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ADJACENT DYADIC SYSTEMS AND THE L^p-BOUNDEDNESS OF SHIFT OPERATORS IN METRIC SPACES REVISITED

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Abstract. With the help of recent adjacent dyadic constructions by Hytönen and the author, we give an alternative proof of results of Lechner, Müller and Passenbrunner about the L^p-boundedness of shift operators acting on functions $f \in L^p(X;E)$ where $1 < p < \infty$, X is a metric space and E is a UMD space.

1. Introduction. During the last three decades, the highly influential $T(1)$ theorem of G. David and J.-L. Journé [7] has been generalized to various settings by different authors (e.g. [10, 11]). One of these generalizations was due to T. Figiel ([9, 8]; a different proof by T. Hytönen and L. Weis [18]) who proved the theorem for UMD-valued functions $f \in L^p(\mathbb{R}^d;E)$ and scalar-valued kernels using a clever observation that any Calderón–Zygmund operator on $\mathbb{R}^d$ can be decomposed into sums and products of Haar shifts (or rearrangements), Haar multipliers and paraproducts. Not long ago, P. F. X. Müller and M. Passenbrunner [25] extended this technique from the Euclidean setting to metric spaces to prove the $T(1)$ theorem for UMD-valued functions $f \in L^p(X;E)$, where X is a normal space of homogeneous type (see [24, Theorems 2 and 3]). One of the key elements of their (and Figiel's) proof—the L^p-boundedness of the shift operators—was revisited and simplified by R. Lechner and Passenbrunner in their recent paper [21] by proving the result in a more general form with different techniques.

Roughly speaking, a shift operator permutes the generating Haar functions in such a way that if $h_Q \mapsto h_P$, then the dyadic cubes P and Q are not too far apart and they belong to the same generation of the given dyadic system. On the real line, this can be expressed in a very simple form: for every $m \in \mathbb{Z}$, the shift operator T_m is the linear extension of the map $h_I \mapsto h_{I+m|I|}$. In [8, Theorem 1], Figiel showed that for UMD-valued

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functions $f\colon [0,1] \to E$ and every $p \in (1,\infty)$ we have the norm estimate

$$\|T_m f\|_p \le C \log(2+|m|)^\alpha \|f\|_p \tag{1.1}$$

where $\alpha < 1$ depends only on E and p, and the constant C depends on E, p and α (the same result was formulated for functions $f\colon \mathbb{R}^d \to E$ in [9, Lemma 1]). In [25, Sections 4.3–4.5], Müller and Passenbrunner generalized the definition of shift operators to Christ-type dyadic systems [5] in quasimetric spaces and proved the corresponding L^p-estimate. Lechner and Passenbrunner then generalized the definition further and gave an alternative proof for this norm estimate by modifying the underlying dyadic system.

In this paper, we revisit and improve some results related to the recent metric adjacent dyadic constructions by Hytönen and the author [17], and give a proof of (1.1) for UMD-valued functions $f\colon X \to E$ as an application. Our central idea is that with the help of adjacent dyadic systems we can split a given dyadic system $\mathscr{D}$ into suitable subcollections $\mathscr{D}_\lambda$ that give a convenient way to approximate certain indicator functions by their conditional expectations. This approximation technique combined with some classical results of UMD-valued analysis give a fairly straightforward proof of the L^p-estimate.

2. Dyadic cubes, conditional expectations and UMD spaces

2.1. Geometrically doubling metric spaces. Let (X,d) be a *geometrically doubling* metric space, that is, there exists a constant M such that every ball $B(x,r) := \{y \in X : d(x,y) < r\}$ can be covered by at most M balls of radius $r/2$. In this subsection we do not assume any measurability of (X,d), but we note that if (Y,d',μ) is a doubling metric measure space, then (Y,d') is a geometrically doubling metric space.

We use the following two standard lemmas repeatedly without explicit mention.

Lemma 2.1 ([12, Lemma 2.3]). *The following properties hold for (X,d):*

(1) *Any ball $B(x,r)$ can be covered by at most $\lfloor M\delta^{-\log_2 M}\rfloor$ balls $B(x_i,\delta r)$ for every $\delta \in (0,1]$.*
(2) *Any ball $B(x,r)$ contains at most $\lfloor M\delta^{-\log_2 M}\rfloor$ centres x_i of pairwise disjoint balls $B(x_i,\delta r)$ for every $\delta \in (0,1]$.*

Lemma 2.2 ([17, Lemma 2.2]). *For any $\delta > 0$ there exists a countable* maximal δ-separated set $\mathscr{A}_\delta \subseteq X$*:*

- $d(x,y) \ge \delta$ *for all* $x,y \in \mathscr{A}_\delta$, $x \ne y$,
- $\min_{x\in\mathscr{A}_\delta} d(x,z) < \delta$ *for every* $z \in X$.

