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Diameter 2 properties and convexity

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Abstract. We present an equivalent midpoint locally uniformly rotund (MLUR) renorming of $C[0,1]$ with the diameter 2 property (D2P), i.e. every non-empty relatively weakly open subset of the unit ball has diameter 2. An example of an MLUR space with the D2P and with convex combinations of slices of arbitrarily small diameter is also given.

1. Introduction. Let X be a (real) Banach space. We say that X (or its norm $\|\cdot\|$) is *midpoint locally uniformly rotund* (MLUR) (resp. *weakly midpoint locally uniformly rotund* (weakly MLUR)) if every x in the unit sphere S_X of X is a strongly extreme point (resp. strongly extreme point in the weak topology), i.e. for every sequence (x_n) in X, we have $x_n \to 0$ in norm (resp. $x_n \to 0$ weakly) whenever $\|x \pm x_n\| \to 1$.

Let $x^* \in S_{X^*}$ and $\varepsilon > 0$. By a *slice* of the unit ball B_X of X we mean a set of the form

$$S(x^*, \varepsilon) := \{x \in B_X : x^*(x) > 1 - \varepsilon\}.$$

Over the last 15 years quite much has been discovered concerning Banach spaces with various kinds of diameter 2 properties (see e.g. [18], [1], [9], [10], [3], [4] to mention but a few).

Definition 1.1. A Banach space X has

(a) the *local diameter 2 property* (LD2P) if every slice of B_X has diameter 2;
(b) the *diameter 2 property* (D2P) if every non-empty relatively weakly open subset of B_X has diameter 2;

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(c) the *strong diameter 2 property* (SD2P) if every finite convex combination of slices of B_X has diameter 2.

By [7, Lemma II.1, p. 26] (c) implies (b), and of course (b) implies (a). None of the reverse implications holds (see [3, Theorem 2.4] and [9, Theorem 1] or [2, Theorem 3.2]). However, note Proposition 1.3 below which is an immediate consequence of Choquet's lemma (see [6, Lemma 3.69, p. 111]).

LEMMA 1.2 (Choquet). *Let C be a compact convex set in a locally convex space X. Then for every $x \in \mathrm{ext}(C)$, the extreme points in C, the slices of C containing x form a neighbourhood base of x in the relative topology of C.*

PROPOSITION 1.3. *If X is weakly MLUR then the LD2P implies the D2P.*

Proof. Simply recall that the points in B_X which are strongly extreme in the weak topology are exactly the extreme points which continue to be extreme in $B_{X^{**}}$ (see [8]), and then use Lemma 1.2 on $B_{X^{**}}$ given the weak* topology. ∎

It is not evident that weakly MLUR spaces with the LD2P exist, but indeed they do. The quotient $C(\mathbb{T})/A$, where $C(\mathbb{T})$ is the space of continuous functions on the complex unit circle $\mathbb{T}$ and where A is the disc algebra, is such an example (see the next paragraph for references). Another example can be constructed as follows: Let Φ be a function on c_0, the space of real valued sequences which converge to 0, defined by $\Phi(x_n) = \sum_{n=1}^{\infty} x_n^{2n}$. Define then a norm on c_0 by $|x| = \inf\{\lambda > 0 : \Phi(x/\lambda) \leq 1\}$ for every $x \in c_0$. Then the space $(c_0, |\cdot|)$ can be shown to be weakly MLUR and to have the LD2P (see the Appendix for details).

The two examples mentioned motivate the following question which we will address in this paper: How rotund can a Banach space be and still have diameter 2 properties? In [11, Remarks 4), p. 286] it is pointed out that $C(\mathbb{T})/A$ is M-embedded and that its dual norm is smooth (see also [17] and [12, p. 167]). Recall that X is *M-embedded* provided we can write $X^{***} = X^* \oplus_1 X^\perp$ where $X^\perp \subset X^{***}$ is the annihilator of X (a good source for the theory of M-embedded spaces is the book [12]). It is well known that M-embedded spaces have the SD2P [1], and so $C(\mathbb{T})/A$ actually furnishes an example of a weakly MLUR space with the SD2P. One can also prove that the space $(c_0, |\cdot|)$ mentioned above has the same properties (see the Appendix). For still more examples see [20].

The unit ball of an M-embedded space (even any proper M-ideal) cannot, however, contain strongly extreme points [12, Sect. II.4], so no MLUR M-embedded space exists. Still, one can ask if there exists an MLUR space with the LD2P (= D2P in this case). Until now, no such example has been known. But, in Section 2 of this paper we construct an equivalent MLUR renorming X of $C[0,1]$ such that for every slice S of B_X and every $x \in S \cap S_X$ there exists $y \in S$ at distance from x as close to 2 as we want, i.e. X has the local diameter 2 property + (LD2P+). In particular this renorming has the LD2P and thus the D2P as it is MLUR. Using this renorming we also construct in Section 2 an example of an MLUR space which has the D2P, the LD2P+, and has convex combinations of slices with arbitrarily small diameter. Section 3 contains a list of open questions together with some comments.

The notation and conventions we use are standard and follow [14]. When considered necessary, notation and concepts are explained as the text proceeds.

