANSITIVE SUBGROUPS OF $\mathrm{Homeo}_+(\mathbb{R})$

BY

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Abstract. Let $\mathbb{R}$ be the real line and let $\mathrm{Homeo}_+(\mathbb{R})$ be the orientation preserving homeomorphism group of $\mathbb{R}$. Then a subgroup G of $\mathrm{Homeo}_+(\mathbb{R})$ is called tightly transitive if there is some point $x \in X$ such that the orbit Gx is dense in X and no subgroups H of G with $|G : H| = \infty$ have this property. In this paper, for each integer $n > 1$, we determine all the topological conjugation classes of tightly transitive subgroups G of $\mathrm{Homeo}_+(\mathbb{R})$ which are isomorphic to $\mathbb{Z}^n$ and have countably many nontransitive points.

1. Introduction and preliminaries. Let X be a topological space and let $\mathrm{Homeo}(X)$ be the homeomorphism group of X. Suppose that G is a subgroup of $\mathrm{Homeo}(X)$. The pair (X, G) is called a *dynamical system*. Recall that the *orbit* of $x \in X$ under G is the set $Gx = \{gx : g \in G\}$. For a subset $A \subseteq X$, define $GA = \bigcup_{x\in A} Gx$. A subset $A \subseteq X$ is said to be *G-invariant* if $GA = A$. If A is G-invariant, by the symbol $G|_A$ we mean the restriction to A of the action of G. If $A = \{x\}$ is G-invariant for some $x \in X$, then x is said to be a *fixed point* of G, that is, $gx = x$ for all $g \in G$. Let f be a homeomorphism on X. A point x is said to be a *fixed point* of f if x is a fixed point of the cyclic group $\langle f\rangle$ generated by f. We use the symbols $\mathrm{Fix}(G)$ and $\mathrm{Fix}(f)$ to denote the fixed point sets of G and f respectively.

For a dynamical system (X, G), G is said to be *topologically transitive* if for any two nonempty open subsets U and V of X, there is some g in G such that $g(U) \cap V \neq \emptyset$. If there is some point $x \in X$ such that the orbit Gx is dense in X then G is said to be *point transitive* and such x is called a *transitive point*. If x is not a transitive point then it is said to be a *nontransitive point*. It is well known that if G is countable and X is a Polish space without isolated points, then the notions of topological transitivity and point transitivity are the same. If for every $x \in X$, Gx is dense in X, then G is called *minimal*. A homeomorphism f on X is said to be

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topologically transitive (resp. *minimal*) if the cyclic group $\langle f\rangle$ is topologically transitive (resp. minimal). Topological transitivity is one of the most basic notions in dynamical systems. One may consult [2] for the discussions about topological transitivity for group actions.

Let $\mathbb{R}$ be the real line. Denote by $\text{Homeo}_+(\mathbb{R})$ the group of all orientation preserving homeomorphisms on $\mathbb{R}$. Let G and H be two subgroups of $\text{Homeo}_+(\mathbb{R})$. If there is a homeomorphism $\phi \in \text{Homeo}_+(\mathbb{R})$ such that $\phi G\phi^{-1} = H$, then G and H are said to be *topologically conjugate* (or *conjugate* for simplicity) by ϕ. If G is topologically transitive and no subgroup F of G with coset index $|G : F| = \infty$ is topologically transitive, then G is said to be *tightly transitive*. In [7], some topologically transitive solvable subgroups H of $\text{Homeo}_+(\mathbb{R})$ are constructed and some relationships between the algebraic structures and the dynamical properties of H are obtained. In this paper, we are interested in the classification problem of topologically transitive subgroups of $\text{Homeo}_+(\mathbb{R})$ up to topological conjugations. One may consult [1, 4, 5] for some surveys on the dynamics of subgroups of $\text{Homeo}_+(\mathbb{R})$.

In Section 2, we give some auxiliary results. In Section 3, for every irrational number $\alpha \in (0, 1)$ and every natural number $n \geq 2$, we construct a tightly transitive subgroup $G_{\alpha,n}$ of $\text{Homeo}_+(\mathbb{R})$ which is isomorphic to $\mathbb{Z}^n$, and we show that $G_{\alpha,n}$ and $G_{\beta,n}$ are topologically conjugate if and only if there are integers m_1, n_1, m_2, n_2 with $|m_1n_2 - n_1m_2| = 1$ such that $(m_1 + n_1\alpha)/(m_2 + n_2\alpha) = \beta$. In Section 4, we show that every tightly transitive subgroup G of $\text{Homeo}_+(\mathbb{R})$ which is isomorphic to $\mathbb{Z}^n$ and has countably many nontransitive points is topologically conjugate to some $G_{\alpha,n}$ constructed in Section 3. Thus we completely determine the topological conjugation classes of tightly transitive subgroups G of $\text{Homeo}_+(\mathbb{R})$ which are isomorphic to $\mathbb{Z}^n$ and have countably many nontransitive points.

