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CELLS AND n-FOLD HYPERSPACES

BY

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In memoriam Professor James T. Rogers, Jr.

Abstract. We prove that X is a hereditarily indecomposable metric continuum if and only if the n-fold hyperspace $\mathcal{C}_n(X)$ does not contain $(n+1)$-cells, for any positive integer n. Also we characterize the class of continua whose n-fold hyperspaces and n-fold hyperspace suspensions are cells.

1. Introduction. In [16], it is shown that the hyperspace of all nonempty compact subsets of a nondegenerate metric continuum X, denoted by 2^X, always contains Hilbert cubes; and we know that 2^X is a Hilbert cube if and only if X is a locally connected continuum [2, Theorem 3.2]. It is also known that the hyperspace $\mathcal{C}(X)$ of all subcontinua of X contains n-cells if and only if X contains n-ods (see [5] and [19]). In studying the structure of hyperspaces, it is useful to know if they contain cells or Hilbert cubes.

Let X be a continuum and let n be a positive integer. Then the *n-fold hyperspace of X*, denoted by $\mathcal{C}_n(X)$, is the family of all nonempty closed subsets of X with at most n components. By [10, Theorem 7.1] and [13, Theorem 3.4], the hyperspace $\mathcal{C}_n(X)$ is a Hilbert cube if and only if X is locally connected without free arcs. Also, it is known that if X contains a decomposable subcontinuum, then $\mathcal{C}_n(X)$ contains $(n+1)$-cells [12, Theorem 6.1.10]. Therefore, we correct [12, Question 7.4.1] and ask:

1.1. Question. *Let X be a continuum. If $\mathcal{C}_n(X)$ contains $(n+1)$-cells, then does X contain a decomposable subcontinuum?*

We give a positive answer to Question 1.1 in Theorem 3.4. First, in Theorem 3.2, we prove that if the n-fold symmetric product $\mathcal{F}_n(X)$ con-

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tains an $(n+1)$-cell, then X contains a simple closed curve. Let us recall [12, Question 7.4.2]:

1.2. QUESTION. *Is* $\mathcal{C}_3([0,1])$ *homeomorphic to* $[0,1]^6$*?*

We provide a negative answer to Question 1.2 in Theorem 4.2. In fact, in this theorem, we characterize the n-fold hyperspaces that are cells. We also characterize the n-fold hyperspace suspensions that are cells (Theorem 4.5).

2. Definitions. A *simple closed curve* is any space homeomorphic to $S^1 = \{z \in \mathbb{R}^2 \mid \|z\| = 1\}$. A *map* is a continuous function.

A *continuum* is a nonempty, compact, connected and metric space. A *subcontinuum* of a space X is a continuum contained in X. A *dendrite* is a locally connected continuum that does not contain a simple closed curve. A continuum X is *decomposable* if there exist two proper subcontinua A and B of X such that $X = A \cup B$. If X is not decomposable, X is *indecomposable.* A continuum is *hereditarily indecomposable* if each of its subcontinua is indecomposable. An *n-cell* is any space homeomorphic to $[0,1]^n$, $n \in \mathbb{N}$. An *arc* is a 1-cell.

Let X be a continuum of dimension n. Then X is a *Cantor manifold* provided that for each subset A of X such that $\dim(A) \leq n-2$ the set $X \setminus A$ is connected.

Given a continuum X, we define several of its *hyperspaces* as the following sets:

$$2^X = \{A \subset X \mid A \text{ is closed and nonempty}\},$$
$$\mathcal{C}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ components}\}, \quad n \in \mathbb{N},$$
$$\mathcal{F}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ points}\}, \quad n \in \mathbb{N}.$$

If $n = 1$, we write $\mathcal{C}(X)$ instead of $\mathcal{C}_1(X)$. The hyperspace 2^X is topologized by the *Vietoris topology*, defined as the topology generated by

$$\beta = \{\langle U_1, \ldots, U_k \rangle \mid U_1, \ldots, U_k \text{ are open subsets of } X,\ k \in \mathbb{N}\},$$

where $\langle U_1, \ldots, U_k \rangle = \{A \in 2^X \mid A \subset \bigcup_{i=1}^k U_i \text{ and } A \cap U_i \neq \emptyset \text{ for each } i \in \{1, \ldots, k\}\}$. By definition, $\mathcal{C}_n(X)$ and $\mathcal{F}_n(X)$ are subspaces of 2^X. We call $\mathcal{C}_n(X)$ the *n-fold hyperspace of* X, and $\mathcal{F}_n(X)$ the *n-fold symmetric product of* X. To simplify notation, $\langle U_1, \ldots, U_m \rangle_n$ stands for the intersection of the open set $\langle U_1, \ldots, U_m \rangle$ with $\mathcal{C}_n(X)$.

