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De Lellis–Topping type inequalities for f-Laplacians

by

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Abstract. We establish an integral geometric inequality on a closed Riemannian manifold with ∞-Bakry–Émery Ricci curvature bounded from below. We also obtain similar inequalities for Riemannian manifolds with totally geodesic boundary. In particular, our results generalize those of Wu (2014) for the ∞-Bakry–Émery Ricci curvature.

1. Introduction. Let (M,g) be an n-dimensional smooth Riemannian manifold with $n\geq 3$ and f be a C^2 function on M. We denote by ∇, Δ and ∇^2 the gradient, Laplacian and Hessian operator on M with respect to g, respectively. Ric and R denote the Ricci curvature and the scalar curvature, respectively. We let

$$\mathrm{Ric}_f=\mathrm{Ric}+\nabla^2 f \tag{1.1}$$

stand for the Bakry–Émery Ricci curvature which is also called the ∞-*Bakry–Émery Ricci curvature*, i.e., the $m=\infty$ case of the m-Bakry–Émery Ricci curvature defined by

$$\mathrm{Ric}_f^m=\mathrm{Ric}_f-\frac{1}{m-n}\nabla f\otimes\nabla f \tag{1.2}$$

with $m\geq n$ a constant, and $m=n$ if and only if f is a constant. We define the f-*Laplacian*

$$\Delta_f:=e^f\,\mathrm{div}(e^{-f}\nabla)=\Delta-\nabla f\nabla,$$

which is a self-adjoint operator with respect to the $L^2(M)$ inner product:

$$\int_M u\Delta_f v e^{-f}\,dv_g=-\int_M \nabla u\nabla v e^{-f}\,dv_g=\int_M v\Delta_f u e^{-f}\,dv_g,\qquad \forall u,v\in C_0^\infty(M),$$

where dv_g is the volume form on M.

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An n-dimensional Riemannian manifold (M,g) is said to be *Einstein* if its traceless Ricci tensor $\mathrm{Ric}-(R/n)g$ is identically zero. The classical Schur lemma states that the scalar curvature of an Einstein manifold of dimension $n\ge 3$ must be constant. Recently, De Lellis and Topping [DT] (and independently Andrews, cf. [CLN, Corollary B. 20]) proved the following *almost-Schur lemma*:

THEOREM 1.1 ([DT]). *If (M,g) is a closed Riemannian manifold of dimension $n\ge 3$ with non-negative Ricci curvature, then*

$$\int_M (R-\overline{R})^2\,dv_g \le \frac{4n(n-1)}{(n-2)^2}\int_M \left|\mathrm{Ric}-\frac{R}{n}g\right|^2 dv_g, \tag{1.3}$$

or equivalently,

$$\int_M \left|\mathrm{Ric}-\frac{\overline{R}}{n}g\right|^2 dv_g \le \frac{n^2}{(n-2)^2}\int_M \left|\mathrm{Ric}-\frac{R}{n}g\right|^2 dv_g, \tag{1.4}$$

where $\overline{R}$ denotes the average of R over M. Moreover, equality holds in (1.3) *or* (1.4) *if and only if M is Einstein.*

Generalizing De Lellis and Topping's results, Cheng [C] proved an almost-Schur lemma for closed manifolds without assuming the non-negativity of the Ricci curvature. That is, he obtained a similar inequality with the coefficient depending not only on the lower bound of the Ricci curvature but also on the value of the first non-zero eigenvalue of the Laplace operator. In the case of dimension $n=3,4$, Ge and Wang [GW1, GW2] proved that Theorem 1.1 holds under the weaker condition of non-negative scalar curvature. However, as pointed out by De Lellis and Topping [DT], the coefficient in (1.3) is optimal and the non-negativity of the Ricci curvature cannot be removed when $n\ge 5$. For the recent research in this direction, see [GW1, B, H, CZ, K, GWX, P] and the references therein.