2. Auxiliary results

LEMMA 2.1. *Suppose that H is a topologically transitive subgroup of* $\text{Homeo}_+(\mathbb{R})$*. If for some $x \in \mathbb{R}$ the set $S = Hx$ is not dense in $\mathbb{R}$, then the closure $K = \overline{S}$ is nowhere dense and* $\inf K = -\infty$, $\sup K = \infty$.

Proof. Assume to the contrary that K contains an open interval (a, b). Then for any open interval $(a', b') \subset \mathbb{R}$, by the topological transitivity of H, there is some $h \in H$ such that $h((a', b')) \cap (a, b) \neq \emptyset$. Thus there is some k in H such that $k(x) \in h((a', b'))$, that is, $h^{-1}k(x) \in (a', b')$. By the arbitrariness of (a', b'), we see that Hx is dense in $\mathbb{R}$. This is a contradiction. So K is nowhere dense in $\mathbb{R}$.

Let $\alpha = \inf K$ and $\beta = \sup K$. If $\beta < \infty$, then β is a fixed point of H. Since each element of H is an orientation preserving homeomorphism,

(β, ∞) is H-invariant, which contradicts the topological transitivity of H. So $\beta = \infty$. Similarly, $\alpha = -\infty$. ∎

LEMMA 2.2. *Let H be a topologically transitive subgroup of* $\mathrm{Homeo}_+(\mathbb{R})$ *which is isomorphic to $\mathbb{Z}^2$. Then H is minimal.*

Proof. Assume to the contrary that there is some $x \in \mathbb{R}$ whose orbit Hx is not dense in $\mathbb{R}$. Let $K = \overline{Hx}$. By Lemma 2.1, $\mathbb{R} \setminus K$ is a disjoint union of countably infinitely many open intervals $\{(a_i, b_i) : i \in \mathbb{Z}\}$. Let $F = \{h \in H : h((a_0, b_0)) = (a_0, b_0)\}$. Since H permutes these intervals (a_i, b_i), the restriction $F|_{(a_0,b_0)}$ must be topologically transitive. So $F \cong \mathbb{Z}^2$ (note that no homeomorphism on (a_0, b_0) is topologically transitive). Thus $m \equiv |H : F| < \infty$. Let $H = h_1F \cup \cdots \cup h_mF$ be the coset decomposition. Then the open set $U = \bigcup_{i=1}^m h_i((a_0, b_0))$ is H-invariant, which contradicts the topological transitivity of H. ∎

LEMMA 2.3. *Let G be a commutative subgroup of* $\mathrm{Homeo}_+(\mathbb{R})$ *and let $f \in$* $\mathrm{Homeo}_+(\mathbb{R})$ *commute with each element of G. If* $\mathrm{Fix}(G), \mathrm{Fix}(f) \neq \emptyset$, *then* $\mathrm{Fix}(G) \cap \mathrm{Fix}(f) \neq \emptyset$.

Proof. If $\mathrm{Fix}(G) \subseteq \mathrm{Fix}(f)$, then the conclusion holds. So we may suppose that there is some $x \in \mathrm{Fix}(G) \setminus \mathrm{Fix}(f)$. This means that there is a maximal interval $(a, b) \subset \mathbb{R} \setminus \mathrm{Fix}(f)$ such that $x \in (a, b)$ (a may be $-\infty$ and b may be ∞). Since $\mathrm{Fix}(f) \neq \emptyset$, either $a \in \mathrm{Fix}(f)$ or $b \in \mathrm{Fix}(f)$. Without loss of generality, we may suppose that a is a fixed point of f. Thus either $\lim_{n\to\infty} f^n(x) = a$ or $\lim_{n\to\infty} f^{-n}(x) = a$. Since x is a fixed point of G and f commutes with each element of G, we see that $f^n(x)$ is a fixed point of G for every $n \in \mathbb{Z}$. It follows that a is a fixed point of G. Therefore $\mathrm{Fix}(G) \cap \mathrm{Fix}(f) \neq \emptyset$. ∎

PROPOSITION 2.4. *Let H be a subgroup of* $\mathrm{Homeo}_+(\mathbb{R})$ *which is isomorphic to $\mathbb{Z}^n$ for some $n \geq 1$, and let $\{e_i : i = 1, \ldots, n\}$ be a basis of H as a $\mathbb{Z}$-module. If* $\mathrm{Fix}(e_i) \neq \emptyset$ *for each $i = 1, \ldots, n$, then* $\mathrm{Fix}(H) \neq \emptyset$.