We also consider the quotient space

$$\mathrm{HS}_n(X) = \mathcal{C}_n(X)/\mathcal{F}_n(X)$$

with the quotient topology. It is called the *n-fold hyperspace suspension* of X and was originally defined in [11]. Let $q_X^n \colon \mathcal{C}_n(X) \to \mathrm{HS}_n(X)$ be the quotient map and denote by F_X^n the point corresponding to $q_X^n(\mathcal{F}_n(X))$.

2.1. REMARK. Note that the space $\mathrm{HS}_n(X) \setminus \{F_X^n\}$ is homeomorphic to $\mathcal{C}_n(X) \setminus \mathcal{F}_n(X)$ via the appropriate restriction of q_X^n.

3. Cells in n-fold hyperspaces

3.1. LEMMA. *Let X be a continuum and let $n \geq 2$. Let $\mathcal{K}$ be a subcontinuum of $\mathcal{C}_n(X) \setminus \mathcal{C}_{n-1}(X)$ and $A \in \mathcal{K}$. If K is a component of $\bigcup \mathcal{K}$ and A has exactly m components in K, then each $D \in \mathcal{K}$ has exactly m components contained in K.*

Proof. Note that $\bigcup \mathcal{K} \in \mathcal{C}_n(X)$ [12, Lemma 6.1.1]. Let K be a component of $\bigcup \mathcal{K}$. Since $\bigcup \mathcal{K}$ has a finite number of components, $K' = \bigcup \mathcal{K} \setminus K$ is a closed subset of X. We define, for each $j \in \{1, \ldots, n\}$,

$$\mathcal{L}_j = \{D \in \mathcal{K} \mid D \text{ has exactly } j \text{ components contained in } K\}.$$

CLAIM. *$\mathcal{L}_j$ is an open subset of $\mathcal{K}$ for each $j \in \{1, \ldots, n\}$.*

Let $j_0 \in \{1, \ldots, n\}$. If $\mathcal{L}_{j_0} = \emptyset$, then $\mathcal{L}_{j_0}$ is open. Suppose that $\mathcal{L}_{j_0} \neq \emptyset$. Let $A \in \mathcal{L}_{j_0}$. Since $A \in \mathcal{C}_n(X) \setminus \mathcal{C}_{n-1}(X)$, we may write $A = A_1 \cup \cdots \cup A_{j_0} \cup A_{j_0+1} \cup \cdots \cup A_n$, where $A_1, \ldots, A_n$ are the components, $A_1 \cup \cdots \cup A_{j_0} \subset K$ and $A_{j_0+1} \cup \cdots \cup A_n \subset K'$. Let $U_1, \ldots, U_n$ be pairwise disjoint open subsets of X such that:

- $A_j \subset U_j$ for each $j \in \{1, \ldots, n\}$;
- $U_1 \cup \cdots \cup U_{j_0} \subset X \setminus K'$;
- $U_{j_0+1} \cup \cdots \cup U_n \subset X \setminus K$.

Note that $A \in \langle U_1, \ldots, U_n \rangle_n$ and $\langle U_1, \ldots, U_n \rangle_n \cap \mathcal{K} \subset \mathcal{L}_{j_0}$. Therefore, $\mathcal{L}_{j_0}$ is an open subset of $\mathcal{K}$.

Now we show that $\mathcal{K} = \bigcup_{j=1}^n \mathcal{L}_j$. It is clear that $\bigcup_{j=1}^n \mathcal{L}_j \subset \mathcal{K}$. Let $B \in \mathcal{K}$. Suppose that $B \notin \mathcal{L}_j$ for any $j \in \{1, \ldots, n\}$. Hence, $B \cap K = \emptyset$. Let U and V be open subsets of X such that $K \subset U$, $K' \subset V$ and $U \cap V = \emptyset$. Let $\mathcal{U} = \langle X, U \rangle_n \cap \mathcal{K}$ and $\mathcal{V} = \langle V \rangle_n \cap \mathcal{K}$ be open subsets of $\mathcal{K}$. It is clear that $\mathcal{U} \neq \emptyset$. Since $B \subset K'$, $B \in \mathcal{V}$ and $\mathcal{V} \neq \emptyset$. Moreover, observe that $\mathcal{U} \cap \mathcal{V} = \emptyset$ and $\mathcal{K} = \mathcal{U} \cup \mathcal{V}$. Since $\mathcal{K}$ is connected, we obtain a contradiction. Therefore, $\mathcal{K} = \bigcup_{j=1}^n \mathcal{L}_j$. Finally, since $\mathcal{L}_j \cap \mathcal{L}_l = \emptyset$ for each $j \neq l$, we have $\mathcal{K} = \mathcal{L}_l$ for some $l \in \{1, \ldots, n\}$. ∎

