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A generalisation of an identity of Lehmer

by

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1. Introduction. For integers $a, q \geq 1$, the *Euler–Briggs constant* $\gamma(a,q)$ (see [3], [10]) is defined as follows:

$$\gamma(a,q) := \lim_{x\to\infty}\Big(\sum_{\substack{n\leq x\\ n\equiv a \bmod q}} \frac{1}{n} - \frac{\log x}{q}\Big).$$

When $q = 1$, one has $\gamma(1,1) = \gamma$, the *Euler constant*. In 1975, Lehmer [10] proved the identity

$$q\gamma(a,q) - \gamma = -\sum_{\substack{\zeta_q\in\mu_q\\ \zeta_q\neq 1}} \zeta_q^{-a}\log(1-\zeta_q), \tag{1}$$

where $a, q > 1$ and μ_q is the group of qth roots of unity in $\overline{\mathbb{Q}}$. In this paper, we generalise Lehmer's identity. In order to state our result, we need to introduce a few definitions and notations. Throughout the paper, we will denote the set of all prime numbers by P, and an arbitrary prime number by p. For any finite subset Ω of primes, we define

$$\mathrm{P}_\Omega := \begin{cases} \prod_{p\in\Omega} p & \text{if } \Omega\neq\emptyset,\\ 1 & \text{otherwise,}\end{cases} \qquad \delta_\Omega := \begin{cases} \prod_{p\in\Omega}(1-1/p) & \text{if } \Omega\neq\emptyset,\\ 1 & \text{otherwise.}\end{cases}$$

Also throughout the paper, empty sums are assumed to be zero.

For natural numbers $a, q \geq 1$ and for a finite set Ω of primes not containing any prime factors of q, the *generalised Euler–Briggs constant* $\gamma(\Omega, a, q)$

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is defined as

$$\gamma(\Omega, a, q) := \lim_{x\to\infty}\Bigg(\sum_{\substack{n\le x\\ (n,\mathrm{P}_\Omega)=1\\ n\equiv a \bmod q}} \frac{1}{n} - \frac{\delta_\Omega \log x}{q}\Bigg).$$

When $q = 1$, one recovers the *generalised Euler constant*

$$\gamma(\Omega) := \lim_{x\to\infty}\Bigg(\sum_{n\le x,\, (n,\mathrm{P}_\Omega)=1} \frac{1}{n} - \delta_\Omega \log x\Bigg),$$

introduced by Diamond and Ford [4] in 2008. Note that $\gamma(\emptyset, 1, 1) = \gamma(\emptyset) = \gamma(1, 1) = \gamma$. In this context, we have the following theorem.

THEOREM 1. *For any finite set Ω of primes and a natural number $q \ge 1$ with $(q, \mathrm{P}_\Omega) = 1$, one has*

$$(2)\qquad \gamma(\Omega, a, q) - \delta_\Omega \frac{\gamma}{q} = \frac{\delta_\Omega}{q} \sum_{p\in\Omega} \frac{\log p}{p-1} - \sum_{\Omega'\subseteq\Omega} \frac{(-1)^{\mathrm{Card}(\Omega')}}{q\mathrm{P}_{\Omega'}} \sum_{\substack{\zeta_q\in\mu_q\\ \zeta_q\ne 1}} \zeta_q^{-a} \log(1-\zeta_q^{\mathrm{P}_{\Omega'}}),$$

where $\mathrm{Card}(\Omega')$ *denotes the cardinality of* Ω'.

Note that when we set $\Omega = \emptyset$ in (2), we recover the identity of Lehmer (1). The techniques involved in our proof are different from that of Lehmer and hence give another proof of Lehmer's original identity.

The above identity together with the celebrated theorem of Baker on linear forms in logarithms (see Preliminaries for the exact statement) allows us to prove the following corollaries.

COROLLARY 1. *For any natural numbers $a, q > 1$ with $(a, q) = 1$ and for any finite set Ω of primes, the number*

$$\gamma(\Omega, a, q) - \delta_\Omega \frac{\gamma}{q}$$

is transcendental.

Further we have the following corollary.

COROLLARY 2. *Let $\mathrm{U} := \{\Omega_i\}_{i\in\mathbb{N}}$ be a sequence of finite subsets of primes and $\mathrm{S} := \{q_j > 1\}_{j\in\mathbb{N}}$ be a sequence of mutually co-prime natural numbers. Also suppose the Ω_i's do not contain any prime divisors of q_j's for all i, j, and let a be a natural number with $(a, q_j) = 1$ for all j. Then the set*

$$T := \{\gamma(\Omega_i, a, q_j) \mid \Omega_i \in \mathrm{U},\ q_j \in \mathrm{S}\}$$

has at most one algebraic element.