Proof. This can be deduced from Lemma 2.3 by induction. ∎

Let $a, b \in \mathbb{R}$. Denote by L_a the translation of $\mathbb{R}$ by a, that is, $L_a(x) = x+a$ for every $x \in \mathbb{R}$; denote by $\langle L_a, L_b\rangle$ the subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ which is generated by L_a and L_b. The lemma below follows from Plante's Theorem (see [6, Theorem 1.3]). For the convenience of the reader, we give a direct proof.

LEMMA 2.5. *Let H be a subgroup of* $\mathrm{Homeo}_+(\mathbb{R})$ *which is isomorphic to $\mathbb{Z}^n$. Then there is an H-invariant Borel measure μ on $\mathbb{R}$ which is finite on compact sets.*

Proof. If every $h \in H$ has a fixed point in $\mathbb{R}$, then $\mathrm{Fix}(H) \neq \emptyset$ by Proposition 2.4. Fix an $x \in \mathrm{Fix}(H)$. Then the Dirac measure δ_x is an H-

invariant Borel measure on $\mathbb{R}$ which is finite on compact sets. So we may as well suppose that there is an $h \in H$ that has no fixed point. Passing to a conjugation if necessary, we may further suppose that h is the unit translation L_1 on $\mathbb{R}$. Let $\pi : \mathbb{R} \to \mathbb{S}^1$, $x \mapsto e^{2\pi i x}$, be the covering map. Since each element in H commutes with h $(= L_1)$, H naturally induces an orientation preserving action on $\mathbb{S}^1$. By the amenability of H and the compactness of $\mathbb{S}^1$, there is an H-invariant finite Borel measure ν on $\mathbb{S}^1$. Define a Borel measure μ on $\mathbb{R}$ by

$$\mu(A) = \sum_{i=-\infty}^{\infty} \nu\big(\pi(A \cap [i, i+1))\big)$$

for any Borel subset A of $\mathbb{R}$. Then ν is the required H-invariant measure on $\mathbb{R}$. ∎

LEMMA 2.6. *Let H be a minimal subgroup of* $\mathrm{Homeo}_+(\mathbb{R})$. *If there is an H-invariant Borel measure μ on $\mathbb{R}$ which is finite on compact sets, then H is topologically conjugate to a subgroup G of* $\mathrm{Homeo}_+(\mathbb{R})$ *which consists of translations.*

Proof. Since H is minimal, the support of μ is the whole real line $\mathbb{R}$ and μ has no atoms. Let $\phi : \mathbb{R} \to \mathbb{R}$ be the map defined by

$$\phi(x) = \begin{cases} \mu([0, x]) & \text{if } x \geq 0, \\ -\mu([x, 0]) & \text{if } x < 0, \end{cases}$$

and let $G = \phi H \phi^{-1}$. Then ϕ is an orientation preserving homeomorphism and the group G consists of translations on $\mathbb{R}$. ∎

LEMMA 2.7. *Let H be a tightly transitive subgroup of* $\mathrm{Homeo}_+(\mathbb{R})$ *which is isomorphic to $\mathbb{Z}^n$ for some $n \geq 3$. Then H cannot be minimal.*

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis of H as a $\mathbb{Z}$-module. Assume to the contrary that H is minimal. By Lemmas 2.5 and 2.6, there is an orientation preserving homeomorphism $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi H \phi^{-1}$ consists of translations on $\mathbb{R}$. Thus we may let $\phi e_i \phi^{-1} = L_{\alpha_i}$ for some real numbers α_i. Since $\phi H \phi^{-1}$ is tightly transitive, $\alpha_1 \neq 0$ and α_k/α_1 is an irrational number for some $k \in \{2, \ldots, n\}$. Thus the group $\langle L_{\alpha_1}, L_{\alpha_k} \rangle$ generated by L_{α_1} and L_{α_k} is minimal. This contradicts the fact that $\phi H \phi^{-1}$ is tightly transitive (note that $n \geq 3$). So H cannot be minimal. ∎

It is well known that if X is a compact metric space and G is a subgroup of $\mathrm{Homeo}(X)$, then there must be a G-invariant closed subset K of X such that $G|_K$ is minimal. In general, this conclusion does not hold when the phase space is not compact.