Recently, Wu [W] established an integral geometric inequality under the assumption that the m-Bakry–Émery Ricci curvature is bounded from below. More precisely, he proved

THEOREM 1.2 ([W]). *If (M,g) is a closed Riemannian manifold of dimension $n\ge 3$ and f is a $C^2(M)$ function with $\mathrm{Ric}_f^m \ge \frac{1}{m-n}|\nabla f|^2$, then*

$$\begin{aligned}\int_M (N_f^m-\overline{N}_f^m)^2 e^{-f}\,dv_g \\ \le \frac{4(m-n+1)(m-n-2)^2}{(m-n)^3}\int_M \left|\mathrm{Ric}_f^m+\frac{\mathrm{tr}\,\mathrm{Ric}_f^m}{m-n-2}g\right|^2 e^{-\frac{m-n+4}{m-n}f}\,dv_g,\end{aligned}\tag{1.5}$$

where

$$T_f^m := R + 2\frac{m-n-1}{m-n}\Delta f - \frac{m-n-1}{m-n}|\nabla f|^2,$$
$$N_f^m := e^{-\frac{2}{m-n}f}T_f^m, \quad \overline{N}_f^m := \int_M N_f^m e^{-f}\,dv_g \Big/ \int_M e^{-f}\,dv_g.$$

Moreover, equality holds in (1.5) *if and only if*

$$\mathrm{Ric}_f^m + \frac{\operatorname{tr}\mathrm{Ric}_f^m}{m-n-2}g = 0. \tag{1.6}$$

In particular, letting $m \to \infty$ in (1.5) yields the following inequality for the ∞-Bakry–Émery Ricci curvature:

Theorem 1.3 ([W]). *If* (M,g) *is a closed Riemannian manifold of dimension* $n \geq 3$ *and* f *is a* $C^2(M)$ *function with* $\mathrm{Ric}_f \geq 0$, *then*

$$\int_M (T_f - \overline{T}_f)^2 e^{-f}\,dv_g \leq 4\int_M |\mathrm{Ric}_f|^2 e^{-f}\,dv_g, \tag{1.7}$$

where

$$T_f := R + 2\Delta f - |\nabla f|^2, \quad \overline{T}_f := \int_M T_f e^{-f} dv_g \Big/ \int_M e^{-f} dv_g.$$

Moreover, equality holds in (1.7) *if and only if* $\mathrm{Ric}_f = 0$.

In this paper, we are interested in manifolds without the assumption of non-negative ∞-Bakry–Émery Ricci curvature. More precisely, we prove

Theorem 1.4. *If* (M,g) *is a closed Riemannian manifold of dimension* $n \geq 3$ *and* f *is a* $C^2(M)$ *function, then*

$$\int_M (N_f - \overline{N_f})^2 e^{-f}\,dv_g \leq 4\left(1 + \frac{K}{\eta_1}\right)\int_M |\mathrm{Ric}_f - \lambda g|^2 e^{-f}\,dv_g, \tag{1.8}$$

where η_1 *denotes the first non-zero eigenvalue of the* f*-Laplacian* Δ_f *on* M, K *is a non-negative constant such that the* ∞*-Bakry–Émery Ricci curvature* Ric_f *satisfies* $\mathrm{Ric}_f \geq -K$, *and* $\lambda \geq -K$ *is a real constant. Here*

$$N_f = R + 2\Delta f - |\nabla f|^2 + 2\lambda f, \quad \overline{N_f} = \int_M N_f e^{-f}\,dv_g \Big/ \int_M e^{-f}\,dv_g.$$

Moreover, equality holds in (1.8) *if and only if* $\mathrm{Ric}_f = \lambda g$.