The earlier results established in [6, 13, 14] follow from Corollary 2; see also [7, 8, 11, 15] for results related to transcendence of Euler's constant γ.

In the article [5], the first and the second author along with Kumar Murty give a non-trivial lower bound for the $\overline{\mathbb{Q}}$-space generated by suitable generalised Euler–Briggs constants.

2. Preliminaries. With the notation of Section 1, one can easily deduce that

$$\begin{aligned} \delta_\Omega &= \sum_{d \mid \mathrm{P}_\Omega} \frac{\mu(d)}{d} = \sum_{\Omega' \subseteq \Omega} \frac{(-1)^{\mathrm{Card}(\Omega')}}{\mathrm{P}_{\Omega'}}, \\ -\sum_{d \mid \mathrm{P}_\Omega} \frac{\mu(d)\log d}{d} &= \delta_\Omega \sum_{p \in \Omega} \frac{\log p}{p-1}. \end{aligned} \tag{3}$$

For natural numbers $q, r \geq 1$, we have

$$\prod_{\substack{\zeta_q \in \mu_q \\ \zeta_q \neq 1}} (1-\zeta_q) = q \quad \text{and} \quad \prod_{\zeta_q \in \mu_q} (1-\zeta_q\zeta_r) = 1-\zeta_r^q, \tag{4}$$

where ζ_r is any fixed rth root of unity. The above identities follow by substituting $X = 1$ in

$$X^{q-1} + X^{q-2} + \cdots + 1 = \prod_{\substack{\zeta_q \in \mu_q \\ \zeta_q \neq 1}} (X-\zeta_q)$$

and $X = 1/\zeta_r$ in

$$X^q - 1 = \prod_{\zeta_q \in \mu_q} (X-\zeta_q).$$

We note the following results which are relevant for proving our corollaries.

LEMMA 1. *Let ζ_q $(\neq 1)$ be a qth root of unity, where $q = p^n$, $n \geq 1$. Then the norm of $1-\zeta_q$ is p.*

For a proof of Lemma 1, see Lang [9, p. 83].

LEMMA 2. *Let ζ_q be a primitive qth root of unity, where $q \geq 1$ has at least two prime factors. Then $1-\zeta_q$ is a unit.*

See Washington [16, p. 12] for a proof of Lemma 2. We end this section by stating the following theorem of Baker [1].

THEOREM 2 (Baker). *If $\alpha_1, \ldots, \alpha_n$ are non-zero algebraic numbers such that $\log\alpha_1, \ldots, \log\alpha_n$ are linearly independent over $\mathbb{Q}$, then $1, \log\alpha_1, \ldots, \log\alpha_n$ are linearly independent over $\overline{\mathbb{Q}}$.*

In particular, one has the following theorem.

Theorem 3. *If* $\alpha_1, \dots, \alpha_n$ *and* $\beta_1, \dots, \beta_n$ *are algebraic numbers with* α_i*'s non-zero, then* $\beta_1 \log \alpha_1 + \dots + \beta_n \log \alpha_n$ *is either zero or transcendental.*

The proofs of the above theorems can be found in [2] (see also [12, p. 101]). Finally, we shall need the following corollary of Baker's theorem.

Corollary 3. *For* $n \geq 2$, *let* $\alpha_1, \dots, \alpha_n$ *be non-zero algebraic numbers such that* $\log \alpha_1$ *belongs to the* $\overline{\mathbb{Q}}$*-vector space* $\overline{\mathbb{Q}}\langle \log \alpha_2, \dots, \log \alpha_n \rangle$ *generated by* $\log \alpha_2, \dots, \log \alpha_n$. *Then* $\log \alpha_1 \in \mathbb{Q}\langle \log \alpha_2, \dots, \log \alpha_n \rangle$.