The following theorem shows that if X does not contain a simple closed curve, then the n-fold symmetric product $\mathcal{F}_n(X)$ cannot contain $(n+1)$-cells.

3.2. THEOREM. *Let X be a continuum and let $n \in \mathbb{N}$. If $\mathcal{F}_n(X)$ contains an $(n+1)$-cell, then X contains a simple closed curve.*

Proof. Suppose that X does not contain a simple closed curve. Since $\mathcal{F}_1(X)$ is homeomorphic to X, $\mathcal{F}_1(X)$ does not contain 2-cells. Suppose, inductively, that $\mathcal{F}_{n-1}(X)$ does not contain n-cells; we will prove that $\mathcal{F}_n(X)$ does not contain $(n+1)$-cells. Suppose that $\mathcal{A}$ is an $(n+1)$-cell contained in $\mathcal{F}_n(X)$. Since $\mathcal{F}_{n-1}(X)$ is closed subset of $\mathcal{F}_n(X)$ and $\mathcal{F}_{n-1}(X)$ does not contain n-cells, we assume that $\mathcal{A} \subset \mathcal{F}_n(X) \setminus \mathcal{F}_{n-1}(X)$.

Note that $\bigcup \mathcal{A} \in \mathcal{C}_n(X)$ [12, Lemma 6.1.1], and $\bigcup \mathcal{A}$ is a compact and locally connected subset of X [1, Lemma 2.2]. Let $A_1, \ldots, A_k$ be the components of $\bigcup \mathcal{A}$ for some $k \leq n$. Since X does not contain a simple closed curve, each A_j is a dendrite, $j \in \{1, \ldots, k\}$.

By Lemma 3.1, $|B \cap A_j| = |D \cap A_j| = m_j$ for any $B, D \in \mathcal{A}$ and $j \in \{1, \ldots, k\}$. Observe that $1 \leq m_j \leq n$ for any $j \in \{1, \ldots, k\}$, and $\sum_{j=1}^k m_j = n$. Let $\varphi\colon \mathcal{A} \to \prod_{j=1}^k \mathcal{F}_{m_j}(A_j)$ be given by

$$\varphi(B) = (B \cap A_1, \ldots, B \cap A_k) \quad \text{for each } B \in \mathcal{A}.$$

To see that φ is continuous, let $\varphi_j\colon \mathcal{A} \to \mathcal{F}_{m_j}(A_j)$ be defined by $\varphi_j(B) = B \cap A_j$ for each $j \in \{1, \ldots, k\}$ and $B \in \mathcal{A}$. Let $B \in \mathcal{A}$ and let $W_1, \ldots, W_s$ be open subsets of X such that $\varphi_j(B) = B \cap A_j \in \langle W_1, \ldots, W_s \rangle \cap \mathcal{F}_{m_j}(A_j)$. Let U and V be open subsets of X such that $A_j \subset U$, $\bigcup_{l \neq j} A_l \subset V$ and $U \cap V = \emptyset$. Since $|B \cap A_j| = m_j$, there exist pairwise disjoint open subsets $U_1, \ldots, U_{m_j}$ of X with $\bigcup_{l=1}^{m_j} U_l \subset U$ and $\varphi_j(B) \in \langle U_1, \ldots, U_{m_j} \rangle \cap \mathcal{F}_{m_j}(A_j)$ $\subset \langle W_1, \ldots, W_s \rangle \cap \mathcal{F}_{m_j}(A_j)$. Note that if $\mathcal{U} = \langle U_1, \ldots, U_{m_j}, V \rangle \cap \mathcal{A}$, then $B \in \mathcal{U}$ and $\varphi_j(\mathcal{U}) \subset \langle U_1, \ldots, U_{m_j} \rangle \cap \mathcal{F}_{m_j}(A_j) \subset \langle W_1, \ldots, W_s \rangle \cap \mathcal{F}_{m_j}(A_j)$. Thus, φ_j is continuous for each $j \in \{1, \ldots, k\}$. Therefore, φ is continuous.