Our second result is that the conclusion of Theorem 1.4 holds for manifolds with totally geodesic boundary:

Theorem 1.5. *Suppose* (M,g) *is a compact Riemannian manifold of dimension* $n \geq 3$ *with totally geodesic boundary* ∂M, *and* f *is a* $C^2(M)$

function. If f satisfies the Dirichlet boundary condition or the Neumann boundary condition, then

$$\int_M (N_f - \overline{N_f})^2 e^{-f}\, dv_g \le 4\left(1 + \frac{K}{\xi_1}\right) \int_M |\mathrm{Ric}_f - \lambda g|^2 e^{-f}\, dv_g, \tag{1.9}$$

where ξ_1 denotes the first non-zero Neumann eigenvalue of Δ_f on M, K is a non-negative constant such that $\mathrm{Ric}_f \ge -K$, and $\lambda \ge -K$ is a real constant.

Moreover, equality holds in (1.9) *if and only if* $\mathrm{Ric}_f = \lambda g$.

REMARK 1.1. In particular, Theorem 1.4 reduces to Theorem 1.3 by letting $K = \lambda = 0$. Therefore, our results generalize those of Wu [W] for the ∞-Bakry–Émery Ricci curvature.

2. Proof of Theorem 1.4

LEMMA 2.1. *Suppose (M, g) is a closed Riemannian manifold of dimension $n \ge 3$ and f is a $C^2(M)$ function. For any vector field X on M,*

$$-\int_M \langle X, \nabla N_f\rangle e^{-f}\, dv_g = \int_M \langle \mathrm{Ric}_f - \lambda g, L_X g\rangle e^{-f}\, dv_g, \tag{2.1}$$

where $N_f = R + 2\Delta f - |\nabla f|^2 + 2\lambda f$.

Proof. Take a local orthonormal frame $\{e_i\}_{1\le i\le n}$ on M. Then $X = X^i e_i$. Noticing that the tensor $\mathrm{Ric}_f - \lambda g$ is symmetric, we have

$$\begin{aligned}\int_M \langle \mathrm{Ric}_f - \lambda g, L_X g\rangle e^{-f}\, dv_g &= 2\int_M (\mathrm{Ric}_{f\,ij} - \lambda g_{ij})\nabla^j X^i e^{-f}\, dv_g \\ &= -2\int_M X^i \nabla^j[(\mathrm{Ric}_{f\,ij} - \lambda g_{ij})e^{-f}]\, dv_g \\ &= -\int_M \langle X, \nabla(R + 2\Delta f - |\nabla f|^2 + 2\lambda f)\rangle e^{-f}\, dv_g,\end{aligned} \tag{2.2}$$

where we have used the contracted second Bianchi identity $2\nabla^j R_{ij} = \nabla^i R$ and the Ricci identity $\nabla^j f_{ij} = \nabla_i(\Delta f) + R_{ij} f^j$. ∎

Proof of Theorem 1.4. We let $u : M \to \mathbb{R}$ be the unique solution to

$$\begin{cases} \Delta_f u = N_f - \overline{N_f}, \\ \int_M u e^{-f}\, dv_g = 0. \end{cases} \tag{2.3}$$

Choosing $X = \nabla u$ and using (2.3), we obtain

$$-\int_M \langle X, \nabla N_f\rangle e^{-f}\,dv_g = -\int_M \langle \nabla u, \nabla(N_f - \overline{N_f})\rangle e^{-f}\,dv_g = \int_M (N_f - \overline{N_f})\Delta_f u e^{-f}\,dv_g = \int_M (N_f - \overline{N_f})^2 e^{-f}\,dv_g. \tag{2.4}$$

On the other hand, by letting $X = \nabla u$ and using the Cauchy inequality, we get

$$\int_M \langle \mathrm{Ric}_f - \lambda g, L_X g\rangle e^{-f}\,dv_g = 2\int_M \langle \mathrm{Ric}_f - \lambda\, g, \nabla^2 u\rangle e^{-f}\,dv_g \le 2\Big(\int_M |\mathrm{Ric}_f - \lambda g|^2 e^{-f}\,dv_g\Big)^{1/2}\Big(\int_M |\nabla^2 u|^2 e^{-f}\,dv_g\Big)^{1/2}. \tag{2.5}$$