2. MLUR renormings of $C[0,1]$ with the D2P. Let $D = (D_n)_{n=1}^{\infty}$ be a base of neighbourhoods in $[0,1]$ (by this we will always mean a base for the usual topology on $[0,1]$). For each $x \in C[0,1]$ set $\|x\|_n = \sup_{d \in D_n} |x(d)|$ and note that each $\|\cdot\|_n$ defines a seminorm on $C[0,1]$. Now define a norm on $C[0,1]$ by

$$\|x\|_D := \Big(\sum_{n=1}^{\infty} 2^{-n}\|x\|_n^2\Big)^{1/2}.$$

By compactness there exists $b > 0$ such that $b\|x\|_\infty \le \|x\|_D \le \|x\|_\infty$, so the norm $\|\cdot\|_D$ is equivalent to the max-norm $\|\cdot\|_\infty$ on $C[0,1]$. The idea to introduce this norm goes back to [16].

Let

$$X_D = (C[0,1], \|\cdot\|_D).$$

PROPOSITION 2.1. *For any base $D = (D_n)_{n=1}^{\infty}$ of neighbourhoods in $[0,1]$ the space X_D is MLUR.*

Proof. Let x and $(y_k)_{k=1}^{\infty}$ in $C[0,1]$ be such that $\lim_{k\to\infty} \|x \pm y_k\|_D = \|x\|_D$. We will show that $\|y_k\|_D \to 0$ to establish that $\|\cdot\|_D$ is MLUR. By a convexity argument (see e.g. [5, Fact II.2.3]) we have

$$\lim_{k\to\infty} \|x \pm y_k\|_n = \|x\|_n, \qquad n = 1, 2, \ldots. \tag{2.1}$$

Let $\varepsilon > 0$. We will first make three simple observations:

- By uniform continuity of x, we can find $\delta = \delta(\varepsilon) > 0$ such that the oscillation over an interval A, $\sup_{t,s\in A}(x(t)-x(s))$, is less than ε whenever $A \subset [0,1]$ is of length less than δ.
- With the δ above, since $(D_n)_{n=1}^{\infty}$ is a base for the topology on the compact space $[0,1]$, there is a finite subset $M \subset \mathbb{N}$ such that $\bigcup_{m\in M} D_m$ covers $[0,1]$ and the length of any D_m, $m \in M$, is less than δ.
- With M and δ as above, having in mind (2.1) which is of course true for each $m \in M$, we can find $K \in \mathbb{N}$ such that $\|x \pm y_k\|_m \le \|x\|_m + \varepsilon$ whenever $k \ge K$ and $m \in M$.

We are now ready to finish the proof. Let $t_0 \in [0,1]$. We will show that $|y_k(t_0)| \leq 2\varepsilon$ for $k \geq K$. Choose $\sigma_k \in \{-1,1\}$ such that

$$|x(t_0) + \sigma_k y_k(t_0)| = |x(t_0)| + |y_k(t_0)|.$$

Since $\bigcup_{m\in M} D_m$ covers $[0,1]$, there is $m' \in M$ such that $t_0 \in D_{m'}$. Now recall that the lengths of all the D_m's are $< \delta$, so that the oscillation of x over $D_{m'}$ is less than ε. We get

$$\begin{aligned}|y_k(t_0)| &= |x(t_0) + \sigma_k y_k(t_0)| - |x(t_0)| \\ &\leq \sup_{t\in D_{m'}} |x(t) + \sigma_k y_k(t)| - |x(t_0)| \\ &\leq \|x + \sigma_k y_k\|_{m'} - (\|x\|_{m'} - \varepsilon) \\ &\leq \|x\|_{m'} + \varepsilon - \|x\|_{m'} + \varepsilon = 2\varepsilon,\end{aligned}$$

provided $k \geq K$. ∎

We will now show that X_D has a rather strong form of the LD2P, to be called LD2P+:

DEFINITION 2.2. A Banach space X has the *local diameter 2 property +* (LD2P+) if for every $\varepsilon > 0$, every slice S of B_X, and every $x \in S \cap S_X$ there exists $y \in S$ such that $\|x - y\| > 2 - \varepsilon$.

PROPOSITION 2.3. *For any base $D = (D_n)_{n=1}^\infty$ of neighbourhoods in $[0,1]$ the space X_D has the LD2P+.*

Proof. We know that the dual of X_D is isomorphic to rca$[0,1]$, the space of regular and countably additive Borel measures on $[0,1]$. Let $\lambda \in$ rca$[0,1]$ be the Lebesgue measure. By Lebesgue's decomposition theorem, any measure $m \in$ rca$[0,1]$ can be decomposed as $m = \mu + \nu$, where μ is absolutely continuous with respect to λ, and ν and λ are mutually singular.

Now, let $m \in S_{X_D^*}$, $\varepsilon > 0$, and denote by S the slice

$$\Big\{x \in B_{X_D} : \int_{[0,1]} x\,dm > 1 - \varepsilon\Big\}.$$

Let $x \in S$ and $\delta > 0$ with $1 - \|x\|_D < \delta < \varepsilon$ and find $N \in \mathbb{N}$ such that

$$\Big(\sum_{n=1}^N 2^{-n}\|x\|_n^2\Big)^{1/2} > 1 - \delta > 1 - \varepsilon.$$

By continuity there exist open intervals $E_n = (r_n, t_n) \subset D_n$, $1 \leq n \leq N$, such that

- $E_i \cap E_j = \emptyset$ for every $i \neq j$,
- $(\sum_{n=1}^N 2^{-n}|x(e_n)|^2)^{1/2} > 1 - \delta$ whenever $e_n \in E_n$.

Moreover, as λ and ν are mutually singular, there exist $s_n \in E_n$ with

- $\nu(\{s_n\}) = 0$ for every $n = 1, \ldots, N$, and thus $m(\{s_n\}) = 0$ for every $n = 1, \ldots, N$.