Since the centre points of dyadic cybes (see Theorem 2.5 below) form sets that are δ^k-separated, the following simple lemma is a convenient tool for splitting dyadic systems into smaller sparse systems. We will use the lemma later in Section 3.

Lemma 2.3. *Let $D_2 \geq D_1 > 0$ and let Z be a D_1-separated set of points in the space X. Then Z is a disjoint union of at most N D_2-separated sets where N depends only on M and D_1/D_2.*

Proof. First, notice that any ball of radius D_2 can contain at most boundedly many, say M_1, points of Z by the second part of Lemma 2.1. By Lemma 2.2, we can choose a maximal D_2-separated subset Z_1 from Z. We claim that if we apply the same lemma M_1 times to choose maximal D_2-separated subsets $Z_k \subseteq Z \setminus \bigcup_{i=1}^{k-1} Z_i$ for every $k = 1, \dots, M_1$, then $Z \setminus \bigcup_{k=1}^{M_1} Z_k = \emptyset$.

For contradiction, suppose that there exists a point $x \in Z \setminus \bigcup_{k=1}^{M_1} Z_k$. By maximality, $B(x, D_2) \cap Z_k \neq \emptyset$ for every $k = 1, \dots, M_1$ since otherwise x would belong to one of the collections Z_k. Thus, the ball $B(x, D_2)$ contains $M_1 + 1$ points of Z, a contradiction. ∎

In the construction of metric dyadic cubes we need maximal δ^k-separated sets for every $k \in \mathbb{Z}$. For this we can use Lemma 2.2 or the following stronger result:

Theorem 2.4 ([17, Theorem 2.4]). *For every $\delta \in (0, 1/2)$ there exist* maximal nested δ^k-separated sets $\mathscr{A}_k := \{z_\alpha^k : \alpha \in \mathcal{N}_k\}$, $k \in \mathbb{Z}$*:*

- $\mathscr{A}_k \subseteq \mathscr{A}_{k+1}$ *for every* $k \in \mathbb{Z}$*;*
- $d(z_\alpha^k, z_\beta^k) \geq \delta^k$ *for* $\alpha \neq \beta$*;*
- $\min_\alpha d(x, z_\alpha^k) < \delta^k$ *for every* $x \in X$ *and every* $k \in \mathbb{Z}$*,*

where $\mathcal{N}_k = \{0, 1, \dots, n_k\}$ *if the space* (X, d) *is bounded, and* $\mathcal{N}_k = \mathbb{N}$ *otherwise.*

2.2. Adjacent dyadic systems in metric spaces. The following theorem is an improved version of the famous constructions of (quasi)metric dyadic cubes by M. Christ [5] and E. Sawyer and R. L. Wheeden [27]. This version was proved by Hytönen and A. Kairema [15, Theorem 2.2] and adapted for different dyadic constructions in [17] (see [17, Theorem 2.9] and Theorem 2.6 below).

Theorem 2.5. *Let (X, d) be a doubling metric space and $\delta \in (0, 1)$ be small enough. Then for given nested maximal sets $\{z_\alpha^k : \alpha \in \mathscr{A}_k\}$, $k \in \mathbb{Z}$, of δ^k-separated points there exist a countable collection $\mathscr{D} := \{Q_\alpha^k : k \in \mathbb{Z}, \alpha \in \mathscr{A}_k\}$ of dyadic cubes such that*

(i) $X = \bigcup_\alpha Q_\alpha^k$ *for every* $k \in \mathbb{Z}$*;*
(ii) $P, Q \in \mathscr{D} \Rightarrow P \cap Q \in \{\emptyset, P, Q\}$*;*

(iii) $B\big(z_\alpha^k, \frac{1}{5}\delta^k\big) \subseteq Q_\alpha^k \subseteq B(z_\alpha^k, 3\delta^k)$;
(iv) $Q_\alpha^k = \bigcup_{\beta : Q_\beta^{k+m} \subseteq Q_\alpha^k} Q_\beta^{k+m}$ *for every* $m \in \mathbb{N}$.