Proof. Let I be a maximal $\overline{\mathbb{Q}}$-linearly independent subset of $\{\log \alpha_2, \dots, \log \alpha_n\}$. By hypothesis, $\log \alpha_1 \in \overline{\mathbb{Q}}(\mathrm{I})$. Hence by Baker's theorem, $\{\log \alpha_1\} \cup \mathrm{I}$ is $\mathbb{Q}$-linearly dependent. Since I is linearly independent over $\mathbb{Q}$, it follows that

$$\log \alpha_1 \in \mathbb{Q}(\mathrm{I}) \subseteq \mathbb{Q}\langle \log \alpha_2, \dots, \log \alpha_n \rangle. \ \blacksquare$$

3. Proof of the main theorem. In order to prove our main theorem, we need the following lemmas.

Lemma 3. *For natural numbers* $a, r > 1$, $q \geq 1$ *with* $(q, r) = 1$, *we have*

$$\lim_{x\to\infty} \Big(\sum_{\substack{n \leq x \\ n \equiv a \bmod q \\ n \equiv 0 \bmod r}} \frac{1}{n} - \frac{1}{qr} \sum_{n \leq x} \frac{1}{n} \Big) = \frac{-1}{qr} \sum_{\zeta_q \in \mu_q} \sum_{\substack{\zeta_r \in \mu_r \\ \zeta_q \zeta_r \neq 1}} \zeta_q^{-a} \log(1 - \zeta_q \zeta_r).$$

Proof. Since $(q, r) = 1$, we have $\zeta_q \zeta_r = 1$ if and only if $\zeta_q = \zeta_r = 1$. Hence

$$\begin{aligned}
&\frac{-1}{qr} \sum_{\zeta_q \in \mu_q} \sum_{\substack{\zeta_r \in \mu_r \\ \zeta_q \zeta_r \neq 1}} \zeta_q^{-a} \log(1 - \zeta_q \zeta_r) \\
&\quad = \frac{-1}{qr} \sum_{\substack{\zeta_r \in \mu_r \\ \zeta_r \neq 1}} \log(1 - \zeta_r) - \frac{1}{qr} \sum_{\substack{\zeta_q \in \mu_q \\ \zeta_q \neq 1}} \sum_{\zeta_r \in \mu_r} \zeta_q^{-a} \log(1 - \zeta_q \zeta_r) \\
&\quad = \frac{1}{qr} \sum_{\substack{\zeta_r \in \mu_r \\ \zeta_r \neq 1}} \sum_{n=1}^{\infty} \frac{\zeta_r^n}{n} + \frac{1}{qr} \sum_{\substack{\zeta_q \in \mu_q \\ \zeta_q \neq 1}} \sum_{\zeta_r \in \mu_r} \sum_{n=1}^{\infty} \frac{\zeta_q^{n-a} \zeta_r^n}{n} \\
&\quad = \frac{1}{qr} \lim_{x\to\infty} \Big(\sum_{n \leq x} \frac{1}{n} \sum_{\substack{\zeta_r \in \mu_r \\ \zeta_r \neq 1}} \zeta_r^n + \sum_{n \leq x} \frac{1}{n} \sum_{\zeta_r \in \mu_r} \zeta_r^n \sum_{\substack{\zeta_q \in \mu_q \\ \zeta_q \neq 1}} \zeta_q^{n-a} \Big) \\
&\quad = \frac{1}{qr} \lim_{x\to\infty} \Big(\sum_{n \leq x} \frac{1}{n} \sum_{\zeta_r \in \mu_r} \zeta_r^n \sum_{\zeta_q \in \mu_q} \zeta_q^{n-a} - \sum_{n \leq x} \frac{1}{n} \Big) \\
&\quad = \lim_{x\to\infty} \Big(\sum_{\substack{n \leq x \\ n \equiv a \bmod q \\ n \equiv 0 \bmod r}} \frac{1}{n} - \frac{1}{qr} \sum_{n \leq x} \frac{1}{n} \Big)
\end{aligned}$$

since

$$\sum_{\zeta_q \in \mu_q} \zeta_q^n = \begin{cases} q & \text{if } n \equiv 0 \bmod q, \\ 0 & \text{otherwise.} \end{cases} \qquad \blacksquare$$

LEMMA 4. *For natural numbers a, q and a finite set Ω of primes, we have*

$$\sum_{\substack{n \le x \\ n \equiv a \bmod q \\ (n, \mathrm{P}_\Omega)=1}} \frac{1}{n} = \sum_{\Omega' \subseteq \Omega} (-1)^{\mathrm{Card}(\Omega')} \sum_{\substack{n \le x \\ n \equiv a \bmod q \\ n \equiv 0 \bmod \mathrm{P}_{\Omega'}}} \frac{1}{n}.$$