For $\eta \in (0, \varepsilon)$, by regularity of m, we can shrink each interval E_n around s_n if necessary so that

- $b^{-1} \sum_{n=1}^{N} |m(E_n)| < \eta$ and for $E = \bigcup_{n=1}^{N} E_n$, $\int_{[0,1]\setminus E} x\,dm - \eta > 1 - \varepsilon$ (see the beginning of Section 2 for the definition of b).

Now, define a continuous function y on $[0,1]$ by letting, for $n = 1, \ldots, N$, $y(r_n) = x(r_n)$, $y(s_n) = -x(s_n)$, $y(t_n) = x(t_n)$, linear on (r_n, s_n) and (s_n, t_n), and otherwise equal to x. Then $y \in X_D$ with $\sup_{d\in E_n} |y(d)| \le \sup_{d\in E_n} |x(d)|$ and $y(d) = x(d)$ for every $d \in [0,1] \setminus E$. Therefore $\|y\|_D \le \|x\|_D \le 1$. Moreover,

$$\begin{aligned}\int\limits_{[0,1]} y\,dm &= \int\limits_{[0,1]\setminus E} y\,dm + \int\limits_{E} y\,dm \\ &\ge \int\limits_{[0,1]\setminus E} x\,dm - \sum_{n=1}^{N} b^{-1}|m(E_n)| > \int\limits_{[0,1]\setminus E} x\,dm - \eta > 1 - \varepsilon,\end{aligned}$$

and

$$\begin{aligned}\|x - y\| &\ge \Big(\sum_{n=1}^{N} 2^{-n}\|x - y\|_n^2\Big)^{1/2} \\ &\ge \Big(\sum_{n=1}^{N} 2^{-n}|x(s_n) - y(s_n)|^2\Big)^{1/2} = 2\Big(\sum_{n=1}^{N} 2^{-n}|x(s_n)|^2\Big)^{1/2} > 2 - 2\delta. \ \blacksquare\end{aligned}$$

From Propositions 2.1, 2.3, and 1.3 we obtain the following result.

THEOREM 2.4. *For any base $D = (D_n)_{n=1}^{\infty}$ of neighbourhoods in $[0,1]$ the space X_D is MLUR and has the D2P and the LD2P+.*

In [10] dual characterizations of the diameter 2 properties in Definition 1.1 were obtained. To formulate these, we need to introduce some concepts.

DEFINITION 2.5. For a Banach space X we say that (the norm on) X is

- *locally octahedral* if for every $\varepsilon > 0$ and every $x \in S_X$ there exists $y \in S_X$ such that $\|x \pm y\| > 2 - \varepsilon$;
- *octahedral* if for every $\varepsilon > 0$ and every finite set $(x_i)_{i=1}^{n} \subset S_X$ there exists $y \in S_X$ such that $\|x_i + y\| > 2 - \varepsilon$ for every $1 \le i \le n$.

For a Banach space X, $x \in S_X$, and $\varepsilon > 0$, by a *weak*-slice* of B_{X^*} we mean a set of the form

$$S(x, \varepsilon) := \{x^* \in B_{X^*} : x^*(x) > 1 - \varepsilon\}.$$

Definition 2.6. A dual Banach space X^* has

- the *weak*-local diameter 2 property* (weak*-LD2P) if every weak*-slice of B_{X^*} has diameter 2;
- the *weak*-strong diameter 2 property* (weak*-SD2P) if every finite convex combination of weak*-slices of B_{X^*} has diameter 2;
- the *weak*-local diameter 2 property +* (weak*-LD2P+) if for every $\varepsilon > 0$, every weak*-slice S of B_{X^*}, and every $x^* \in S \cap S_{X^*}$ there exists $y^* \in S$ such that $\|x^* - y^*\| > 2 - \varepsilon$.

Theorem 2.7 ([10, Theorems 3.1, 3.3, and 3.5]). *For a Banach space X we have*

(a) *X is locally octahedral $\Leftrightarrow$ X^* has the weak*-LD2P.*
(b) *X is octahedral $\Leftrightarrow$ X^* has the weak*-SD2P.*

From [13, Theorem 1.5], a Banach space X has the LD2P+ if and only if every rank one, norm one projection P on X satisfies $\|I - P\| = 2$ (I is the identity operator on X). Using this formulation of the LD2P+ and a similar argument to the proof of [15, Lemma 2.1] one can also prove that a Banach space has the LD2P+ if and only if its dual has the weak*-LD2P+. From this and Theorems 2.4 and 2.7 we then see that for any base $D = (D_n)_{n=1}^{\infty}$ of neighbourhoods in $[0,1]$ the space X_D is locally octahedral. However, X_D is never octahedral. To see this, we will use the following lemma.