For every dyadic system $\mathscr{D}$ and cube $Q := Q_\alpha^j \in \mathscr{D}$ we use the following notation:

$$\begin{aligned}
\operatorname{lev}(Q) &:= j && \text{(level/generation of the cube } Q\text{)},\\
\mathscr{D}^k &:= \{Q_\alpha^k \in \mathscr{D} : \alpha \in \mathscr{A}_k\} && \text{(cubes of level } k\text{)},\\
B_Q &:= B(z_\alpha^j, 3\delta^j) && \text{(a ball containing } Q\text{)},\\
x_Q &:= z_\alpha^j && \text{(the centre point of } Q\text{)}.
\end{aligned}$$

As mentioned earlier, the central idea of our techniques in Section 4 is to split a given dyadic system into suitable subcollections that help us approximate certain indicators by their conditional expectations. For this we use *adjacent dyadic systems* which have turned out to be a convenient tool for approximating arbitrary balls and other objects by cubes both in $\mathbb{R}^n$ and in more abstract settings (see e.g. [20, 23]). In quasimetric spaces they were first constructed by Hytönen and Kairema [15, Theorem 4.1] (based on the ideas of Hytönen and H. Martikainen [16]), but by restricting ourselves to a strictly metric setting we can use systems with more powerful properties. The following theorem was recently proved by Hytönen and the author for $n = 1$:

THEOREM 2.6. *Let (X, d) be a doubling metric space with doubling constant M and let $n \in \mathbb{N}$. Then for $\delta < 1/(n \cdot 168M^8)$ there exist a bounded number of* adjacent *dyadic systems $\mathscr{D}(\omega)$, $\omega = 1, \dots, K = K(\delta)$, such that:*

(I) *each $\mathscr{D}(\omega)$ is a dyadic system in the sense of Theorem* 2.5*;*
(II) *for a fixed $p \in \mathbb{N}$ and n fixed balls $B_1, \dots, B_n$ there exist $\omega \in \{1, \dots, K\}$ and cubes $Q_{B_1}, \dots, Q_{B_n} \in \mathscr{D}(\omega)$ such that for every $i \in \{1, \dots, n\}$ we have*
   (i) $B_i \subseteq Q_{B_i}$,
   (ii) $\ell(Q_{B_i}) \le \delta^{-2} r(B_i)$,
   (iii) $\delta^{-p} B_i \subseteq Q_{B_i}^{(p)}$,

   where $\ell(Q) = \delta^k$ if $Q = Q_\alpha^k$, $r(B)$ is the radius of the ball B and $Q_{B_i}^{(p)}$ is the unique dyadic ancestor of Q_{B_i} of generation $\operatorname{lev}(Q_{B_i}) - p$.

Proof. Let $\Omega := \{0, 1, \dots, \lfloor 1/\delta \rfloor\}$ and let $\mathbb{P}_\omega$ be the natural probability measure Ω. Also denote

$$\begin{aligned}
\partial_\varepsilon A &:= \{x \in A : d(x, A^c) < \varepsilon\} \cup \{x \in A^c : d(x, A) < \varepsilon\},\\
\mathcal{B}_k(\omega) &:= \bigcup_\alpha \partial_{\delta^{k+1}} Q_\alpha^k(\omega)
\end{aligned}$$

for every $k \in \mathbb{Z}$, where $Q(\omega)$ is a cube of $\mathscr{D}(\omega)$.

In [17, Theorem 5.9] the case $n = 1$ was proved by showing that if $B(x, r)$ is a ball such that $\delta^{k+2} < r \le \delta^{k+1}$, then

$$\mathbb{P}_\omega(\{\omega \in \Omega : x \in \mathcal{B}_k(\omega) \cup \mathcal{B}_{k-p}(\omega)\}) \le 168M^8\delta < 1. \tag{2.7}$$

Given (2.7), the proof for general $n \in \mathbb{N}$ is simple. Let $B_1, \ldots, B_n$ be balls such that $B_i := B(x_i, r_i)$, $\delta^{k_i+2} < r_i \le \delta^{k_i+1}$. Then

$$\mathbb{P}_\omega(\{\omega \in \Omega : x_i \in \mathcal{B}_{k_i}(\omega) \cup \mathcal{B}_{k_i-p}(\omega) \text{ for some } i\}) \le n \cdot 168M^8 < 1.$$

In particular, there exists $\omega \in \Omega$ such that $x_i \notin \mathcal{B}_{k_i}(\omega) \cup \mathcal{B}_{k_i-p}(\omega)$ for every $i = 1, \ldots, n$, which is enough to prove the claim. ∎

Remark 2.8. (1) In the previous theorem, the constant K is roughly $1/\delta$ [17, Section 5.2]. Thus, for a large n both the number of systems $\mathscr{D}(\omega)$ and the change of length scale between two consecutive levels of cubes become large.

(2) We will use the previous theorem only for $n = 2$, in the following way. Let $Q_1, Q_2 \in \mathscr{D}^k$ and $m > 1$ be fixed. Then by Theorem 2.6 there exist an index ω and cubes $P_1, P_2 \in \mathscr{D}(\omega)^{k-3}$ such that

$$Q_1 \subseteq B_{Q_1} \subseteq P_1, \quad Q_2 \subseteq B_{Q_2} \subseteq P_2, \quad 2mB_{Q_1} \subseteq P_1^{(p_m)}$$

for $p_m \in \mathbb{N}$ with $2m\delta^{p_m} \le 1$.