PROPOSITION 2.8. *Let H be a subgroup of* $\mathrm{Homeo}_+(\mathbb{R})$ *which is isomorphic to $\mathbb{Z}^n$ for some $n \geq 1$. Suppose A is an H-invariant closed subset* *of* $\mathbb{R}$ *such that* $\inf A = -\infty$ *and* $\sup A = \infty$. *If* $H|_A$ *is topologically transitive, then* A *contains a minimal* H*-invariant closed subset* M *of* $\mathbb{R}$ *with* $\inf M = -\infty$ *and* $\sup M = \infty$.

Proof. First we have the following claim:

CLAIM A. *There is an* $f \in H$ *that has no fixed point in* $\mathbb{R}$.

In fact, if $\mathrm{Fix}(g) \neq \emptyset$ for every $g \in H$, then $\mathrm{Fix}(H) \neq \emptyset$ by Proposition 2.4. Fix $c \in \mathrm{Fix}(H)$. Then $(-\infty, c)$ and (c, ∞) are both H-invariant. Since $\inf A = -\infty$ and $\sup A = \infty$, both $A \cap (-\infty, c)$ and $A \cap (c, \infty)$ are H-invariant nonempty open subset of A in the relative topology. But this contradicts the topological transitivity of $H|_A$. This completes the proof of the claim.

Now use Claim A to fix an $f \in H$ that has no fixed point in $\mathbb{R}$. Then f or f^{-1} is conjugate to the unit translation L_1 on $\mathbb{R}$ by an orientation preserving homeomorphism on $\mathbb{R}$. Without loss of generality, we may suppose that $f = L_1$. This implies the following claim:

CLAIM B. *For any* H*-invariant nonempty closed subset* B *of* $\mathbb{R}$, *we have* $B \cap [0, 2] \neq \emptyset$.

Let $\mathcal{F}$ be the family of all H-invariant nonempty closed subsets of A. Then $\mathcal{F} \neq \emptyset$ for $A \in \mathcal{F}$. Let $\succeq$ be a partial order in $\mathcal{F}$ defined by $B \succeq C$ if and only if $B \subseteq C$ for $B, C \in \mathcal{F}$. If $\{A_\lambda : \lambda \in \Lambda\}$ is a chain in $\mathcal{F}$, then from Claim B and the compactness of $[0, 2]$ we obtain $\bigcap_{\lambda \in \Lambda} A_\lambda \supset \bigcap_{\lambda \in \Lambda}(A_\lambda \cap [0, 2]) \neq \emptyset$. This means that $\bigcap_{\lambda \in \Lambda} A_\lambda$ is an upper bound of $\{A_\lambda : \lambda \in \Lambda\}$. By Zorn's Lemma there is a maximal element M in $\mathcal{F}$. Thus M is a minimal H-invariant closed subset of A. Since L_1 belongs to H, we have $\inf M = -\infty$ and $\sup M = \infty$. ■

3. Construction and properties of $G_{\alpha,n}$. For every irrational α in $(0, 1)$ and every positive integer $n \geq 2$, we will construct by induction a tightly transitive subgroup $G_{\alpha,n}$ of $\mathrm{Homeo}_+(\mathbb{R})$ which is isomorphic to $\mathbb{Z}^n$.

When $k = 2$, take $G_{\alpha,2} = \langle L_1, L_\alpha \rangle$. Then $G_{\alpha,2}$ is tightly transitive and is isomorphic to $\mathbb{Z}^2$.

Suppose that $G_{\alpha,k}$ has been constructed for $2 \leq k \leq n - 1$. Then we construct $G_{\alpha,n}$ as follows. Let $\hbar : \mathbb{R} \to (0, 1)$ be the homeomorphism defined by

$$\hbar(x) = \frac{1}{\pi}(\arctan x + \pi/2) \quad \text{for } x \in \mathbb{R}.$$

For $g \in G_{\alpha,n-1}$, define $\tilde{g} \in \mathrm{Homeo}_+(\mathbb{R})$ by

$$\tilde{g}(x) = \begin{cases} \hbar g \hbar^{-1}(x - i) + i & \text{for } x \in (i, i + 1),\ i \in \mathbb{Z}, \\ x & \text{for } x \in \mathbb{Z}. \end{cases}$$

Clearly $\tilde{g}$ and L_1 commute for each $g \in G_{\alpha,n-1}$. Let $G_{\alpha,n}$ be the group generated by $\{\tilde{g} : g \in G_{\alpha,n-1}\} \cup \{L_1\}$.