Integrating the Bochner formula for u (for the elementary proof, see [L, LI]),

$$\frac{1}{2}\Delta_f|\nabla u|^2 = |\nabla^2 u|^2 + \langle \nabla u, \nabla \Delta_f u\rangle + \mathrm{Ric}_f(\nabla u, \nabla u), \tag{2.6}$$

and applying the Stokes formula, we obtain

$$\int_M |\nabla^2 u|^2 e^{-f}\,dv_g = \int_M (\Delta_f u)^2 e^{-f}\,dv_g - \int_M \mathrm{Ric}_f(\nabla u, \nabla u) e^{-f}\,dv_g \le \int_M [(\Delta_f u)^2 + K|\nabla u|^2] e^{-f}\,dv_g. \tag{2.7}$$

Here we have used $\mathrm{Ric}_f \ge -K$.

Let η_1 denote the first non-zero eigenvalue of Δ_f on M, i.e.,

$$\eta_1 = \inf\Big\{\frac{\int_M |\nabla\varphi|^2 e^{-f}\,dv_g}{\int_M \varphi^2 e^{-f}\,dv_g} : \varphi \not\equiv 0 \text{ and } \int_M \varphi e^{-f}\,dv_g = 0\Big\}.$$

We have

$$\int_M |\nabla u|^2 e^{-f}\,dv_g = -\int_M u\Delta_f u e^{-f}\,dv_g = -\int_M u(N_f - \overline{N_f}) e^{-f}\,dv_g \le \Big(\int_M u^2 e^{-f}\,dv_g\Big)^{1/2}\Big(\int_M (N_f - \overline{N_f})^2 e^{-f}\,dv_g\Big)^{1/2} \le \Big(\frac{\int_M |\nabla u|^2 e^{-f}\,dv_g}{\eta_1}\Big)^{1/2}\Big(\int_M (N_f - \overline{N_f})^2 e^{-f}\,dv_g\Big)^{1/2}. \tag{2.8}$$

Thus

$$\int_M |\nabla u|^2 e^{-f}\, dv_g \le \frac{1}{\eta_1} \int_M (N_f - \overline{N_f})^2 e^{-f}\, dv_g. \tag{2.9}$$

Substituting (2.9) into (2.7) and noting that $K \ge 0$ gives

$$\int_M |\nabla^2 u|^2 e^{-f}\, dv_g \le \left(1 + \frac{K}{\eta_1}\right) \int_M (N_f - \overline{N_f})^2 e^{-f}\, dv_g. \tag{2.10}$$

Combining (2.4) with (2.5) and (2.10) yields (1.8).

Next, we consider the case of equality in (1.8). Obviously, if $\mathrm{Ric}_f = \lambda g$, then equality holds. Conversely, if equality holds, we claim that $u \equiv 0$. Thus $N_f - \overline{N_f} = \Delta_f u = 0$ on M and so $\mathrm{Ric}_f = \lambda g$ on M.

To the contrary, suppose u is not identically zero. By the proof of (1.8), the following assertions must hold:

(1) $(\mathrm{Ric}_f + Kg)(\nabla u, \nabla u) = 0$;
(2) $\mathrm{Ric}_f - \lambda g$ and $\nabla^2 u$ are linearly dependent;
(3) u and $N_f - \overline{N_f}$ are linearly dependent;
(4) $\int_M (|\nabla u|^2 - \eta_1 u^2) e^{-f}\, dv_g = 0$.