Proof. We have

$$\sum_{\substack{n \le x \\ n \equiv a \bmod q \\ (n, \mathrm{P}_\Omega)=1}} \frac{1}{n} = \sum_{\substack{n \le x \\ n \equiv a \bmod q}} \frac{1}{n} \sum_{d \mid (n, \mathrm{P}_\Omega)} \mu(d) = \sum_{d \mid \mathrm{P}_\Omega} \mu(d) \sum_{\substack{n \le x \\ n \equiv a \bmod q \\ d \mid n}} \frac{1}{n}$$

$$= \sum_{d \mid \mathrm{P}_\Omega} (-1)^{\mathrm{Card}(\Omega_d)} \sum_{\substack{n \le x \\ n \equiv a \bmod q \\ d \mid n}} \frac{1}{n},$$

where Ω_d is the set of prime divisors of d. Hence

$$\sum_{\substack{n \le x \\ n \equiv a \bmod q \\ (n, \mathrm{P}_\Omega)=1}} \frac{1}{n} = \sum_{\Omega' \subseteq \Omega} (-1)^{\mathrm{Card}(\Omega')} \sum_{\substack{n \le x \\ n \equiv a \bmod q \\ n \equiv 0 \bmod \mathrm{P}_{\Omega'}}} \frac{1}{n}. \qquad \blacksquare$$

We now prove our main theorem, which generalises the identity (1) of Lehmer.

Proof of Theorem 1. Using Lemma 4 and equation (3), we can write

$$\gamma(\Omega, a, q) - \delta_\Omega \frac{\gamma}{q} = \lim_{x \to \infty} \Bigg(\sum_{\substack{n \le x \\ n \equiv a \bmod q \\ (n, \mathrm{P}_\Omega)=1}} \frac{1}{n} - \frac{\delta_\Omega}{q} \sum_{n \le x} \frac{1}{n} \Bigg)$$

$$= \lim_{x \to \infty} \Bigg(\sum_{\Omega' \subseteq \Omega} (-1)^{\mathrm{Card}(\Omega')} \sum_{\substack{n \le x \\ n \equiv a \bmod q \\ n \equiv 0 \bmod \mathrm{P}_{\Omega'}}} \frac{1}{n} - \frac{1}{q} \sum_{\Omega' \subseteq \Omega} \frac{(-1)^{\mathrm{Card}(\Omega')}}{\mathrm{P}_{\Omega'}} \sum_{n \le x} \frac{1}{n} \Bigg).$$

Thus

$$\gamma(\Omega, a, q) - \delta_\Omega \frac{\gamma}{q}$$

$$= \lim_{x \to \infty} \Bigg(\sum_{\Omega' \subseteq \Omega} (-1)^{\mathrm{Card}(\Omega')} \Bigg(\sum_{\substack{n \le x \\ n \equiv a \bmod q \\ n \equiv 0 \bmod \mathrm{P}_{\Omega'}}} \frac{1}{n} - \frac{1}{q \mathrm{P}_{\Omega'}} \sum_{n \le x} \frac{1}{n} \Bigg) \Bigg)$$

$$= \sum_{\Omega' \subseteq \Omega} \frac{(-1)^{\mathrm{Card}(\Omega')+1}}{q\mathrm{P}_{\Omega'}} \sum_{\zeta_q \in \mu_q} \sum_{\substack{\zeta_{\mathrm{P}_{\Omega'}} \in \mu_{\mathrm{P}_{\Omega'}} \\ \zeta_q \zeta_{\mathrm{P}_{\Omega'}} \neq 1}} \zeta_q^{-a} \log(1 - \zeta_q \zeta_{\mathrm{P}_{\Omega'}}) \quad \text{(by Lemma 3)}$$

$$= \sum_{\Omega' \subseteq \Omega} \frac{(-1)^{\mathrm{Card}(\Omega')+1}}{q\mathrm{P}_{\Omega'}} \times \Bigg(\sum_{\substack{\zeta_{\mathrm{P}_{\Omega'}} \in \mu_{\mathrm{P}_{\Omega'}} \\ \zeta_{\mathrm{P}_{\Omega'}} \neq 1}} \log(1 - \zeta_{\mathrm{P}_{\Omega'}}) + \sum_{\substack{\zeta_q \in \mu_q \\ \zeta_q \neq 1}} \sum_{\zeta_{\mathrm{P}_{\Omega'}} \in \mu_{\mathrm{P}_{\Omega'}}} \zeta_q^{-a} \log(1 - \zeta_q \zeta_{\mathrm{P}_{\Omega'}}) \Bigg)$$