Lemma 2.8. *Let u and v be continuous functions on the unit interval. Suppose $\|u\|_n = \|v\|_n$ for every $n \in \mathbb{N}$. Then*

$$|u(t)| = |v(t)| \quad \text{for every } t \in [0,1].$$

Proof. Let $\varepsilon > 0$, and choose $\delta > 0$ such that

$$|u(s') - u(s'')| < \varepsilon \quad \text{and} \quad |v(s') - v(s'')| < \varepsilon \tag{2.2}$$

whenever $|s' - s''| < \delta$. Fix $t \in [0,1]$. There exists $n \in \mathbb{N}$ such that t belongs to D_n and $\operatorname{diam}(D_n) < \delta$. Now find t', t'' in D_n such that $\big|\|u\|_n - |u(t')|\big| < \varepsilon$ and $\big|\|v\|_n - |v(t'')|\big| < \varepsilon$. Then $\big||u(t')| - |v(t'')|\big| < 2\varepsilon$, and thus by (2.2) we have $\big||u(t)| - |v(t)|\big| < 4\varepsilon$. ∎

Proposition 2.9. *For any base $D = (D_n)_{n=1}^{\infty}$ of neighbourhoods in $[0,1]$ the space X_D fails to be octahedral.*

Proof. Choose two different non-negative norm 1 functions u and v in X_D. Assume there exists a sequence $(y_k)_{k=1}^{\infty} \subset S_X$ such that

$$\lim_{k\to\infty} \|u + y_k\|_D = 2 \quad \text{and} \quad \lim_{k\to\infty} \|v + y_k\|_D = 2. \tag{2.3}$$

Using (2.3) and [5, Fact II.2.3] we find that for every $n \in \mathbb{N}$,

$$\|u\|_n = \lim_k \|y_k\|_n = \|v\|_n.$$

Now we get a contradiction from Lemma 2.8, as u and v are non-negative and different. ■

The final part of this section will be devoted to showing that there exists a Banach space which is MLUR, has the D2P, the LD2P+, but has convex combinations of slices with arbitrarily small diameter. First we will show that for any given $\delta > 0$ there exists a base $D = (D_n)_{n=1}^{\infty}$ of neighbourhoods in $[0, 1]$ for which B_{X_D} contains convex combinations of slices with diameter $< \delta$. For this we will use the following lemma.

Let $t \in [0, 1]$. Set $J(t) = \{n \in \mathbb{N} : t \in D_n\}$ and $w(t) = \sum_{n \in J(t)} 2^{-n}$.

LEMMA 2.10. *Let $D = (D_n)_{n=1}^{\infty}$ be a base of neighbourhoods in $[0, 1]$, $t \in [0, 1]$, and δ_t the point measure in X_D^*. If $\overline{D_n} \cap \{t\} = \emptyset$ for every $n \notin J(t)$, then*

$$\|\delta_t\|_D^* = \frac{1}{\sqrt{w(t)}},$$

where $\|\cdot\|_D^*$ is the norm in X_D^*.

Proof. Let $x \in X_D$ with norm 1. Then

$$1 = \sum_{n=1}^{\infty} 2^{-n} \|x\|_n^2 \geq \sum_{n \in J(t)} 2^{-n} |x(t)|^2 = w(t) |\delta_t(x)|^2.$$

Thus $\|\delta_t\|_D^* \leq 1/\sqrt{w(t)}$. Moreover, by the assumptions it is always possible to find for $i \notin J(t)$ an open set which contains t and does not intersect $\overline{D_i}$. Thus we can always find an $x_i \in S_{X_D}$ which takes its maximum value at t and is zero on D_i. It follows that for any $\varepsilon > 0$ we can find $x \in S_{X_D}$ which takes its maximum value at t and $\sum_{n \notin J(t)} 2^{-n} \|x\|_n^2 < \varepsilon$. From the inequality

$$\begin{aligned} 1 = \|x\|_D &= \sum_{n \in J(t)} 2^{-n} x(t)^2 + \sum_{n \notin J(t)} 2^{-n} \|x\|_n^2 \\ &< \sum_{n \in J(t)} 2^{-n} x(t)^2 + \varepsilon \end{aligned}$$

we get $\delta_t(x)^2 > (1 - \varepsilon)/w(t)$. Thus we conclude that $\|\delta_t\|_D^* = 1/\sqrt{w(t)}$. ■

Let $(\varepsilon_n)_{n=1}^{\infty}$ (with ε_1 small!) be a strictly decreasing sequence of positive real numbers converging fast to 0. For each $i \in \mathbb{N}$ let us define a base of neighbourhoods $(D_{i,n})_{n=1}^{\infty}$ in $[0, 1]$: Let $i = 1$ and

$$D_{1,1} = [0, 2^{-1} + \varepsilon_1), \quad D_{1,2} = (2^{-1} - \varepsilon_2, 1].$$

We call this the first level. For the second level, set

$$\begin{aligned} D_{1,3} &= [0, 2^{-2} + \varepsilon_3), & D_{1,4} &= (2^{-2} - \varepsilon_4, 2 \cdot 2^{-2} + \varepsilon_4), \\ D_{1,5} &= (2 \cdot 2^{-2} - \varepsilon_5, 3 \cdot 2^{-2} + \varepsilon_5), & D_{1,6} &= (3 \cdot 2^{-2} - \varepsilon_6, 1]. \end{aligned}$$

Continue in this fashion to obtain the base $(D_{1,n})_{n=1}^{\infty}$ consisting of open intervals in $[0,1]$. Finally, let $D_i = (D_{i,n})_{n=1}^{\infty}$ be the base of $[0,1]$ consisting of the intervals in $(D_{1,n})_{n=1}^{\infty}$ starting from level i. For $t \in [0,1]$ and $i \in \mathbb{N}$, set $J_i(t) = \{n \in \mathbb{N} : t \in D_{i,n}\}$.