By (1) and the assumption that $\mathrm{Ric}_f + Kg \ge 0$, we have

$$(\mathrm{Ric}_f + Kg)(\nabla u, \cdot) = 0. \tag{2.11}$$

By (3), there exists a constant b such that $N_f - \overline{N_f} = bu$. Then by (4),

$$\begin{aligned} \int_M |\nabla u|^2 e^{-f} dv_g &= -\int_M u \Delta_f u e^{-f} dv_g = -\int_M u(N_f - \overline{N_f}) e^{-f} dv_g \\ &= -\int_M b u^2 e^{-f} dv_g = \int_M \eta_1 u^2 e^{-f} dv_g, \end{aligned}$$

and

$$b = -\eta_1, \qquad N_f - \overline{N_f} = -\eta_1 u. \tag{2.12}$$

By non-triviality of u, $\mathrm{Ric}_f - \lambda g$ and $\nabla^2 u$ must be non-trivial. [If $\mathrm{Ric}_f = \lambda g$, then $u = -\frac{1}{\eta_1}(N_f - \overline{N_f}) = 0$. If $\nabla^2 u = 0$, by (2.4) and (2.5) we deduce that $N_f - \overline{N_f} = 0$ on M, which contradicts the non-triviality of u.] So we can suppose there exists a non-zero constant μ such that

$$\mu(\mathrm{Ric}_f - \lambda g) = \nabla^2 u. \tag{2.13}$$

By the Ricci identity and (2.13), we have

$$\begin{aligned} 0 &= \nabla(\Delta_f u) - \mathrm{div}_f(\nabla^2 u) + \mathrm{Ric}_f(\nabla u, \cdot) \\ &= \nabla(\Delta_f u) - \mu\, \mathrm{div}_f(\mathrm{Ric}_f - \lambda g) + \mathrm{Ric}_f(\nabla u, \cdot). \end{aligned} \tag{2.14}$$

A direct calculation yields

$$\mathrm{div}_f(\mathrm{Ric}_f - \lambda g) = \frac{1}{2} \nabla N_f. \tag{2.15}$$

Hence, inserting (2.15) into (2.14), we get

$$\nabla N_f - \frac{1}{2}\mu\nabla N_f + \frac{K}{\eta_1}g(\nabla N_f,\cdot) = 0.$$

Then

$$\left(1+\frac{K}{\eta_1}-\frac{1}{2}\mu\right)|\nabla N_f|^2 = 0. \tag{2.16}$$

If $\mu \neq 2(1+K/\eta_1)$, then $\nabla N_f = 0$ and hence $N_f = \overline{N}_f$, $u=0$. So we must have $\mu = 2(1+K/\eta_1)$. Then (2.13) turns into

$$2\left(1+\frac{K}{\eta_1}\right)(\mathrm{Ric}_f - \lambda g) = \nabla^2 u. \tag{2.17}$$

Combining (2.11) with (2.17) we infer

$$\nabla^2 u(\nabla u,\cdot) = -2\left(1+\frac{K}{\eta_1}\right)(K+\lambda)g(\nabla u,\cdot),$$

which we can rewrite as

$$\nabla\frac{|\nabla u|^2}{2} = -2\left(1+\frac{K}{\eta_1}\right)(K+\lambda)\nabla u. \tag{2.18}$$

Fix $x_0 \in M$ and let $\gamma : [0,\infty) \to M$ be the solution of $\dot\gamma(t) = -\nabla u(\gamma(t))$ with $\gamma(0) = x_0$. Consider $\alpha(t) = u(\gamma(t))$. Then $\alpha'(t) = -|\nabla u(\gamma(t))|^2$ and, by (2.18),

$$\alpha''(t) = -4\left(1+\frac{K}{\eta_1}\right)(K+\lambda)|\nabla u|^2.$$

Since $\lambda + K \geq 0$, we have $\alpha''(t) \leq 0$, hence α is a bounded nonincreasing concave function on $[0,\infty)$ and therefore it must be constant. We conclude that $-|\nabla u(x_0)|^2 = \alpha'(0) = 0$. The arbitrariness of x_0 implies that u is constant, which completes the proof.