Observe that if $B \neq D$, then there exists $j \in \{1, \ldots, k\}$ with $B \cap A_j \neq D \cap A_j$. Hence, $\varphi_j(B) \neq \varphi_j(D)$ and $\varphi(B) \neq \varphi(D)$. Thus, φ is one-to-one. Therefore, φ is an embedding.

Note that $\dim(\mathcal{A}) = \dim(\varphi(\mathcal{A}))$. By [4, Theorem III.1, p. 26], we have $\dim(\mathcal{A}) \leq \dim(\prod_{j=1}^k \mathcal{F}_{m_j}(A_j))$. Moreover, by [4, Theorem III.4, p. 33],

$$\dim\Big(\prod_{j=1}^{k} \mathcal{F}_{m_j}(A_j)\Big) \leq \sum_{i=j}^{k} \dim(\mathcal{F}_{m_j}(A_j)).$$

In the proof of [1, Lemma 3.1], it is shown that $\dim(\mathcal{F}_{m_j}(A_j)) \leq m_j \dim(A_j)$ for each $j \in \{1, \ldots, k\}$. Since A_j is a dendrite, $\dim(A_j) = 1$. Thus, $\dim(\mathcal{A}) \leq \sum_{j=1}^k m_j \dim(A_j) = \sum_{j=1}^k m_j = n$, a contradiction. Therefore, $\mathcal{F}_n(X)$ does not contain $(n+1)$-cells. ∎

The following result gives an interesting property of the n-fold hyperspace of a hereditarily indecomposable continuum.

3.3. Proposition. *Let X be a hereditarily indecomposable continuum and let $n \geq 2$. If Γ is a locally connected subcontinuum in $\mathcal{C}_n(X) \setminus \mathcal{C}_{n-1}(X)$, then $\bigcup \Gamma \in \mathcal{C}_n(X) \setminus \mathcal{C}_{n-1}(X)$.*

Proof. Let $\gamma\colon [0,1] \to \Gamma \subset \mathcal{C}_n(X)$ be an onto map [18, Theorem 8.14]. Let $\sigma\colon [0,1] \to \mathcal{C}_n(X)$ be given by

$$\sigma(t) = \bigcup \gamma([0,t]) \quad \text{for each } t \in [0,1].$$

Note that σ is a map, by [12, Lemma 6.1.1] and [17, Lemma (1.48)]. Observe that $\sigma(0) \in \Gamma$, and $\sigma(t) \subset \sigma(s)$ whenever $t \leq s$.

Claim. *If $s < t$, then each component of $\sigma(t)$ intersects $\sigma(s)$.*

Let $s < t$. Since γ is a map, $\gamma([0,t])$ is a subcontinuum of $\mathcal{C}_n(X)$. Suppose D is a component of $\sigma(t)$ such that $D \cap \sigma(s) = \emptyset$. Since $\sigma(t) = \bigcup \gamma([0,t])$, we have $\gamma(r) \subset \sigma(t)$ for each $r \in [0,t]$. Observe that $\sigma(t) \setminus D$ is a closed subset of X. Let U and V be disjoint open subsets of X with $D \subset U$ and $\sigma(t) \setminus D \subset V$. Let $\langle X, U\rangle_n$ and $\langle V\rangle_n$ be open subsets of $\mathcal{C}_n(X)$. If $r < s$, then $\gamma(r) \subset \sigma(s)$. Hence, $\gamma(r) \cap D = \emptyset$ and $\gamma(r) \in \langle V\rangle_n$. So, $\langle V\rangle_n \cap \gamma([0,t]) \neq \emptyset$. Moreover, D is a component of $\sigma(t)$. So, there exists $l \in [0,t]$ such that $\gamma(l) \cap D \neq \emptyset$, and $\langle X, U\rangle_n \cap \gamma([0,t]) \neq \emptyset$. Since $U \cap V = \emptyset$, we have $\langle X, U\rangle_n \cap \langle V\rangle_n = \emptyset$. Finally, since $\sigma(t) \subset U \cup V$, we find that $\gamma([0,t]) \subset \langle X, U\rangle_n \cup \langle V\rangle_n$, a contradiction. Therefore, each component of $\sigma(t)$ intersects $\sigma(s)$ whenever $s < t$.