By the above construction, we immediately have

LEMMA 3.1.

(i) *$G_{\alpha,n}$ is tightly transitive and is isomorphic to $\mathbb{Z}^n$.*
(ii) *Let $\operatorname{intr} G_{\alpha,n}$ denote the set of nontransitive points of $G_{\alpha,n}$. Then $\operatorname{intr} G_{\alpha,3} = \mathbb{Z}$ and $\operatorname{intr} G_{\alpha,n} = \bigcup_{i\in\mathbb{Z}} \hbar(\operatorname{intr} G_{\alpha,n-1}) + i$ for $n \geq 4$.*
(iii) *Suppose (a, b) is a connected component of $\mathbb{R} \setminus \operatorname{intr} G_{\alpha,n}$ and $F = \{g \in G_{\alpha,n} : g((a, b)) = (a, b)\}$. Then $F|_{(a,b)}$ is minimal and is homeomorphic to $\mathbb{Z}^2$.*

For $a \in \mathbb{R}$, define $M_a : \mathbb{R} \to \mathbb{R}$ by $M_a(x) = ax$ for all $x \in \mathbb{R}$. If $a > 0$, then $M_a \in \mathrm{Homeo}_+(\mathbb{R})$.

PROPOSITION 3.2. *Let H be a topologically transitive subgroup of the group $\mathrm{Homeo}_+(\mathbb{R})$ and assume that H is isomorphic to $\mathbb{Z}^2$. Then H is conjugate to $G_{\delta,2}$ for some irrational $\delta \in (0, 1)$.*

Proof. From Lemmas 2.2 and 2.5, H is minimal and there is an H-invariant Borel measure μ on $\mathbb{R}$ which is finite on compact sets. Lemma 2.6 yields an orientation preserving homeomorphism ϕ on $\mathbb{R}$ such that $\phi H \phi^{-1}$ consists of translations on $\mathbb{R}$. Select two generators $L_\alpha, L_\beta \in \phi H \phi^{-1}$ such that $0 < \alpha < \beta$. Then $M_{\beta^{-1}} L_\beta M_\beta = L_1$ and $M_{\beta^{-1}} L_\alpha M_\beta = L_{\alpha/\beta}$. Thus H is conjugate to the group $G_{\alpha/\beta,2} = \langle L_1, L_{\alpha/\beta}\rangle$ by the orientation preserving homeomorphism $M_{\beta^{-1}}\phi$. Notice that $0 < \alpha/\beta < 1$. ■

LEMMA 3.3. *Let $0 < \alpha, \beta < 1$ be irrational. Then the subgroups $G_{\alpha,2}$ and $G_{\beta,2}$ are conjugate by an orientation preserving homeomorphism if and only if there are integers m_1, n_1, m_2, n_2 with $|m_1 n_2 - n_1 m_2| = 1$ such that $(m_1 + n_1\beta)/(m_2 + n_2\beta) = \alpha$.*

Proof. Suppose that $G_{\alpha,2}$ and $G_{\beta,2}$ are conjugate by an $h \in \mathrm{Homeo}_+(\mathbb{R})$, so $G_{\beta,2} = hG_{\alpha,2}h^{-1}$. Let $hL_1h^{-1} = L_u$ and $hL_\alpha h^{-1} = L_v$ for some $u, v \in \mathbb{R}$.

We may assume that $h(0) = 0$, otherwise we need only replace h by $L_{-h(0)} \circ h$. Since h preserves the orientation of $\mathbb{R}$, we have

$$v = L_v(0) = hL_\alpha h^{-1}(0) = h(\alpha) < h(1) = hL_1h^{-1}(0) = L_u(0) = u$$

and

$$0 = h(0) < h(\alpha) = v.$$

Thus $0 < v < u$. Let $f = M_{1/u} \circ h$. Then f is an orientation preserving homeomorphism on $\mathbb{R}$, and

$$f \circ L_\alpha = L_{v/u} \circ f \quad \text{and} \quad f \circ L_1 = L_1 \circ f. \tag{3.1}$$