From Theorem 1.4, we have the following corollary.

Corollary 2.2. *Suppose (M,g) is a closed Riemannian manifold of dimension $n \geq 3$ and f is a $C^2(M)$ function. Then*

$$\int_M (N_f - \overline{N_f})^2 e^{-f}\,dv_g \leq C_{n,K,d}\int_M |\mathrm{Ric}_f - \lambda g|^2 e^{-f}\,dv_g, \tag{2.19}$$

where K is a non-negative constant such that $\mathrm{Ric}_f \geq -K$, $\lambda \geq -K$ is a real constant, d denotes the diameter of M, and $C_{n,K,d}$ is a constant only depending on n, K, d.

Proof. When $\mathrm{Ric}_f \geq -K$, Futaki, Li and Li [FLL] proved a lower bound for the first non-zero eigenvalue η_1:

$$\eta_1 \geq \sup_{s\in(0,1)} \left\{ 4s(1-s)\frac{\pi^2}{d^2 - sK} \right\}, \tag{2.20}$$

where d denotes the diameter of M. Obviously, we can make the right hand side of (2.20) positive by choosing s small enough. Therefore, inserting (2.20) into (1.8) completes the proof of Corollary 2.2.

3. Proof of Theorem 1.5. Let $u : M \to \mathbb{R}$ be the unique solution to

$$\begin{cases} \Delta_f u = N_f - \overline{N_f} & \text{in } M, \\ \partial u/\partial\nu = 0 & \text{on } \partial M, \end{cases} \tag{3.1}$$

where $\partial u/\partial \nu$ is the normal derivative of u with respect to the metric g.

Using the Stokes formula and the Cauchy inequality, we obtain

$$\begin{aligned} \int_M (N_f - \overline{N_f})^2 e^{-f}\, dv_g &= \int_M (\Delta_f u)(N_f - \overline{N_f}) e^{-f}\, dv_g \\ &= -\int_M \langle \nabla u, \nabla N_f\rangle e^{-f}\, dv_g + \int_{\partial M} (N_f - \overline{N_f})\frac{\partial u}{\partial \nu} e^{-f} dA_g \\ &= -\int_M \langle \nabla u, \nabla N_f\rangle e^{-f}\, dv_g \\ &= -\int_M \langle \nabla u, \nabla N_f\rangle e^{-f}\, dv_g + 2\int_{\partial M} [\mathrm{Ric}_f(\nabla u, \nu) - \lambda g(\nabla u, \nu)] e^{-f} dA_g \\ &= 2\int_M \langle \mathrm{Ric}_f - \lambda\, g, \nabla^2 u\rangle e^{-f}\, dv_g \\ &\leq 2\Big(\int_M |\mathrm{Ric}_f - \lambda g|^2 e^{-f}\, dv_g\Big)^{1/2}\Big(\int_M |\nabla^2 u|^2 e^{-f}\, dv_g\Big)^{1/2}, \end{aligned} \tag{3.2}$$

where in the last equality we have used [CN, Lemma 2.10] and the fact that f satisfies the Dirichlet boundary condition or Neumann boundary condition.

Recall the Bochner formula:

$$\frac{1}{2}\Delta_f |\nabla u|^2 = |\nabla^2 u|^2 + \langle \nabla u, \nabla \Delta_f u\rangle + \mathrm{Ric}_f(\nabla u, \nabla u). \tag{3.3}$$