Suppose $\bigcup \Gamma \in \mathcal{C}_{n-1}(X)$, that is, $\sigma(1) \in \mathcal{C}_{n-1}(X)$. Since $\mathcal{C}_{n-1}(X)$ is a closed subset of $\mathcal{C}_n(X)$, there exists $t_0 = \min\{t \in [0,1] \mid \sigma(t) \in \mathcal{C}_{n-1}(X)\}$. Let $L_1, \ldots, L_k$ be the components of $\sigma(t_0)$ for some $k \leq n-1$.

Let $A \in \Gamma$. Assume that $A = A_1 \cup \cdots \cup A_n$, where $A_1, \ldots, A_n$ are the components of A. Since $k \leq n-1$, there exists $j \in \{1, \ldots, k\}$ such that A has at least two components in L_j. Without loss of generality, we assume that $j = 1$ and A has exactly m $(1 < m \leq n)$ components, say $A_1, \ldots, A_m$, contained in L_1. Since $\sigma(0) \in \Gamma$, by Lemma 3.1, $\sigma(0)$ has exactly m components contained in L_1. Furthermore, $\sigma(t)$ has exactly m components in L_1 for each $t < t_0$, because $\sigma(0), \sigma(t) \in \mathcal{C}_n(X) \setminus \mathcal{C}_{n-1}(X)$ and by the Claim. Let $R_t = \bigcup \gamma([t, t_0]) \in \mathcal{C}_n(X)$ for each $t \in [0, t_0]$. Note that $R_t \subset \sigma(t_0)$ and $\sigma(t) \cup R_t = \sigma(t_0)$. Hence, $L_1 = (\bigcup_{j=1}^{m} A_{j,t}) \cup (\bigcup_{j=1}^{l} R_{j,t})$, where $A_{1,t}, \ldots, A_{m,t}$ are the m components of $\sigma(t)$ in L_1, and $R_{1,t}, \ldots, R_{l,t}$ are components of R_t. Since L_1 is indecomposable and $\bigcup_{j=1}^{m} A_{j,t} \subsetneq L_1$, we have $R_{s,t} = L_1$ for some $s \in \{1, \ldots, l\}$. Thus, L_1 is a component of R_t for all $t \in [0, t_0]$; in particular, L_1 is a component of $R_{t_0} = \gamma(t_0) \in \Gamma$, contrary to the fact that R_{t_0} has exactly m components contained in L_1 (Lemma 3.1). Therefore, $\bigcup \Gamma \in \mathcal{C}_n(X) \setminus \mathcal{C}_{n-1}(X)$. ∎

The following result gives a positive answer to Question 1.1.

3.4. Theorem. *Let X be a continuum and let $n \in \mathbb{N}$. If X is hereditarily indecomposable, then $\mathcal{C}_n(X)$ does not contain $(n+1)$-cells.*

Proof. By [5, Theorem 1.9], $\mathcal{C}(X)$ does not contain 2-cells. Suppose, inductively, that $\mathcal{C}_{n-1}(X)$ does not contain n-cells; we will prove that $\mathcal{C}_n(X)$ does not contain $(n+1)$-cells. Suppose that there exists an $(n+1)$-cell $\mathcal{A}$ contained in $\mathcal{C}_n(X)$. Since $\mathcal{C}_{n-1}(X)$ is a closed subset of $\mathcal{C}_n(X)$ and $\mathcal{C}_{n-1}(X)$ does not contain n-cells, we have $\mathcal{A} \subset \mathcal{C}_n(X) \setminus \mathcal{C}_{n-1}(X)$. Let $\varphi\colon \mathcal{C}_n(X) \setminus \mathcal{C}_{n-1}(X) \to \mathcal{F}_n(\mathcal{C}(X))$ be given by $\varphi(A) = \{A_1, \ldots, A_n\}$ for each $A \in \mathcal{C}_n(X) \setminus \mathcal{C}_{n-1}(X)$, where $A_1, \ldots, A_n$ are the components of A. By [12, Theorem 6.1.21], φ is an embedding. Thus, $\varphi(\mathcal{A})$ is an $(n+1)$-cell such that $\varphi(\mathcal{A}) \subset \mathcal{F}_n(\mathcal{C}(X))$. Note that, by [17, Theorem (1.61)], $\mathcal{C}(X)$ is uniquely arcwise connected. Hence, $\mathcal{C}(X)$ does not contain a simple closed curve. Therefore, $\mathcal{F}_n(\mathcal{C}(X))$ does not contain $(n+1)$-cells, by Theorem 3.2, a contradiction. Hence, $\mathcal{C}_n(X)$ does not contain $(n+1)$-cells. ∎

The next theorem follows from [12, Theorem 6.1.10] and Theorem 3.4.