Since L_α, $L_{v/u}$ and f commute with L_1, we get three naturally induced orientation preserving homeomorphisms $\tilde{L}_\alpha, \tilde{L}_{v/u}, \tilde{f} : \mathbb{S}^1 \to \mathbb{S}^1$ on the unit circle $\mathbb{S}^1$. By the first equation in (3.1) we immediately get

$$\tilde{f} \circ \tilde{L}_\alpha = \tilde{L}_{v/u} \circ \tilde{f},$$

that is, the rotations $\tilde{L}_\alpha$ and $\tilde{L}_{v/u}$ are conjugate by the orientation preserving homeomorphism $\tilde{f}$. Since $0 < \alpha, v/u < 1$, we obtain $\alpha = v/u$, i.e., $v = \alpha u$. Thus

$$\langle L_1, L_\beta\rangle = G_{\beta,2} = hG_{\alpha,2}h^{-1} = \langle hL_1h^{-1}, hL_\alpha h^{-1}\rangle = \langle L_u, L_v\rangle = \langle L_u, L_{\alpha u}\rangle.$$

Since $G_{\beta,2} \cong \mathbb{Z}^2$, there are integers m_1, n_1, m_2, n_2 with $|m_1n_2 - n_1m_2| = 1$ such that

$$L_1^{m_1} \circ L_\beta^{n_1} = L_{\alpha u}, \qquad L_1^{m_2} \circ L_\beta^{n_2} = L_u.$$

So

$$L_1^{m_1} \circ L_\beta^{n_1}(0) = L_{\alpha u}(0), \qquad L_1^{m_2} \circ L_\beta^{n_2}(0) = L_u(0),$$

that is,

$$m_1 + n_1\beta = \alpha u, \qquad m_2 + n_2\beta = u.$$

Thus $(m_1 + n_1\beta)/(m_2 + n_2\beta) = \alpha$.

On the other hand, if there are integers m_1, n_1, m_2, n_2 such that we have $|m_1n_2 - n_1m_2| = 1$ and $(m_1 + n_1\beta)/(m_2 + n_2\beta) = \alpha$, then let $u = m_2 + n_2\beta$. We may suppose $u > 0$, otherwise we need only replace m_1, m_2, n_1, n_2 by $-m_1, -m_2, -n_1, -n_2$ respectively. Thus

$$L_1^{m_1} \circ L_\beta^{n_1} = L_{\alpha u}, \qquad L_1^{m_2} \circ L_\beta^{n_2} = L_u.$$

Since $|m_1n_2 - n_1m_2| = 1$, we obtain

$$\mathbb{Z}^2 \cong G_{\beta,2} = \langle L_1, L_\beta\rangle = \langle L_u, L_{\alpha u}\rangle. \tag{3.2}$$

Noting that

$$M_uL_1M_u^{-1} = L_u \quad \text{and} \quad M_uL_\alpha M_u^{-1} = L_{\alpha u}, \tag{3.3}$$

we have $M_uG_{\alpha,2}M_u^{-1} = G_{\beta,2}$ from (3.2), that is, $G_{\alpha,2}$ and $G_{\beta,2}$ are topologically conjugate by the orientation preserving homeomorphism M_u. ∎

THEOREM 3.4. *For any $n \geq 2$ and any irrational numbers $0 < \alpha, \beta < 1$, $\alpha \neq \beta$, the subgroups $G_{\alpha,n}$ and $G_{\beta,n}$ are conjugate by an orientation preserving homeomorphism if and only if there are integers m_1, n_1, m_2, n_2 with $|m_1n_2 - n_1m_2| = 1$ such that $(m_1 + n_1\alpha)/(m_2 + n_2\alpha) = \beta$.*

Proof. Necessity. For each $G_{\alpha,n}$, by Lemma 3.1, there is an open interval (a, b) in $\mathbb{R}$ such that the restriction to (a, b) of the group $F \equiv \{g \in G_{\alpha,n} : g((a, b)) = (a, b)\}$ is minimal. We call such an open interval (a, b) a *minimal interval* of $G_{\alpha,n}$. (When $n = 2$, $(a, b) = \mathbb{R}$, and when $n > 2$, (a, b) is a proper subinterval of $\mathbb{R}$.) By Lemma 2.7 and Proposition 3.2, F is isomorphic to $\mathbb{Z}^2$

and the restriction $F|_{(a,b)}$ is conjugate to $G_{\alpha,2}$ on $\mathbb{R}$. So, if $G_{\alpha,n}$ and $G_{\beta,n}$ are conjugate by an orientation preserving homeomorphism $h:\mathbb{R}\to\mathbb{R}$, then h maps a minimal interval (a,b) of $G_{\alpha,n}$ to a minimal interval $(h(a),h(b))$ of $G_{\beta,n}$. Thus h also induces an orientation preserving conjugation between $G_{\alpha,2}$ and $G_{\beta,2}$ on $\mathbb{R}$. So the conclusion holds by Lemma 3.3.