We will prove that for $i \geq 2$ the space X_{D_i} fails to have the SD2P. In fact, we will prove the following:

Proposition 2.11. *For each $i \geq 2$ let X_{D_i} be the space $C[0,1]$ with the norm $\|\cdot\|_{D_i}$. Then for every $\varepsilon > 0$ there exist finite convex combinations of slices of $B_{X_{D_i}}$ with diameter at most $2\sqrt{i+\varepsilon}/i$.*

Proof. First suppose $i = 2$, choose $t_1 = 0$, $t_2 = 1$, and note that $J_2(t_1) \cap J_2(t_2) = \emptyset$, $\{t_1\} \cap \overline{D_{2,n}} = \emptyset$ for every $n \notin J_2(t_1)$, and $\{t_2\} \cap \overline{D_{2,n}} = \emptyset$ for every $n \notin J_2(t_2)$. Set $M = \sup\{\|x\|_\infty : x \in B_{X_{D_2}}\} < \infty$. Since for any $\alpha > 0$ we can find $\beta > 0$ such that

$$\sum_{n \notin J_2(t_1)} 2^{-n}\|x\|_{2,n}^2 < \alpha \quad \text{and} \quad \sum_{n \notin J_2(t_2)} 2^{-n}\|y\|_{2,n}^2 < \alpha$$

whenever $x \in S(\delta_{t_1}/\|\delta_{t_1}\|_{D_2}^*, \beta)$ and $y \in S(\delta_{t_2}/\|\delta_{t_2}\|_{D_2}^*, \beta)$, we can find $\eta > 0$ such that

$$\sum_{n \notin J_2(t_1)} 2^{-n}2M\|x\|_{2,n} < \frac{\varepsilon}{7}, \qquad \sum_{n \notin J_2(t_2)} 2^{-n}2M\|y\|_{2,n} < \frac{\varepsilon}{7},$$
$$\sum_{n \notin J_2(t_1)} 2^{-n}\|x\|_{2,n}^2 < \frac{\varepsilon}{7}, \qquad \sum_{n \notin J_2(t_2)} 2^{-n}\|y\|_{2,n}^2 < \frac{\varepsilon}{7}$$

whenever $x \in S(\delta_{t_1}/\|\delta_{t_1}\|_{D_2}^*, \eta)$ and $y \in S(\delta_{t_2}/\|\delta_{t_2}\|_{D_2}^*, \eta)$. Now, if we write $h = \frac{1}{2}x + \frac{1}{2}y$ and use the fact that $J_2(t_1) \cap J_2(t_2) = \emptyset$, we get

$$\begin{aligned}
2^2\|h\|_{D_2}^2 &= \sum_{n=1}^{\infty} 2^{-n}\|x+y\|_{2,n}^2 \\
&\leq \sum_{n \in J_2(t_1)} 2^{-n}(\|x\|_{2,n}^2 + 2\|x\|_{2,n}\|y\|_{2,n} + \|y\|_{2,n}^2) \\
&\quad + \sum_{n \in J_2(t_2)} 2^{-n}(\|x\|_{2,n}^2 + 2\|x\|_{2,n}\|y\|_{2,n} + \|y\|_{2,n}^2) \\
&\quad + \sum_{n \notin J_2(t_1) \cup J_2(t_2)} 2^{-n}(\|x\|_{2,n}^2 + 2\|x\|_{2,n}\|y\|_{2,n} + \|y\|_{2,n}^2) \\
&\leq 2(1 + 2\varepsilon/7) + 3\varepsilon/7 \leq 2 + \varepsilon.
\end{aligned}$$

Thus the set $S = \frac{1}{2}S(\delta_{t_1}/\|\delta_{t_1}\|_{D_2}^*, \eta) + \frac{1}{2}S(\delta_{t_2}/\|\delta_{t_2}\|_{D_2}^*, \eta)$ has diameter at most $2\sqrt{2+\varepsilon}/2$.

For an arbitrary $i \geq 2$ we can choose i points $(t_k)_{k=1}^{i}$ in $[0,1]$ such that $J_i(t_j) \cap J_i(t_k) = \emptyset$ for any $j \neq k$ and $\{t_k\} \cap \overline{D_{i,n}} = \emptyset$ for every $n \notin J_i(t_k)$.

Using a similar argument to that for $i = 2$ we deduce that for any $\varepsilon > 0$ and $k = 1, \dots, i$ there exists a slice $S(\delta_{t_k}/\|\delta_{t_k}\|^*_{D_i}, \eta)$ of $B_{X_{D_i}}$ such that the convex combination

$$\sum_{k=1}^{i} \frac{1}{i} S(\delta_{t_k}/\|\delta_{t_k}\|^*_{D_i}, \eta)$$

has diameter at most $2\sqrt{i+\varepsilon}/i$. ∎

THEOREM 2.12. *The space ℓ_2-$\bigoplus_{i=1}^{\infty} X_{D_i}$ is MLUR, has the D2P, the LD2P+, and has convex combinations of slices of arbitrarily small diameter.*

Proof. The properties of being MLUR, having the D2P, and having the LD2P+ are all stable by taking ℓ_2-sums (see [1, Theorem 3.2] and [13, Theorem 3.2] for the last two). Thus ℓ_2-$\bigoplus_{k=1}^{\infty} X_{D_k}$ has all these properties, since each X_{D_k} does.