3.5. Theorem. *Let X be a continuum and let $n \in \mathbb{N}$. Then X is a hereditarily indecomposable continuum if and only if $\mathcal{C}_n(X)$ does not contain $(n+1)$-cells.*

3.6. Theorem. *Let X be a continuum and let $n, k \in \mathbb{N}$, where $k \geq 2$. Then $\mathcal{C}_n(X)$ contains a k-cell if and only if $\mathrm{HS}_n(X)$ contains a k-cell.*

Proof. Since it is always the case that $\mathcal{C}_n(X)$ contains an n-cell [12, 6.1.9] and $\mathrm{HS}_n(X)$ also contains an n-cell [11, Theorem 3.7], we only need to consider the case when $k > n$.

Suppose $\mathrm{HS}_n(X)$ contains a k-cell $\mathfrak{K}$. Without loss of generality we assume that $\mathfrak{K} \subset \mathrm{HS}_n(X) \setminus \{F_X^n\}$. Hence, by Remark 2.1, $(q_X^n)^{-1}(\mathfrak{K})$ is a k-cell in $\mathcal{C}_n(X)$.

Now, suppose $\mathcal{K}$ is a k-cell contained in $\mathcal{C}_n(X)$. If $\mathcal{K} \cap (\mathcal{C}_n(X) \setminus \mathcal{F}_n(X)) \neq \emptyset$, then there exists a k-cell $\mathcal{K}_0 \subset \mathcal{K} \cap (\mathcal{C}_n(X) \setminus \mathcal{F}_n(X))$. Thus, by Remark 2.1, $q_X^n(\mathcal{K}_0)$ is a k-cell in $\mathrm{HS}_n(X)$. We assume now that $\mathcal{K}$ is contained in $\mathcal{F}_n(X)$. Let $K = \{k_1, \ldots, k_m\}$ be a point of $\mathcal{K}$, and let $U_1, \ldots, U_m$ be pairwise disjoint open subsets of X such that $K \in \langle U_1, \ldots, U_m \rangle \cap \mathcal{F}_n(X)$. Since $\mathcal{K}$ is a k-cell, without loss of generality we assume $\mathcal{K} \subset \langle U_1, \ldots, U_m \rangle \cap \mathcal{F}_n(X)$. Thus, $\bigcup \mathcal{K} \subset \bigcup_{j=1}^m U_j \neq X$.

If $\mathcal{K} \subset \mathcal{F}_{n-1}(X)$, then, by [12, 6.1.1], $\bigcup \mathcal{K} \in \mathcal{C}_{n-1}(X)$. Also, by the previous paragraph, we assume that $\bigcup \mathcal{K} \neq X$. Let D be a nondegenerate subcontinuum of X such that $D \cap \bigcup \mathcal{K} = \emptyset$ [18, 5.5]. Thus, $\mathcal{K}_0 = \{K \cup D \mid K \in \mathcal{K}\}$ is homeomorphic to $\mathcal{K}$, and $\mathcal{K}_0 \subset \mathcal{C}_n(X) \setminus \mathcal{F}_n(X)$. Therefore, $q_X^n(\mathcal{K}_0)$ is a k-cell contained in $\mathrm{HS}_n(X)$.