2.3. Conditional expectations. Conditional expectations are mostly used in the field of probability theory, but they have turned out to be extremely useful also in many questions related to more classical analysis (see e.g. [13]). It is well known among specialists that most of the results involving conditional expectations remain true in more general measure spaces but, unfortunately, it is difficult to find a comprehensive presentation of this extended theory in the literature. We refer to [28] for some basic properties of conditional expectations in σ-finite measure spaces, and to [29, Chapter 9] for a presentation of the classical probabilistic theory of conditional expectations.

Let $(X, \mathscr{F}, \mu, d)$ be a metric measure space such that μ is a *doubling Borel measure*, i.e. there exists a constant $D := D_\mu$ such that

$$\mu(2B) \le D\mu(B) < \infty$$

for every ball B. By construction we know that if $\mathscr{D}$ is a dyadic system given by Theorem 2.5, then $\mathscr{D} \subseteq \operatorname{Bor} X$. In particular, the σ-algebra generated by any subcollection of $\mathscr{D}$ is a subset of $\mathscr{F}$.

Set $\mathscr{G}_0 := \{G \in \mathscr{G} : \mu(G) < \infty\}$ for every σ-algebra $\mathscr{G} \subseteq \mathscr{F}$, and let $L^1_\sigma(\mathscr{G})$ be the space of functions that are integrable over all $G \in \mathscr{G}_0$.

Definition 2.9. Let $\mathscr{G}$ be a σ-finite sub-σ-algebra of $\mathscr{F}$ and let $f\colon X \to E$ be an $\mathscr{F}$-measurable function where E is a Banach space. Then a $\mathscr{G}$-measurable function g is a *conditional expectation of f with respect to $\mathscr{G}$* if

$$\int_G f\,d\mu = \int_G d\mu \qquad \text{for every } G \in \mathscr{G}_0.$$

It is not difficult to prove that if the conditional expectation exists, it is unique a.e. Thus, we denote $\mathbb{E}[f|\mathscr{G}] := g$ if g is a conditional expectation of f with respect to $\mathscr{G}$. Concerning existence, we only need the following elementary case.

LEMMA 2.10. *Let $\mathcal{A} := \{A_i : i \in \mathbb{N}\} \subseteq \mathscr{F}$ be a countable partition of the space X such that $\mu(A_i) < \infty$ for every $i \in \mathbb{N}$ and let $\mathscr{A}$ be the σ-algebra generated by $\mathcal{A}$. Then for every $f \in L^1_\sigma(\mathscr{F})$ we have*

$$\mathbb{E}[f|\mathscr{A}] = \sum_{A\in\mathcal{A}} 1_A \langle f\rangle_A.$$

Proof. Let $G \in \mathscr{A}_0$. Then there exist pairwise disjoint sets $A_1^G, A_2^G, \ldots$ in $\mathcal{A}$ such that $G = \bigcup_i A_i^G$. Now

$$\begin{aligned}\int_G f\,d\mu &= \sum_i \int_{A_i^G} \Big(\fint_{A_i^G} f\,d\mu\Big)\,d\mu = \int_G \sum_i 1_{A_i^G} \Big(\fint_{A_i^G} f\,d\mu\Big)\,d\mu \\ &= \int_G \Big(\sum_{A\in\mathcal{A}} 1_A \fint_A f\,d\mu\Big)\,d\mu,\end{aligned}$$

which proves the claim. ∎

2.4. UMD spaces; type and cotype of Banach spaces. Suppose that $(X, d, \mathscr{F}, \mu)$ is a metric measure space and let $(\mathscr{F}_k)$, $k = 0, 1, \ldots, N$, be a sequence of sub-σ-algebras of $\mathscr{F}$ such that $\mathscr{F}_k \subseteq \mathscr{F}_{k+1}$ for all k. For simplicity, let

$$\|\cdot\|_p := \|\cdot\|_{L^p(X;E)}$$

where $\|\cdot\|_{L^p(X;E)}$ is the L^p-Bochner norm.

DEFINITION 2.11. A sequence $(d_k)_{k=1}^N$ of functions is a *martingale difference sequence* if d_k is $\mathscr{F}_k$-measurable and $\mathbb{E}[d_k|\mathscr{F}_{k-1}] = 0$ for every k.