Sufficiency. We proceed by induction. Suppose that there are integers m_1,n_1,m_2,n_2 with $|m_1n_2-n_1m_2|=1$ such that $(m_1+n_1\alpha)/(m_2+n_2\alpha) = \beta$. By Lemma 3.3, $G_{\alpha,2}$ and $G_{\beta,2}$ are conjugate by an orientation preserving homeomorphism, that is, the theorem holds for $n=2$. Assume that $G_{\alpha,n-1}$ and $G_{\beta,n-1}$ are conjugate by an orientation preserving homeomorphism $h:\mathbb{R}\to\mathbb{R}$. Now define $\tilde{h}:\mathbb{R}\to\mathbb{R}$ by

$$\tilde{h}(x)=\begin{cases}\hbar h\hbar^{-1}(x-i)+i & \text{for } x\in(i,i+1),\ i\in\mathbb{Z},\\ x & \text{for } x\in\mathbb{Z},\end{cases}$$

where $\hbar$ is defined at the beginning of this section. Then it is a direct check that $\tilde{h}$ is an orientation preserving conjugation between $G_{\alpha,n}$ and $G_{\beta,n}$. ∎

4. The main result

THEOREM 4.1. *Let H be a tightly transitive subgroup of* $\mathrm{Homeo}_+(\mathbb{R})$ *which is isomorphic to $\mathbb{Z}^n$ for some $n\geq 2$ and has countably many nontransitive points. Then H is conjugate to $G_{\alpha,n}$ by an orientation preserving homeomorphism for some irrational $\alpha\in(0,1)$.*

Proof. We use induction. From Proposition 3.2, the theorem holds for $n=2$. Assume that it holds for $n=k$ where $k\geq 2$. Let H be a tightly transitive subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ which is isomorphic to $\mathbb{Z}^{k+1}$ and has countably many nontransitive points. From Lemma 2.7, we know that H is not minimal. So there is some $x\in\mathbb{R}$ such that $S=Hx$ is not dense in $\mathbb{R}$. Let $K=\overline{S}$. From Lemma 2.1, K is nowhere dense, $\inf K=-\infty$ and $\sup K=\infty$. By Proposition 2.8, there is a minimal H-invariant closed subset M of K such that $\inf M=-\infty$ and $\sup M=\infty$. Since M is countable by assumption (because every point in M is nontransitive), M must have an isolated point. Since M is also minimal, every point of M is isolated. Thus we may suppose that

$$M=\{\cdots<a_{-1}<a_0<a_1<\cdots\}\quad\text{with }\lim_{i\to\infty}a_i=\infty,\ \lim_{i\to-\infty}a_i=-\infty.$$

Let $F=\{g\in H: g((a_0,a_1))=(a_0,a_1)\}$. Since H permutes these intervals (a_i,a_{i+1}) transitively, we have

$$|H:F|=\infty. \tag{4.1}$$

Clearly the restricted action $F|_{(a_0,a_1)}$ is topologically transitive. Let $f\in H$ be such that $f(a_0)=a_1$. Then $f(a_i)=a_{i+1}$ for all $i\in\mathbb{Z}$, as f is orientation preserving. Hence the group $\langle F,f\rangle$ generated by f and F is topologically transitive. By the tight transitivity of H, we see that $\langle F, f\rangle$ has finite index in H, that is,

$$|H : \langle F, f\rangle| < \infty. \tag{4.2}$$

From (4.1) and (4.2) we obtain $F \cong \mathbb{Z}^k$. For any $g \in H \setminus F$, since g preserves the orientation of $\mathbb{R}$, we have $g^i((a_0, a_1)) \cap (a_0, a_1) = \emptyset$ for all $i \neq 0$, that is, $g^i \notin F$ for all $i \neq 0$. Thus H/F is torsion free, which means that H/F is an infinite cyclic group. This implies that $H = F \oplus \langle f\rangle$.