Suppose $\mathcal{K} \cap (\mathcal{F}_n(X) \setminus \mathcal{F}_{n-1}(X)) \neq \emptyset$. As before, $\mathcal{K} \subset \langle U_1, \ldots, U_n \rangle \cap \mathcal{F}_n(X)$, where $U_1, \ldots, U_n$ are pairwise disjoint and nonempty open subsets of X. Thus, $\bigcup \mathcal{K}$ has exactly n components [3, Lemma 3.1]. Since $\mathcal{K}$ is locally connected, $\bigcup\mathcal{K}$ is a locally connected subset of X [1, Lemma 2.2]. Let $K_1,\dots,K_n$ be the components of $\bigcup\mathcal{K}$. Note that $\prod_{j=1}^n K_j$ is homeomorphic to $\langle K_1,\dots,K_n\rangle\cap\mathcal{F}_n(X)$. Thus, $\prod_{j=1}^n K_j$ contains a k-cell. Since $k>n$, there exists $j_0\in\{1,\dots,n\}$ with $\dim(K_{j_0})\geq 2$ [4, Theorem III.4, p. 33]. Let $p\in K_{j_0}$ with $\dim_p(K_{j_0})\geq 2$, that is, there exists an open neighborhood U of p such that for each open subset V of K_{j_0} with $p\in V\subset U$ the boundary $\mathrm{Bd}(V)$ is of dimension at least one. Hence, $|\mathrm{Bd}(V)|=\infty$. Thus, $\mathrm{ord}_p(X)\geq m$ for each $m\in\mathbb{N}$ (see [8, p. 274] for the definition of $\mathrm{ord}_p(X)$). In particular, $\mathrm{ord}_p(X)\geq k$. Hence, there exist k arcs $\alpha_1,\dots,\alpha_k$ in K_{j_0} such that $\alpha_j\cap\alpha_l=\{p\}$ for $j\neq l$ and $j,l\in\{1,\dots,k\}$ [8, p. 277]. By [17, (1.100)], $\mathcal{C}(K_{j_0})$ contains a k-cell $\mathcal{K}'$ such that $\mathcal{K}'\subset\mathcal{C}(K_{j_0})\setminus\mathcal{F}_1(K_{j_0})\subset\mathcal{C}_n(X)\setminus\mathcal{F}_n(X)$. Therefore, $q_X^n(\mathcal{K}')$ is homeomorphic to $\mathcal{K}'$, and $\mathrm{HS}_n(X)$ contains a k-cell. ∎

3.7. Corollary. *Let X be a continuum and let $n\in\mathbb{N}$. The following are equivalent:*

(1) *X is hereditarily indecomposable;*
(2) *$\mathcal{C}_n(X)$ does not contain $(n+1)$-cells;*
(3) *$\mathrm{HS}_n(X)$ does not contain $(n+1)$-cells.*

4. n-fold hyperspaces that are cells. We characterize the n-fold hyperspaces that are cells (Theorem 4.2) and n-fold hyperspace suspensions that are cells too (Theorem 4.5).

We begin with the characterization of graphs whose n-fold hyperspace is a Cantor manifold.

4.1. Theorem. *Let X be a graph and let n be a positive integer. Then $\mathcal{C}_n(X)$ is a Cantor manifold if and only if X is either an arc or a simple closed curve.*

Proof. Suppose X is neither an arc nor a simple closed curve. Then X has a ramification point p. Hence, if A is an element of $\mathcal{C}_n(X)$ with exactly n components and such that $p\in A$, we find that $\dim_A(\mathcal{C}_n(X))\geq(2n-1)+\mathrm{ord}_p(X)\geq(2n-1)+3=2n+1$. Also, if x is an element of X that is not a ramification point, then there exists a subarc L of X such that $x\in\mathrm{Int}_X(L)$ and L does not contain a ramification point of X. Thus, $\mathcal{C}_n(L)$ is a closed $2n$-dimensional neighborhood of $\{x\}$ [12, 6.8.10] that is a Cantor manifold [14, Theorem 4.6]. This implies that $\dim_{\{x\}}(\mathcal{C}_n(X))=2n$. Since a Cantor manifold has the same dimension at each of its points [4, A), pp. 93 and 94], we deduce that $\mathcal{C}_n(X)$ is not a Cantor manifold.

If X is either an arc or a simple closed curve, then by [14, Theorem 4.6], $\mathcal{C}_n(X)$ is a $2n$-dimensional Cantor manifold. ∎

4.2. THEOREM. *Let X be a continuum and let n and k be positive integers. Then $\mathcal{C}_n(X)$ is homeomorphic to $[0,1]^k$ if and only if X is an arc or a simple closed curve, $n \in \{1,2\}$ and $k \in \{2,4\}$. Moreover, if $n = 2$, then X is an arc and $k = 4$.*

Proof. Suppose $\mathcal{C}_n(X)$ is homeomorphic to $[0,1]^k$. Then, by [12, 6.1.4], X is a locally connected continuum. Since $\dim(\mathcal{C}_n(X)) = k < \infty$, by [12, 6.8.3], X is a graph. Since $[0,1]^k$ is a Cantor manifold [4, Example VI.11, p. 93], by Theorem 4.1, X is an arc or a simple closed curve. Hence, by [12, 6.8.10], $k = 2n$. Suppose $n \geq 3$. Note that each point of $[0,1]^k$ has a local basis of closed neighborhoods homemorphic to $[0,1]^k$. Hence, by [6, Lemma 3.4], $\mathcal{C}_n(X)$ cannot be homeomorphic to $[0,1]^k$. Thus, $n \in \{1,2\}$ and $k \in \{2,4\}$.