It remains to prove that the unit ball of ℓ_2-$\bigoplus_{k=1}^{\infty} X_{D_k}$ has finite convex combinations of slices with arbitrarily small diameter. To this end let $Z = X_{D_i} \oplus_2 Y_i$ where $Y_i = \ell_2$-$\bigoplus_{k\neq i} X_{D_k}$. Let $x_i^* \in S_{X^*_{D_i}}$, let $S_i(x_i^*, \delta)$ be a slice of $B_{X_{D_i}}$, and let $0 < \delta < \eta$. Now, if (x_i, y_i) is in the slice $S((x_i^*, 0), \delta)$ of B_Z, then $x_i^*(x_i) > 1 - \delta$, and so $\|x_i\| > 1 - \delta$. Thus $\|y_i\|^2 \le 2\delta - \delta^2$. But this means that

$$S((x_i^*, 0), \delta) \subset S_i(x_i^*, \delta) \times (2\delta - \delta^2)^{1/2} B_{Y_i}.$$

From this we see that if $z \in \sum_{j=1}^{i} \frac{1}{i} S_j(B_Z, (x_{i,j}^*, 0), \delta)$, then we can write

$$z = x + y \quad \text{where} \quad x \in \sum_{j=1}^{i} \frac{1}{i} S_{i,j}(B_{X_{D_i}}, x_{i,j}^*, \delta), \qquad y \in (2\delta - \delta^2)^{1/2} B_{Y_i}.$$

Now, if the convex combination $\sum_{j=1}^{i} \frac{1}{i} S_{i,j}(B_{X_{D_i}}, x_{i,j}^*, \delta)$ is chosen so that its diameter is at most $2\sqrt{i+\eta}/i$, which is possible by Proposition 2.11, we get $\|x\| \le \sqrt{i+\eta}/i$ and $y \in (2\delta - \delta^2)^{1/2} B_{Y_i}$. As $\|z\| \le \|x\| + \|y\|$ and i can be chosen as large as desired and $\eta > 0$ as small as desired, we are done. ∎

3. Questions and remarks.

Let us end the paper with some questions suggested by the current work, together with some remarks we think are relevant:

QUESTION 1. *Does there exist an equivalent MLUR norm on c_0 with the LD2P?*

A Banach space X is said to be *weakly uniformly rotund* (*wUR*) if whenever (x_n) and (y_n) are two sequences in S_X with $\|x_n + y_n\| \to 2$, we have $x_n - y_n \to 0$ weakly.

QUESTION 2. *Does there exist a Banach space with the LD2P and which is weakly locally uniformly rotund?*

Regarding this question we note that there does exist a Banach space X which is wUR, but not locally uniformly rotund, and which has the property that for every $\varepsilon > 0$ and every weak*-null sequence $(f_n) \subset S_{X^*}$ the diameter of the slices $S(f_n, \varepsilon)$ tends to 2. Such a Banach space can be constructed as follows: Let $1 < p_1 < p_2 < \cdots$ be a sequence such that

$$\prod_{i\in\mathbb{N}} \|\mathrm{I}\colon \ell_\infty(2) \to \ell_{p_i}(2)\| < 2 \tag{3.1}$$

(operator norms of the formal identity mappings between 2-dimensional ℓ_p spaces). Then one can form a Banach sequence space as follows:

$$X = \mathbb{R} \oplus_{p_1} (\mathbb{R} \oplus_{p_2} (\mathbb{R} \oplus_{p_3} (\ldots)))$$

where $\mathbb{R}$ is considered a 1-dimensional Banach space and the space is normed by first defining seminorms in finite-dimensional initial parts according to the above schema and then taking a limit of the seminorms, much as in the construction of the variable exponent spaces introduced in [19]. We will now show that this space X has the above mentioned properties.

Set $Y = \overline{\mathrm{span}}(e_n\colon n \in \mathbb{N}) \subset X$ and $Y_k := \overline{\mathrm{span}}(e_n\colon n \in \mathbb{N},\ n \geq k) \subset X$. It can be seen from arguments in [19] that X and Y are isomorphic to ℓ_∞ and c_0, respectively. Also the tail spaces Y_k become asymptotically isometric to c_0, i.e., for each $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that the tail spaces Y_j, $j \geq k$, are $1+\varepsilon$-isomorphic to c_0 via a linear mapping which identifies the canonical unit vector bases of Y_j and of c_0.

The wUR part: Let $(x_n), (y_n) \in B_Y$ be such that $\|x_n+y_n\|_Y \to 2$. Denote by P_n the basis projection to the first n coordinates and let $Q_n = \mathrm{I} - P_n$. Then, according to the definition of the space,

$$(|P_1(x_n + y_n)|^{p_1} + \|Q_1(x_n + y_n)\|^{p_1})^{1/p_1} \to 2, \tag{3.2}$$

so by the triangle inequality

$$\big((|P_1(x_n)| + |P_1(y_n)|)^{p_1} + (\|Q_1(x_n)\| + \|Q_1(y_n)\|)^{p_1}\big)^{1/p_1} \to 2,$$

and by the uniform convexity of $\ell_{p_1}(2)$ we get

$$|P_1(x_n)| - |P_1(y_n)| \to 0, \qquad \|Q_1(x_n)\| - \|Q_1(y_n)\| \to 0. \tag{3.3}$$

By inspecting (3.2) we obtain $|P_1(x_n - y_n)| \to 0$. By continuing inductively, using the right-hand side of (3.3), we see that $P_k(x_n - y_n) \to 0$ for each k. Recall that Y is isomorphic to c_0, thus Y^* is isomorphically ℓ_1. Therefore $x_n - y_n \to 0$ weakly.