$$= \sum_{\Omega' \subseteq \Omega} \frac{(-1)^{\mathrm{Card}(\Omega')+1}}{q\mathrm{P}_{\Omega'}} \times \Bigg(\log \prod_{\substack{\zeta_{\mathrm{P}_{\Omega'}} \in \mu_{\mathrm{P}_{\Omega'}} \\ \zeta_{\mathrm{P}_{\Omega'}} \neq 1}} (1 - \zeta_{\mathrm{P}_{\Omega'}}) - \sum_{\substack{\zeta_q \in \mu_q \\ \zeta_q \neq 1}} \zeta_q^{-a} \sum_{\zeta_{\mathrm{P}_{\Omega'}} \in \mu_{\mathrm{P}_{\Omega'}}} \sum_{m \geq 1} \frac{\zeta_q^m \zeta_{\mathrm{P}_{\Omega'}}^m}{m} \Bigg)$$

$$= \sum_{\Omega' \subseteq \Omega} \frac{(-1)^{\mathrm{Card}(\Omega')+1}}{q\mathrm{P}_{\Omega'}} \Bigg(\log \mathrm{P}_{\Omega'} - \sum_{\substack{\zeta_q \in \mu_q \\ \zeta_q \neq 1}} \zeta_q^{-a} \sum_{\substack{m \geq 1 \\ m \equiv 0 \bmod \mathrm{P}_{\Omega'}}} \frac{\zeta_q^m \mathrm{P}_{\Omega'}}{m} \Bigg)$$

$$= \sum_{\Omega' \subseteq \Omega} \frac{(-1)^{\mathrm{Card}(\Omega')+1}}{q\mathrm{P}_{\Omega'}} \Bigg(\log \mathrm{P}_{\Omega'} - \sum_{\substack{\zeta_q \in \mu_q \\ \zeta_q \neq 1}} \zeta_q^{-a} \sum_{k=1}^{\infty} \frac{\zeta_q^{k\mathrm{P}_{\Omega'}}}{k} \Bigg)$$

$$= \sum_{\Omega' \subseteq \Omega} \frac{(-1)^{\mathrm{Card}(\Omega')+1}}{q\mathrm{P}_{\Omega'}} \Bigg(\log \mathrm{P}_{\Omega'} + \sum_{\substack{\zeta_q \in \mu_q \\ \zeta_q \neq 1}} \zeta_q^{-a} \log(1 - \zeta_q^{\mathrm{P}_{\Omega'}}) \Bigg) \quad \text{(by (4))}$$

$$= \sum_{d \mid \mathrm{P}} \frac{-\mu(d) \log d}{qd} + \sum_{\Omega' \subseteq \Omega} \frac{(-1)^{\mathrm{Card}(\Omega')+1}}{q\mathrm{P}_{\Omega'}} \sum_{\substack{\zeta_q \in \mu_q \\ \zeta_q \neq 1}} \zeta_q^{-a} \log(1 - \zeta_q^{\mathrm{P}_{\Omega'}})$$

$$= \frac{\delta_\Omega}{q} \sum_{p \in \Omega} \frac{\log p}{p-1} - \sum_{\Omega' \subseteq \Omega} \frac{(-1)^{\mathrm{Card}(\Omega')}}{q\mathrm{P}_{\Omega'}} \sum_{\substack{\zeta_q \in \mu_q \\ \zeta_q \neq 1}} \zeta_q^{-a} \log(1 - \zeta_q^{\mathrm{P}_{\Omega'}}) \quad \text{(by (3))}.$$

This completes the proof of the theorem. ∎

4. Proofs of the corollaries

Proof of Corollary 1. We know from Theorem 1 that

$$\gamma(\Omega, a, q) - \delta_\Omega \frac{\gamma}{q} = \frac{\delta_\Omega}{q} \sum_{p \in \Omega} \frac{\log p}{p-1} - \sum_{\Omega' \subseteq \Omega} \frac{(-1)^{\mathrm{Card}(\Omega')}}{q\mathrm{P}_{\Omega'}} \sum_{\substack{\zeta_q \in \mu_q \\ \zeta_q \neq 1}} \zeta_q^{-a} \log(1 - \zeta_q^{\mathrm{P}_{\Omega'}}).$$

Now by Theorem 3, $\gamma(\Omega, a, q) - \delta_\Omega \gamma/q$ is either 0 or transcendental. Assume that it is equal to zero.