DEFINITION 2.12. A Banach space $(E, \|\cdot\|_E)$ is a *UMD* (*unconditional martingale difference*) *space* if for every $p \in (1,\infty)$ there exists a constant β_p such that

$$\Big\|\sum_{i=1}^N \varepsilon_i d_i\Big\|_p \le \beta_p \Big\|\sum_{i=1}^N d_i\Big\|_p$$

for all E-valued L^p-martingale difference sequences $(d_i)_{i=1}^N$ (i.e. (d_i) is a martingale difference sequence such that $d_i \in L^p(X, \mathscr{F}_i; E)$ for every i) and all choices of signs $(\varepsilon_i)_{i=1}^N \in \{-1,+1\}^N$.

UMD spaces are crucial in Banach-space-valued harmonic analysis due to their many good properties; for example, a Banach space E is UMD if and only if the Hilbert transform is bounded on $L^p(\mathbb{R};E)$ [4, 3]. They give us a natural setting for analysis that is based on techniques used in probability spaces in the following way. Let (d_i) be a martingale difference sequence and let (ε_i) be a sequence of *random signs*, i.e. independent random variables on some probability space $(\Omega,\mathbb{P})$, with distribution $\mathbb{P}(\varepsilon_i=-1)=\mathbb{P}(\varepsilon_i=+1)=1/2$. Then for every $\eta\in\Omega$ the sequence $(\varepsilon_i(\eta)d_i)$ is a martingale difference sequence. In particular, the UMD property gives us

$$\Big\|\sum_{i=1}^N d_i\Big\|_p \eqsim_E \Big(\int_\Omega \Big\|\sum_{i=1}^N \varepsilon_i(\eta)d_i\Big\|_p^p\,d\mathbb{P}(\eta)\Big)^{1/p} =: \Big\|\sum_{i=1}^N \varepsilon_i d_i\Big\|_{\Omega,p} \tag{2.13}$$

for every $p\in(1,\infty)$.

The following inequality by J. Bourgain is a standard tool in UMD-valued analysis. Its original scalar-valued version was due to E. Stein.

THEOREM 2.14 (see e.g. [6, Proposition 3.8]). *Let (f_k) be a sequence of functions in $L^p(X,\mathscr{F};E)$ and $(\mathscr{F}_k)$ a sequence of σ-finite σ-algebras such that $\mathscr{F}_k\subseteq\mathscr{F}_{k+1}\subseteq\mathscr{F}$ for every $k\in\mathbb{N}$. Then for any sequence (ε_k) of random signs we have*

$$\Big\|\sum_k \varepsilon_k\mathbb{E}[f_k|\mathscr{F}_k]\Big\|_{\Omega,p} \lesssim_{p,\beta_p} \Big\|\sum_k \varepsilon_k f_k\Big\|_{\Omega,p}.$$

In our proofs we also need the following version of the well-known principle of contraction by J.-P. Kahane. It holds in all Banach spaces.

THEOREM 2.15 ([19, Section 2.6, Theorem 5]). *Suppose that (ε_i) is a sequence of random signs and the series $\sum_i\varepsilon_i x_i$ converges in E almost surely. Then for any bounded sequence (c_i) of scalars the series $\sum_i\varepsilon_i c_i x_i$ converges in E almost surely and*

$$\int_\Omega\Big\|\sum_i\varepsilon_i c_i x_i\Big\|_E^p\,d\mathbb{P} \le \Big(\sup_i|c_i|\Big)^p\int_\Omega\Big\|\sum_i\varepsilon_i x_i\Big\|_E^p\,d\mathbb{P}.$$

2.4.1. *Type and cotype of Banach spaces*

DEFINITION 2.16. Let $(E,\|\cdot\|)$ be a Banach space. We say that E has *type* $t\in[1,2]$ if there exists a constant $C_t>0$ such that for every finite sequence (x_i) in E and finite sequence (ε_i) of random signs we have

$$\int_\Omega\Big\|\sum_i\varepsilon_i x_i\Big\|_E\,d\mathbb{P} \le C_t\Big(\sum_i\|x_i\|^t\Big)^{1/t}.$$

In a similar fashion, we say that E has *cotype* $q\in[2,\infty]$ if there exists a constant $C_q > 0$ such that

$$\Big(\sum_i \|x_i\|^q\Big)^{1/q} \le C_q \int_\Omega \Big\|\sum_i \varepsilon_i x_i\Big\|_E \, d\mathbb{P}.$$

The notion of type and cotype of Banach spaces was introduced by B. Maurey and G. Pisier in the 1970's and it has become an important part of analysis on Banach spaces. From this rich theory, we need the following results:

(i) If Y is a σ-finite measure space and E is a Banach space of type r and cotype s, then $L^p(X;E)$ has type $\min\{p,r\}$ and cotype $\max\{p,s\}$.
(ii) If E is a UMD space, then E has a non-trivial type $s > 1$ and non-trivial cotype $t < \infty$.