Since $F|_{(a_0,a_1)}$ is tightly transitive with countably many nontransitive points and $F \cong \mathbb{Z}^k$, by the inductive hypothesis there is an orientation preserving homeomorphism $h : (a_0, a_1) \to \mathbb{R}$ such that $hF|_{(a_0,a_1)}h^{-1} = G_{\alpha,k}$ for some irrational $\alpha \in (0, 1)$. Now define a homeomorphism $\tilde{h} : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{h}(x) = \begin{cases} \hbar h f^{-i}(x) + i & \text{for } x \in (a_i, a_{i+1}),\ i \in \mathbb{Z}, \\ i & \text{for } x = a_i,\ i \in \mathbb{Z}, \end{cases}$$

($\hbar$ is defined at the beginning of Section 3). Then $\tilde{h} f \tilde{h}^{-1}(x) = x + 1$ for all $x \in \mathbb{R}$. Let

$$\widetilde{G_{\alpha,k+1}} = \{g \in G_{\alpha,k+1} : g((0, 1)) = (0, 1)\}.$$

Then

$$\hbar h F|_{(a_0,a_1)} h^{-1} \hbar^{-1} = \hbar G_{\alpha,k} \hbar^{-1} = \widetilde{G_{\alpha,k+1}}|_{(0,1)},$$

which implies that $\tilde{h} F \tilde{h}^{-1} = \widetilde{G_{\alpha,k+1}}$ by the definition of $\tilde{h}$. Since $H = F \oplus \langle f\rangle$, $\tilde{h}$ is an orientation preserving conjugacy between H and $G_{\alpha,k+1}$. ■

We finish by constructing a tightly transitive subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ which is isomorphic to $\mathbb{Z}^4$ and has uncountably many nontransitive points. This example shows in particular that the class of tightly transitive subgroups of $\mathrm{Homeo}_+(\mathbb{R})$ which are isomorphic to $\mathbb{Z}^n$ and have countably many nontransitive points is a proper subclass of all tightly transitive subgroups of $\mathrm{Homeo}_+(\mathbb{R})$ which are isomorphic to $\mathbb{Z}^n$.

EXAMPLE 4.2. First we construct a Denjoy homeomorphism on the circle (see e.g. [3, p. 107]). Let $\rho_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$, $e^{2\pi \mathrm{i} x} \mapsto e^{2\pi \mathrm{i}(x+\alpha)}$, where α is irrational. Take any $\theta_0 \in \mathbb{S}^1$. At the point $\rho_\alpha^n(\theta_0)$, we cut the circle and glue in a small interval I_n which satisfies $\sum_{n=-\infty}^{\infty} l(I_n) < \infty$, where $l(I_n)$ denotes the length of I_n. The result of this operation is still a circle. Then extend the rotation ρ_α to the union of the I_n's by choosing any orientation preserving homeomorphism h_n taking I_n to I_{n+1}. We obtain a homeomorphism f on the new circle, which is called a *Denjoy homeomorphism*.

Now fix two orientation preserving homeomorphisms g, h on I_0 such that $fg = gf$ and $\langle f, g\rangle$ is topologically transitive. Define two homeomorphisms

$\bar{g}$ and $\bar{h}$ on the new circle by

$$\bar{g}(x) = \begin{cases} f^n g f^{-n}(x) & \text{for } x \in I_n \ (= f^n(I_0)),\ n \in \mathbb{Z}, \\ x & \text{otherwise,} \end{cases}$$

$$\bar{h}(x) = \begin{cases} f^n h f^{-n}(x) & \text{for } x \in I_n \ (= f^n(I_0)),\ n \in \mathbb{Z}, \\ x & \text{otherwise.} \end{cases}$$

Then the group $\langle f, \bar{g}, \bar{h} \rangle$ is tightly transitive and isomorphic to $\mathbb{Z}^3$. Passing to a conjugation, we may suppose f, $\bar{g}$ and $\bar{h}$ are defined on the unit circle $\mathbb{S}^1$ in the complex plane. Let $\tilde{f}$, $\tilde{g}$ and $\tilde{h}$ be some fixed liftings to the line of f, $\bar{g}$ and $\bar{h}$ respectively via the quotient map $\pi : \mathbb{R} \to \mathbb{S}^1$, $x \mapsto e^{2\pi i x}$. Thus the group $\langle \tilde{f}, \tilde{g}, \tilde{h}, L_1 \rangle$ is tightly transitive, is isomorphic to $\mathbb{Z}^4$, and has uncountably many nontransitive points.

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