Case I. Suppose that Ω is non-empty and $p_0 \in \Omega$. Consider the set

$$\mathrm{I} := \{\log p : p \in \Omega,\ p \neq p_0\} \cup \{\log(1-\zeta_q) : \zeta_q \in \mu_q,\ \zeta_q \neq 1\}.$$

Since $\delta_\Omega \neq 0$, we get $\log p_0 \in \overline{\mathbb{Q}}(\mathrm{I})$. Then by Corollary 3, there are integers a_0 ($\neq 0$), a_p, a_{ζ_q} such that

$$a_0 \log p_0 = \sum_{\substack{p\in\Omega\\ p\neq p_0}} a_p \log p + \sum_{\substack{\zeta_q\in\mu_q\\ \zeta_q\neq 1}} a_{\zeta_q} \log(1-\zeta_q),$$

which implies that

$$p_0^{a_0} = \prod_{\substack{p\in\Omega\\ p\neq p_0}} p^{a_p} \prod_{\substack{\zeta_q\in\mu_q\\ \zeta_q\neq 1}} (1-\zeta_q)^{a_{\zeta_q}}.$$

Since $(q, p_0) = 1$, by taking the norm on both sides and applying Lemmas 1 and 2, we get a contradiction.

Case II. Suppose that $\Omega = \emptyset$. Then by Theorem 1,

$$\gamma(a,q) - \frac{\gamma}{q} = \frac{-1}{q} \sum_{\substack{\zeta_q\in\mu_q\\ \zeta_q\neq 1}} \zeta_q^{-a} \log(1-\zeta_q) = \frac{-1}{q} \sum_{b=1}^{q-1} \eta_q^{-ab} \log(1-\eta_q^b), \tag{5}$$

where η_q is a primitive qth root of unity. Let $\mathrm{I} := \{\log\alpha_1, \ldots, \log\alpha_t\}$ be a maximal $\mathbb{Q}$-linearly independent subset of $\{\log(1-\eta_q^b) \mid 1 \le b \le q-1\}$. Write

$$\log(1-\eta_q^b) = \sum_{c=1}^{t} u_{b,c} \log\alpha_c$$

for some $u_{b,c} \in \mathbb{Q}$. Set $\gamma_a := \gamma(a,q) - \gamma/q$. Then by (5), we have

$$\gamma_a = \frac{-1}{q} \sum_{b=1}^{q-1} \eta_q^{-ab} \sum_{c=1}^{t} u_{b,c} \log\alpha_c = \frac{-1}{q} \sum_{c=1}^{t} u_c \log\alpha_c,$$

where

$$u_c = \sum_{b=1}^{q-1} \eta_q^{-ab} u_{b,c} \in \mathbb{Q}(\eta_q).$$

Without loss of generality we assume that $\gamma_a = 0$, since by Theorem 3 the above quantity is either zero or transcendental. Then by our assumption, $u_c = 0$ for all c. Let σ_ℓ be an element of the Galois group of $\mathbb{Q}(\eta_q)$ over $\mathbb{Q}$

which sends η_q to η_q^ℓ. Then

$$\begin{aligned}\gamma_{a\ell} &= \frac{-1}{q}\sum_{b=1}^{q-1}\eta_q^{-ab\ell}\sum_{c=1}^{t}u_{b,c}\log\alpha_c\\ &= \frac{-1}{q}\sum_{b=1}^{q-1}\sigma_\ell(\eta_q^{-ab})\sum_{c=1}^{t}u_{b,c}\log\alpha_c\\ &= \frac{-1}{q}\sum_{c=1}^{t}\sigma_\ell(u_c)\log\alpha_c.\end{aligned}$$