For proofs, see e.g. [22, Chapter 9] for (i), and [2, Theorem 11.1.14], [26, Proposition 3] for (ii).

2.5. Structural constants. We say that c is a *structural constant* if it depends only on the doubling constant D, the UMD constant β_p for a fixed $p \in (1,\infty)$ and the type and cotype constants C_t and C_q. We do not track the dependences of our bounds on the structural constants, and thus we use the notation $a \lesssim b$ if $a \le cb$ for some structural constant c, and $a \eqsim b$ if $a \lesssim b \lesssim a$.

3. Embedding cubes into larger cubes. In this section we prove a decomposition result for dyadic systems using Theorem 2.6. We formulate the result in such a way that it is easy to apply it in Section 4, but we note that it is easy to modify the proof for other similar decompositions.

Let $\mathscr{D}$ be a dyadic system with $\delta < 1/(2\cdot 168M^8)$ and $\{\mathscr{D}(\omega)\}_\omega$ be adjacent dyadic systems for the same δ given by Theorem 2.6. Fix $m \ge 1$ and an injective function $\tau\colon \mathscr{D} \to \mathscr{D}$ such that $\tau(Q) \subseteq mB_Q$ for every $Q \in \mathscr{D}$, and $\tau\mathscr{D}^k \subseteq \mathscr{D}^k$ for every $k \in \mathbb{Z}$.

PROPOSITION 3.1. *The system $\mathscr{D}$ is a disjoint union of a bounded number of subcollections $\mathscr{D}_\lambda \subseteq \mathscr{D}$, $\lambda = (i,j,\omega)$, with the following property: for every $Q \in \mathscr{D}_\lambda$ there exist cubes $P_Q, P_{\tau(Q)} \in \mathscr{D}(\omega)^{k-3}$ and $P_Q^* \in \mathscr{D}(\omega)^{k-3-T}$, where $2m\delta^T \le 1$, such that*

$$Q \subseteq P_Q, \quad \tau(Q) \subseteq P_{\tau(Q)}, \quad P_Q \cup P_{\tau(Q)} \cup 2mB_Q \subseteq P_Q^*; \tag{3.2}$$

$$\text{if } Q_1, Q_2 \in \mathscr{D}_\lambda \cap \mathscr{D}^k, \text{ then } (P_{Q_1} \cup P_{\tau(Q_1)}) \cap (P_{Q_2} \cup P_{\tau(Q_2)}) = \emptyset; \tag{3.3}$$

$$\text{if } Q_1, Q_2 \in \mathscr{D}_\lambda,\ Q_1 \subsetneq Q_2, \text{ then } P_{Q_1}^* \subseteq P_{Q_2}. \tag{3.4}$$

In other words, we split the collection $\mathscr{D}$ into sparse subcollections $\mathscr{D}_\lambda$ such that we can embed every cube $Q \in \mathscr{D}_\lambda$ and its image $\tau(Q)$ into some larger cubes P_Q and $P_{\tau(Q)}$ such that P_Q and $P_{\tau(Q)}$ belong to the same dyadic system and they have a mutual dyadic ancestor P_Q^*.

We form the sets $\mathscr{D}_\lambda$ with the help of the next technical lemma.

Lemma 3.5. *The collection $\mathscr{D}$ is a disjoint union of $L = L(X)$ subcollections $\mathscr{Q}_i$ such that for every $k \in \mathbb{Z}$ and $Q_1, Q_2 \in \mathscr{Q}_i \cap \mathscr{D}^k$ we have*

$$3\delta^{-3} B_{R_1} \cap 3\delta^{-3} B_{R_2} = \emptyset$$

where $R_1 \in \{Q_1, \tau(Q_1)\}$ and $R_2 \in \{Q_2, \tau(Q_2)\}$, $R_1 \neq R_2$, and the number L is independent of m.

Proof. We only need to use basic properties of geometrically doubling metric spaces and the observation that if $Q, P \in \mathscr{D}^k$ and $d(x(Q), x(P)) \geq 12\delta^{k-3}$, then $3\delta^{-3}B_Q \cap 3\delta^{-3}B_P = \emptyset$.

Let $k \in \mathbb{Z}$ be fixed. For any subcollection $\mathscr{Q} \subseteq \mathscr{D}^k$ and any set A of centre points of cubes, set

$$Y_{\mathscr{Q}} := \{x(Q) : Q \in \mathscr{Q}\}, \qquad \mathscr{D}_A := \{Q \in \mathscr{D} : x(Q) \in A\}.$$

We split $Y_{\mathscr{D}^k}$ into smaller sets in three steps. To keep our notation simple, i is an index whose role may change from one occurrence to the next.