Note that

$$\frac{1}{2}\int_M \Delta_f |\nabla u|^2 e^{-f}\, dv_g = \frac{1}{2}\int_{\partial M} \frac{\partial |\nabla u|^2}{\partial \nu} e^{-f}\, dv_g = 0, \tag{3.4}$$

where the last equality follows from [CN, Lemma 2.10] which says that $\partial|\nabla u|^2/\partial\nu = 0$ if $\partial u/\partial \nu = 0$ and (M, g) has totally geodesic boundary. Combining (3.3) and (3.4) gives

$$
\begin{aligned}
(3.5)\qquad &\int_M |\nabla^2 u|^2 e^{-f}\, dv_g \\
&= -\int_M \langle \nabla u, \nabla \Delta_f u\rangle e^{-f}\, dv_g - \int_M \mathrm{Ric}_f(\nabla u, \nabla u) e^{-f}\, dv_g \\
&= -\int_{\partial M} (\Delta_f u)\frac{\partial u}{\partial \nu} e^{-f}\, dv_g + \int_M (\Delta_f u)^2 e^{-f}\, dv_g - \int_M \mathrm{Ric}_f(\nabla u, \nabla u) e^{-f}\, dv_g \\
&\le \int_M [(\Delta_f u)^2 + K|\nabla u|^2] e^{-f}\, dv_g.
\end{aligned}
$$

Here we have used $\mathrm{Ric}_f \ge -K$.

Let ξ_1 denote the first non-zero Neumann eigenvalue of Δ_f on M, i.e.,

$$
\xi_1 = \inf\left\{ \frac{\int_M |\nabla \psi|^2 e^{-f}\, dv_g}{\int_M \psi^2 e^{-f}\, dv_g} : \psi \not\equiv 0 \text{ and } \frac{\partial \psi}{\partial \nu} = 0 \text{ on } \partial M \right\}.
$$

We have

$$
\begin{aligned}
(3.6)\quad \int_M |\nabla u|^2 e^{-f}\, dv_g &= \int_{\partial M} u \frac{\partial u}{\partial \nu} e^{-f} dA_g - \int_M u \Delta_f u e^{-f}\, dv_g \\
&= -\int_M u(N_f - \overline{N_f}) e^{-f}\, dv_g \\
&\le \Big(\int_M u^2 e^{-f}\, dv_g\Big)^{1/2} \Big(\int_M (N_f - \overline{N_f})^2 e^{-f}\, dv_g\Big)^{1/2} \\
&\le \left(\frac{\int_M |\nabla u|^2 e^{-f}\, dv_g}{\xi_1}\right)^{1/2} \Big(\int_M (N_f - \overline{N_f})^2 e^{-f}\, dv_g\Big)^{1/2}.
\end{aligned}
$$

Thus

$$
(3.7)\qquad \int_M |\nabla u|^2 e^{-f}\, dv_g \le \frac{1}{\xi_1} \int_M (N_f - \overline{N_f})^2 e^{-f}\, dv_g.
$$

Substituting (3.7) into (3.5) and noting that $K \ge 0$ gives

$$
(3.8)\qquad \int_M |\nabla^2 u|^2 e^{-f}\, dv_g \le \left(1 + \frac{K}{\xi_1}\right) \int_M (N_f - \overline{N_f})^2 e^{-f}\, dv_g.
$$

Combining (3.2) and (3.8) yields (1.9)

Next, we consider the case of equality in (1.9). Obviously, if $\mathrm{Ric}_f = \lambda g$, then equality holds. Conversely, if equality holds, we claim that $u \equiv 0$. Thus $N_f - \overline{N_f} = \Delta_f u = 0$ on M and so $\mathrm{Ric}_f = \lambda g$ on M.

To the contrary, suppose u is not identically zero. By the proof of (1.9), the following assertions must hold:

(1) $(\mathrm{Ric}_f + Kg)(\nabla u, \nabla u) = 0$;
(2) $\mathrm{Ric}_f - \lambda g$ and $\nabla^2 u$ are linearly dependent;
(3) u and $N_f - \overline{N_f}$ are linearly dependent;
(4) $\int_M (|\nabla u|^2 - \xi_1 u^2) e^{-f} dv_g = 0$.

Using similar arguments to those for Theorem 1.4 (see also [H]), we complete the proof of Theorem 1.5. ∎

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