Hence $\gamma_{a\ell}=0$ for all ℓ with $(\ell,q)=1$. Then

$$\begin{aligned}0 = \sum_{\substack{1\le r<q\\(r,q)=1}}\gamma_r &= \lim_{x\to\infty}\sum_{\substack{1\le r<q\\(r,q)=1}}\Bigg(\sum_{\substack{n\le x\\ n\equiv r \bmod q}}\frac{1}{n}-\frac{1}{q}\sum_{n\le x}\frac{1}{n}\Bigg)\\ &= \lim_{x\to\infty}\Bigg(\sum_{\substack{n\le x\\(n,q)=1}}\frac{1}{n}-\frac{\varphi(q)}{q}\sum_{n\le x}\frac{1}{n}\Bigg)\\ &= \gamma(\Omega_q)-\delta_{\Omega_q}\gamma,\end{aligned}$$

where Ω_q is the set of all prime divisors of q. Substituting Ω_q in place of Ω and 1 in place of q in Theorem 1, we get

$$\gamma(\Omega_q)-\delta_{\Omega_q}\gamma = \delta_{\Omega_q}\sum_{p\in\Omega_q}\frac{\log p}{p-1},$$

a contradiction since the set $\{\log p : p\in\Omega_q\}$ is linearly independent over $\mathbb{Q}$. This completes the proof of Corollary 1. ∎

Proof of Corollary 2. Suppose that $\gamma(\Omega_1,a,q_1),\gamma(\Omega_2,a,q_2)\in\overline{\mathbb{Q}}$. Then

$$\begin{aligned}(6)\quad &\frac{\delta_{\Omega_2}}{q_2}\gamma(\Omega_1,a,q_1)-\frac{\delta_{\Omega_1}}{q_1}\gamma(\Omega_2,a,q_2)\\ &= \frac{\delta_{\Omega_1}\delta_{\Omega_2}}{q_1q_2}\Bigg(\sum_{p\in\Omega_1}\frac{\log p}{p-1}-\sum_{p\in\Omega_2}\frac{\log p}{p-1}\Bigg)\\ &\quad-\delta_{\Omega_2}\sum_{\Omega_1'\subseteq\Omega_1}\frac{(-1)^{\mathrm{Card}(\Omega_1')}}{q_1q_2\mathrm{P}_{\Omega_1'}}\sum_{b=1}^{q_1-1}\eta_{q_1}^{-ab}\log(1-\eta_{q_1}^{bP'})\\ &\quad+\delta_{\Omega_1}\sum_{\Omega_2'\subseteq\Omega_2}\frac{(-1)^{\mathrm{Card}(\Omega_2')}}{q_1q_2\mathrm{P}_{\Omega_2'}}\sum_{c=1}^{q_2-1}\eta_{q_2}^{-ac}\log(1-\eta_{q_2}^{cP'})\in\overline{\mathbb{Q}},\end{aligned}$$

where η_{q_1} and η_{q_2} are primitive q_1th and q_2th roots of unity respectively.

Hence by Theorem 3, we know that

$$\frac{\delta_{\Omega_2}}{q_2}\gamma(\Omega_1,a,q_1)-\frac{\delta_{\Omega_1}}{q_1}\gamma(\Omega_2,a,q_2)=0.$$

CASE I. Suppose that $\Omega_1\neq\Omega_2$. Choose p_0 either from $\Omega_1\setminus\Omega_2$ or from $\Omega_2\setminus\Omega_1$. Then arguing as in Case I of Corollary 1, and using Lemma 3, we get the assertion.

CASE II. Suppose that $\Omega_1=\Omega_2=\Omega$, say. Set

$$\gamma_a:=\frac{1}{q_1}\gamma(\Omega,a,q_2)-\frac{1}{q_2}\gamma(\Omega,a,q_1).$$

Then from Theorem 1, we see that

$$\gamma_a=\sum_{\Omega'\subseteq\Omega}\frac{(-1)^{\mathrm{Card}(\Omega')}}{q_1q_2\mathrm{P}_{\Omega'}}\Bigg(\sum_{b=1}^{q_1-1}\eta_{q_1}^{-ab}\log(1-\eta_{q_1}^{bP'})-\sum_{c=1}^{q_2-1}\eta_{q_2}^{-ac}\log(1-\eta_{q_2}^{cP'})\Bigg),$$

where η_{q_1} and η_{q_2} are primitive q_1th and q_2th roots of unity respectively. Let $\{\log\alpha_1,\dots,\log\alpha_t\}$ be a maximal $\mathbb{Q}$-linearly independent subset of

$$\{\log(1-\eta_{q_1}^{b}),\,\log(1-\eta_{q_2}^{c})\mid 1\le b\le q_1-1,\ 1\le c\le q_2-1\}.$$