The large slices part: First note that if $(f_n) \subset Y^*$ is a normalized sequence then $\|f_n\|_{\ell_1} \geq 1$ because $\|\cdot\|_{c_0} \leq \|\cdot\|_Y$. Fix $\varepsilon > 0$. Let $k \in \mathbb{N}$ be such that

$$\sum_{i=1}^{\infty} a_i e_{k+i} \mapsto \sum_{i=1}^{\infty} a_i e_i$$

defines a $(1+\varepsilon/4)$-isomorphism $Y_k \to c_0$. Note that then

$$\frac{1}{1+\varepsilon/4}\|f\circ Q_k\|_{\ell_1} \le \|f\circ Q_k\|_Y \le (1+\varepsilon/4)\|f\circ Q_k\|_{\ell_1}, \qquad f\in \ell_1.$$

Because (f_n) is weak*-null, we may choose $m_0 \in \mathbb{N}$ such that a sufficiently large part of the mass is supported on the domain of Q_k, more precisely,

$$\frac{1-\varepsilon}{\|f_m\circ Q_k\|_{\ell_1}} < \frac{1}{1+\varepsilon/3}$$

for $m\in\mathbb{N}$, $m\ge m_0$.

Set $g = f_m\circ Q_k/\|f_m\circ Q_k\|_{\ell_1}$. Then

$$\left\{x\in c_0\colon g(x) > \frac{1}{1+\varepsilon/3}\right\} \subset \{x\in c_0\colon (f_m\circ Q_k)(x) > 1-\varepsilon\}.$$

Note that $\frac{1}{1+\varepsilon/4}B_{c_0}\cap Y_k \subset B_{Y_k}$. Therefore the above inclusion implies that we may pick $x, y\in\{z\in B_{Y_k}\colon f_m(z) > 1-\varepsilon\}$ with

$$\|x-y\|_Y \ge \|x-y\|_\infty > \frac{2}{1+\varepsilon/3},$$

finishing the proof.

It is known from [3] that there exists a Banach space with the LD2P which fails the D2P. In fact, it is proved in [3, Theorem 2.4] that any Banach space containing a copy of c_0 can be equivalently renormed to have the LD2P, but with non-empty relatively weakly open subsets of the unit ball with arbitrarily small diameter.

QUESTION 3. *Does there exist a Banach space with the LD2P+ which fails the D2P?*

The LD2P+ can be viewed as a weak version of the Daugavet property. It is then natural to ask:

QUESTION 4. *Does every Banach space with the LD2P+ contain a copy of ℓ_1?*

Appendix. Denote by $\|\cdot\|_\infty$ the canonical sup-norm on ℓ_∞.

PROPOSITION A.1. *Let $\Phi:\ell_\infty\to[0,\infty]$ be the Musielak–Orlicz function given by*

$$\Phi(x_n) = \sum_{n=1}^{\infty} x_n^{2n}.$$

Give ℓ_∞ the Luxemburg norm

$$\|x\| = \inf\{\lambda>0 : \Phi(x/\lambda)\le 1\} \qquad \textit{for } x\in\ell_\infty.$$

Let

$$A = \{x\in\ell_\infty : \Phi(x) < \infty\}.$$

Then:

(a) $(\ell_\infty, \|\cdot\|)$ *is the bidual of* $(c_0, \|\cdot\|)$.

(b) *The function* Φ *is convex on* A. *Moreover, if* $x \in A$, $y \in \ell_\infty$, *and*

$$\frac{\Phi(x+y)+\Phi(x-y)}{2} = \Phi(x) < \infty, \tag{A.1}$$

then $y = 0$.

(c) *The space* $(c_0, \|\cdot\|)$ *is M-embedded.*

(d) *The dual of* $(c_0, \|\cdot\|)$ *is smooth. In particular,* $(c_0, \|\cdot\|)$ *is weakly MLUR.*

We will need the following result of Å. Lima in the proof of the proposition.

THEOREM A.2 ([12, Theorem 2.2]). *Let* X *be a closed subspace of a Banach space* Y. *Then the following are equivalent.*

(a) X *is an M-ideal in* Y.

(b) *For all* $\varepsilon > 0$, *all* $(x_i)_{i=1}^3 \subset B_X$, *and all* $y \in B_Y$, *there exists* $x \in X$ *such that*

$$\|y + x_i - x\| \le 1 + \varepsilon \quad \textit{for every } i = 1, 2, 3.$$

Proof of Proposition A.1. (a) It is straightforward to show that $\|\cdot\|$ is indeed a norm equivalent to $\|\cdot\|_\infty$ on ℓ_∞.

Now, to prove that $\|\cdot\|$ is a bidual norm on ℓ_∞, it suffices to prove that $\|\cdot\|$ is lower semicontinuous on A with respect to the topology on A given by ℓ_1 (from here on termed the *weak* topology*). To this end, start by noting that Φ is continuous with respect to $\|\cdot\|$ at every $x \in c_0$ and that for every $x \in \ell_\infty$ we have

$$\Phi(x) = \sup_n \Phi(P_n x), \tag{A.2}$$

where P_n is the projection of ℓ_∞ onto the first n coordinates. Now, let $x \in A$ and $(x_k)_{k=1}^\infty \subset (\ell_\infty, \|\cdot\|)$, such that $x_k \to x$ weak*. Then

$$\|P_n x_k - P_n x\|_\infty \to_k 0 \quad \forall n \in \mathbb{N}.$$

Hence

$$\Phi(P_n x_k) \to_k \Phi(P_n x) \quad \forall n \in \mathbb{N}.$$

Taking (A.2) into account we get

$$\liminf_k \Phi(x_k) \ge \Phi(x).$$

Thus Φ is lower semicontinuous at any $x \in A$ with respect to the weak* topology. It follows that the same is true for $\|\cdot\|$.

(b) Clearly Φ is convex on A since the functions $f_n(t) = t^{2n}$, $n \in \mathbb{N}$, are convex.