(1) By Lemma 2.3, we can split the δ^k-separated set $Y_{\mathscr{D}^k}$ into a bounded number of $12\delta^{k-3}$-separated subsets $Y^1_{i,k}$.
(2) For every $Q \in \mathscr{D}_{Y^1_{i,k}}$, the ball $3\delta^{-3}B_Q$ intersects at most a bounded number of balls $3\delta^{-3}B_{\tau(P)}$ where $P \in \mathscr{D}_{Y^1_{i,k}}$. Thus, we can split the set $Y^1_{i,k}$ into a bounded number of subsets $Y^2_{i,k}$ such that we have $3\delta^{-3}B_Q \cap 3\delta^{-3}B_{\tau(P)} = \emptyset$ for all $Q, P \in \mathscr{D}_{Y^2_{i,k}}$, $Q \neq \tau(P)$.
(3) For every $Q \in \mathscr{D}_{Y^2_{i,k}}$, the ball $3\delta^{-3}B_{\tau(Q)}$ intersects at most a bounded number of balls $3\delta^{-3}B_{\tau(P)}$, $P \in \mathscr{D}_{Y^2_{i,k}}$. In particular, we can split the set $Y^2_{i,k}$ into a bounded number of subsets $Y^3_{i,k}$ such that we have $3\delta^{-3}B_{\tau(Q)} \cap 3\delta^{-3}B_{\tau(P)} = \emptyset$ for any $Q, P \in \mathscr{D}_{Y^3_{i,k}}$, $Q \neq P$.

Now we set $\mathscr{Q}_i := \bigcup_{k\in\mathbb{Z}} \mathscr{D}_{Y^3_{i,k}}$ for every i. ∎

Let $\{\mathscr{Q}_i\}_i$ be the partition of $\mathscr{D}$ given by the previous lemma and let $T \in \mathbb{N}$, $T \geq 1$, be the smallest number such that

$$2m\delta^T \leq 1.$$

Recall Theorem 2.6 and denote

$$\gamma(R) := \min\{\omega : Q_{B_R}, Q_{B_{\tau(R)}} \in \mathscr{D}(\omega),\ \delta^{-T} B_R \subseteq Q^{(T)}_{B_R}\}$$

for every cube $R \in \mathscr{D}$ and

$$\mathscr{Q}_{i,\omega} := \{R \in \mathscr{Q}_i : \gamma(R) = \omega\}$$

for $i = 1, \dots, L$ and $\omega = 1, \dots, K$. Then the collections $\mathscr{Q}_{i,\omega}$ satisfy properties (3.2) and (3.3) but they are still not suitable for property (3.4). Thus, we split the collections $Q_{i,\omega}$ into smaller ones whose cubes have large enough generation gaps: we set

$$\mathscr{D}_{i,j,\omega} := \bigcup_{k \in \mathbb{Z}} (\mathscr{Q}_{i,\omega} \cap \mathscr{D}^{j+4kT})$$

for every $j = 0, 1, \dots, 4T-1$. Notice that i, j and ω are independent of each other.

Proof of Proposition 3.1. Clearly we only need to show the claim for the collections $\mathscr{D}_{i,0,\omega} =: \mathscr{D}_i$.

Notice first that

$$2m \cdot r(B_Q) = 6m\delta^{4kT} \le \delta^{-T} 3\delta^{4kT} = \delta^{-T} \cdot r(B_Q)$$

for every $Q := Q_\alpha^{4kT} \in \mathscr{D}_i$. Thus, by Remark 2.8 and the definition of $\mathscr{D}_i$, for every cube $Q \in \mathscr{D}_i$ there exist cubes $P_Q, P_{\tau(Q)} \in \mathscr{D}(\omega)^{4kT-3}$ such that

$$B_Q \subseteq P_Q, \quad B_{\tau(Q)} \subseteq P_{\tau(Q)}, \quad 2mB_Q \subseteq P_Q^{(T)} =: P_Q^*.$$

Let us show that the cubes P_Q, $P_{\tau(Q)}$ and P_Q^* have properties (3.2)–(3.4).

For (3.2), since $Q, \tau(Q) \subseteq 2mB_Q$, we know that $P_Q \cap P_Q^* \neq \emptyset$ and $P_{\tau(Q)} \cap P_Q^* \neq \emptyset$. Thus, as $\mathscr{D}(\omega)$ is a dyadic system and $\operatorname{lev}(P_Q^*) < \operatorname{lev}(P_Q) = \operatorname{lev}(P_{\tau(Q)})$, we have $P_Q \cup P_{\tau(Q)} \subseteq P_Q^*$.