If we write $\log(1-\eta_{q_1}^{b})=\sum_{r=1}^{t}d_{b,r}\log\alpha_r$ and $\log(1-\eta_{q_2}^{c})=\sum_{r=1}^{t}e_{c,r}\log\alpha_r$ where $d_{b,r},e_{c,r}$ are in $\mathbb{Q}$, then we get $\gamma_a=\sum_{r=1}^{t}\beta_r\log\alpha_r$, where

$$\beta_r:=\sum_{\Omega'\subseteq\Omega}\frac{(-1)^{\mathrm{Card}(\Omega')}}{q_1q_2\mathrm{P}_{\Omega'}}\Bigg(\sum_{b=1}^{q_1-1}d_{b,r}\eta_{q_1}^{-ab}-\sum_{c=1}^{q_2-1}e_{c,r}\eta_{q_2}^{-ac}\Bigg).$$

Hence by Theorem 3, $\beta_r=0$ for all r since by assumption $\gamma_a=0$. Arguing as in Case II, Corollary 1 and by applying Galois elements of $\mathbb{Q}(\eta_{q_1q_2})$ over $\mathbb{Q}$, we find that $\gamma_a=0$ for all $(a,q_1q_2)=1$. Hence

$$\sum_{\substack{1\le a<q_1q_2\\(a,q_1q_2)=1}}\gamma_a=0.$$

Note that by orthogonality of characters, we have

$$\frac{1}{q_1}\sum_{\substack{1\le a<q_1q_2\\(a,q_1q_2)=1}}\sum_{\substack{k\le x\\(k,\mathrm{P}_\Omega)=1\\k\equiv a \bmod q_2}}\frac{1}{k}=\frac{1}{q_1\phi(q_2)}\sum_{\substack{1\le a<q_1q_2\\(a,q_1q_2)=1}}\sum_{\substack{k\le x\\(k,\mathrm{P}_\Omega)=1}}\frac{1}{k}\sum_{\chi \bmod q_2}\chi(k)\bar{\chi}(a) \tag{7}$$

$$=\frac{\phi(q_1)}{q_1}\sum_{\substack{k\le x\\(k,q_2\mathrm{P}_\Omega)=1}}\frac{1}{k}=\delta_{\Omega_{q_1}}\sum_{\substack{k\le x\\(k,\mathrm{P}_{\Omega\cup\Omega_{q_2}})=1}}\frac{1}{k},$$

where Ω_{q_1} is the set of all prime divisors of q_1. Thus using (7), we get

$$\begin{aligned}\sum_{\substack{1\le a<q_1q_2\\(a,q_1q_2)=1}}\gamma_a &= \lim_{x\to\infty}\sum_{\substack{1\le a<q_1q_2\\(a,q_1q_2)=1}}\Bigg(\frac{1}{q_1}\sum_{\substack{k\le x\\(k,\mathrm{P}_\Omega)=1\\k\equiv a \bmod q_2}}\frac{1}{k}-\frac{1}{q_2}\sum_{\substack{k\le x\\(k,\mathrm{P}_\Omega)=1\\k\equiv a \bmod q_1}}\frac{1}{k}\Bigg)\\ &= \lim_{x\to\infty}\Bigg(\delta_{\Omega_{q_1}}\sum_{\substack{k\le x\\(k,\mathrm{P}_{\Omega\cup\Omega_{q_2}})=1}}\frac{1}{k}-\delta_{\Omega_{q_2}}\sum_{\substack{k\le x\\(k,\mathrm{P}_{\Omega\cup\Omega_{q_1}})=1}}\frac{1}{k}\Bigg)\\ &= \delta_{\Omega_{q_1}}\gamma(\Omega\cup\Omega_{q_2})-\delta_{\Omega_{q_2}}\gamma(\Omega\cup\Omega_{q_1}).\end{aligned}$$

Here $\Omega_{q_1}, \Omega_{q_2}$ denote the set of all prime divisors of q_1 and q_2 respectively. Now using Theorem 1, we know that

$$\delta_{\Omega_{q_1}}\gamma(\Omega\cup\Omega_{q_2})-\delta_{\Omega_{q_2}}\gamma(\Omega\cup\Omega_{q_1}) = \delta_{\Omega\cup\Omega_{q_1}\cup\Omega_{q_2}}\Bigg(\sum_{p\in\Omega_{q_2}}\frac{\log p}{p-1}-\sum_{p\in\Omega_{q_1}}\frac{\log p}{p-1}\Bigg).$$

Since $(q_1, q_2) = 1$, by Theorem 3 the above expression is transcendental. This completes the proof of Corollary 2. ∎

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