Now, let $x = (x_n)_{n=1}^{\infty} \in A$ and $y = (y_n)_{n=1}^{\infty} \in \ell_\infty$ and assume that (A.1) holds. Then

$$\frac{1}{2}[\Phi(x+y)+\Phi(x-y)]-\Phi(x) = \sum_{n=1}^{\infty}\left[\frac{(x_n+y_n)^{2n}+(x_n-y_n)^{2n}}{2} - x_n^{2n}\right] \geq 0.$$

Since the functions $f_n(t) = t^{2n}$, $n \in \mathbb{N}$, are convex, all expressions in the brackets [ ] are non-negative. If $y \neq 0$, then there is an $n \in \mathbb{N}$ such that $y_n \neq 0$. Since $f_n(t) = t^{2n}$ is strictly convex, we have

$$\frac{(x_n+y_n)^{2n}+(x_n-y_n)^{2n}}{2} - x_n^{2n} > 0,$$

so

$$\frac{1}{2}[\Phi(x+y)+\Phi(x-y)]-\Phi(x) > 0,$$

and we are done.

(c) Set $Z = (c_0, \|\cdot\|)$. We will prove that Z is M-embedded. To this end, it suffices to prove that statement (b) in Theorem A.2 holds for all $(z_i)_{i=1}^3$ in a norm dense subspace of Z. So, let $\varepsilon > 0$, $(z_i)_{i=1}^3 \subset B_Z$, each z_i with finite support, and let $z^{**} = (z_n^{**}) \in B_{Z^{**}}$. Now find $N \in \mathbb{N}$ such that $N > \max\{k \in \mathbb{N} : k \in \bigcup_{i=1}^3 \operatorname{supp}(z_i)\}$ and $\sum_{n>N}(z_n^{**})^{2n} < \varepsilon$. Set $z = P_N z^{**}$ where $P_N : Z^{**} \to Z^{**}$ is the projection onto $\operatorname{span}\{e_n : n = 1, \dots, N\}$ and e_n is the nth standard unit vector in ℓ_∞. Then

$$\Phi(z^{**}+z_i-z) = \sum_{n\in \operatorname{supp}\{z_i\}_{i=1}^3} (z_{i,n}^{**})^{2n} + \sum_{n>N} (z_{i,n})^{2n} < 1+\varepsilon,$$

so $\|z^{**}+z_i-z\| \leq 1+\varepsilon$, and thus we are done.

(d) We will now prove that Z^* is Gateaux smooth. First we show that

$$\Phi(z) = 1 \tag{A.3}$$

whenever $z = (z_n)_{n=1}^{\infty} \in S_{Z^{**}}$ and $z(y) = 1$ for some $y = (y_n)_{n=1}^{\infty} \in S_{Z^*}$. Assume the contrary, i.e. $\Phi(z) < 1$. Then $|z_n| < 1$ for $n \in \mathbb{N}$. Choose k such that $|z_k| \neq 0$ and find $t > |z_k|$ with

$$\Phi(z) + t^{2k} - z_k^{2k} \leq 1.$$

Set $u = (u_n)_{n=1}^{\infty}$ where $u_n = z_n$ for $n \neq k$ and $u_k = t \cdot \operatorname{sign}(y_k)$. Since

$$\Phi(u) = \Phi(z) + t^{2k} - z_k^{2k} \leq 1,$$

we get $\|u\| \leq 1$. On the other hand

$$u(y) - z(y) = |y_k|t - y_k z_k > 0.$$

So $u(y) > z(y) = 1$, which contradicts $\|u\| \leq 1 = \|y\|$.

Now we are ready to prove that Z^* is Gateaux smooth. We will show that for every y with $\|y\| = 1$, there is only one supporting functional. Indeed, let $x, z \in S_{Z^{**}}$ be such that

$$x(y) = z(y) = 1.$$

By (A.3) we have $\Phi(x) = \Phi(z) = 1$. Since

$$1 \geq \left\| \frac{x+z}{2} \right\| \geq \frac{x+z}{2}(y) = 1,$$

we get $\left\|\frac{x+z}{2}\right\| = 1$. Using (A.3) again, we get $\Phi((x+z)/2) = 1$. So,

$$\Phi(x) = \Phi(z) = \Phi((x+z)/2),$$

and thus

$$\frac{\Phi((x+z)/2 + (x-z)/2) + \Phi((x+z)/2 - (x-z)/2)}{2} = \frac{\Phi(x) + \Phi(z)}{2} = \Phi\Big(\frac{x+z}{2}\Big).$$

From (A.1) we now get $x = z$.

For the particular case, note first that if a dual space X^* is smooth, then every extreme point x in B_X is strongly extreme in the weak topology. Indeed, assume that $x = (x_1^{**} + x_2^{**})/2$ where $x_1^{**}, x_2^{**} \in B_{X^{**}}$. There exists $x^* \in S_{X^*}$ such that

$$1 = x^*(x) = \frac{1}{2}\big(x_1^{**}(x^*) + x_2^{**}(x_2)\big).$$

Thus $x_1^{**}(x^*) = x_2^{**}(x^*) = 1$, and by the smoothness of X^* we must have $x_1^{**} = x_2^{**} = x$.

To finish the proof of the particular case, one can use the facts that a Banach space is strictly convex provided its dual is smooth, and that a point in the unit ball of a Banach space is a strongly extreme point in the weak topology if and only if it continues to be an extreme point in the unit ball of the bidual (see [8, p. 674]). ■

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