If $n = 1$, then $\mathcal{C}_1(X)$ is homeomorphic to $[0,1]^2$ [17, (0.54) and (0.55)]. Now, if $n = 2$, then $\mathcal{C}_2([0,1])$ is homeomorphic to $[0,1]^4$ [12, 6.8.11] and $\mathcal{C}_2(\mathcal{S}^1)$ is homemorphic to the cone over a solid torus [7]. ∎

4.3. REMARK. Observe that Theorem 4.2 gives a negative answer to [12, Question 7.4.2].

The next theorem characterizes the graphs whose n-fold hyperspace suspensions are Cantor manifolds.

4.4. THEOREM. *Let X be a graph and let n be a positive integer. Then* $\mathrm{HS}_n(X)$ *is a Cantor manifold if and only if X is either an arc or a simple closed curve.*

Proof. Suppose X is neither an arc nor a simple closed curve. Then X has a ramification point p. Hence, by the proof of Theorem 4.1, there exists an element A of $\mathcal{C}_n(X)$ with exactly n components such that $\dim_A(\mathcal{C}_n(X)) \geq 2n+1$. Since $q_X^n|_{\mathcal{C}_n(X)\setminus\mathcal{F}_n(X)}$ is a homeomorphism (Remark 2.1), we obtain $\dim_{q_X^n(A)}(\mathrm{HS}_n(X)) \geq 2n+1$. Now, by [15, Lemma 4.1], there exists an element χ in $\mathrm{HS}_n(X)$ such that $\dim_\chi(\mathrm{HS}_n(X)) = 2n$. Since a Cantor manifold has the same dimension at each of its points [4, A), pp. 93 and 94], we see that $\mathrm{HS}_n(X)$ is not a Cantor manifold.

If X is either an arc or a simple closed curve, then by [9, Corollary 3.1], $\mathrm{HS}_n(X)$ is a $2n$-dimensional Cantor manifold. ∎

4.5. THEOREM. *Let X be a continuum and let n and k be positive integers. Then* $\mathrm{HS}_n(X)$ *is homeomorphic to $[0,1]^k$ if and only if X is an arc, $n \in \{1,2\}$ and $k \in \{2,4\}$.*

Proof. Suppose $\mathrm{HS}_n(X)$ is homeomorphic to $[0,1]^k$. Then, by [9, Theorem 5.2], X is a locally connected continuum. Since $\dim(\mathrm{HS}_n(X)) = k < \infty$ and $\dim(\mathrm{HS}_n(X)) = \dim(\mathcal{C}_n(X))$ [9, Theorem 3.6], by [12, 6.8.3], X is a graph. Since $[0,1]^k$ is a Cantor manifold [4, Example VI.11, p. 93], by Theorem 4.4, X is an arc or a simple closed curve. Hence, k is an even number [9, Corollary 3.1]. Suppose $n \geq 3$. Note that each point of $[0,1]^k$ has a local basis of closed neighborhoods homemorphic to $[0,1]^k$. Hence, since $q_X^n|_{\mathcal{C}_n(X)\setminus\mathcal{F}_n(X)}$ is a homeomorphism (Remark 2.1), by [6, Lemma 3.4], $\mathrm{HS}_n(X)$ cannot be homeomorphic to $[0,1]^k$. Thus, $n \in \{1,2\}$ and $k \in \{2,4\}$.

If $n = 1$, then $\mathrm{HS}_1([0,1])$ is homeomorphic to a 2-cell and $\mathrm{HS}_1(\mathcal{S}^1)$ is homeomorphic to a 2-sphere. Thus, if $n = 1$, then X is an arc and $k = 2$.

If $n = 2$, then $\mathrm{HS}_2([0,1])$ is homeomorphic to a 4-cell [15, Theorem 4.6] and $\mathrm{HS}_2(\mathcal{S}^1)$ cannot be homeomorphic to a 4-cell since $T^2_{\mathcal{S}^1}$ does not have a 4-cell neighborhood in $\mathrm{HS}_2(\mathcal{S}^1)$ [15, Lemma 4.7]. Hence, if $n = 2$, then X is an arc and $k = 4$. ■

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