For (3.3), since $x(Q) \in P_Q$ for every cube $Q \in \mathscr{D}$, we have

$$P_Q \subseteq B(x(P_Q), 3\delta^{4kT-3}) \subseteq B(x(Q), 6\delta^{4kT-3}) = 2\delta^{-3} B_Q$$

for every $Q \in \mathscr{D}$. Thus, (3.3) follows directly from Lemma 3.5.

For (3.4) suppose that $R \subsetneq Q := Q_\alpha^{4kT}$. Then $\operatorname{lev}(R) \ge (4k+4)T$ and thus $\operatorname{lev}(P_R) \ge (4k+4)T - 3$ and

$$\operatorname{lev}(P_R^*) \ge (4k+4)T - 3 - T \ge 4kT = \operatorname{lev}(Q) \ge \operatorname{lev}(P_Q)$$

since $T \ge 1$. In particular, $P_R^* \subseteq P_Q$ since $P_R^*, P_Q \in \mathscr{D}(\omega)$ and $\mathscr{D}(\omega)$ is a dyadic system. ∎

4. L^p-boundedness of shift operators. In this section, we show that with the help of Proposition 3.1 we can give a straightforward proof for the L^p-boundedness of the shift operators in doubling metric measure spaces. We follow some ideas of [8] and [21] but mostly we rely on our own dyadic constructions.

Let (X, d, μ) be a metric measure space, where μ is a doubling Borel measure on X, and let $(E, \|\cdot\|)$ be a UMD space. Since the doubling property of μ implies the geometrical doubling property of d, there exists a finite geometrical doubling constant M. Thus, we may fix a dyadic system $\mathscr{D}$ for

$\delta < 1/(2 \cdot 168 M^8)$ and adjacent dyadic systems $\{\mathscr{D}(\omega)\}_\omega$ given by Theorem 2.5 for the same δ.

4.1. Haar functions. There are various ways to construct Haar functions in metric spaces (see e.g. [1, Section 5]), and thus we do not want to fix any particular construction. We do, however, refer to the construction in [14, Section 4] (with the choice $b \equiv 1$) for a system of Haar functions that satisfy the properties in the following definition. In [14] the construction is done in $\mathbb{R}^n$ for a non-doubling measure but it is easy to generalize the result to our setting.

Definition 4.1. A collection of functions $h_Q^\theta \colon X \to \mathbb{R}$, where $Q := Q_\alpha^k \in \mathscr{D}$ and $\theta = 1, \ldots, n(Q) \leq \Theta$, is a *system of Haar functions* if it has the following properties: for all Q and θ we have

- $\operatorname{supp} h_Q^\theta \subseteq Q$;
- h_Q^θ is constant on every child cube $Q_\beta^{k+1} \subseteq Q$;
- $\int h_Q^\theta = 0 = \int h_Q^\theta h_Q^{\theta'}$ if $\theta \neq \theta'$;
- $\|h_Q^\theta\|_2 = 1$;

and the space of finite linear combinations of the functions h_Q^θ is dense in $L^2(X; E)$.

The number Θ above depends only on M or, more precisely, on the maximum number of children Q_β^{k+1} a cube Q_α^k can have. Henceforth, we fix some $\theta = \theta(Q)$ for each $Q \in \mathscr{D}$ and drop the dependence on θ in notation.

Let $h_Q = \sum_k v_k 1_{Q_k}$ be a Haar function, where Q_k are the children of Q. The following properties are straightforward consequences of the above definition:

$$\|h_Q\|_\infty = \max |v_k| \eqsim \frac{1}{\mu(Q)^{1/2}}, \qquad \|h_Q\|_1 \eqsim \mu(Q)^{1/2}.$$

In particular,

$$\frac{1_{Q_k}(x)}{\mu(Q_k)^{1/2}} \lesssim |h_Q(x)| \lesssim \frac{1_Q(x)}{\mu(Q)^{1/2}} \quad \text{for every } x \in Q \text{ and some } Q_k. \tag{4.2}$$

The previous properties give us the following lemma:

Lemma 4.3. *For every $p \in (1, \infty)$ and any finite collection of cubes Q we have*

$$\Big\| \sum_Q x_Q h_Q \Big\|_p \eqsim \Big\| \sum_Q \varepsilon_Q x_Q \frac{1_Q}{\mu(Q)^{1/2}} \Big\|_{\Omega, p}.$$

Proof. Let $\sum_Q x_Q h_Q = \sum_k \sum_\alpha x_{Q_\alpha^k} h_{Q_\alpha^k}$ and let (ε_Q) be a sequence of random signs. Then for all $y \in X$ and $k \in \mathbb{Z}$ there exists at most one $Q_{\alpha,y}^k$ such that $h_{Q_{\alpha,y}^k}(y) \neq 0$. For any $y \in X$ and $k \in \mathbb{Z}$, let $\sigma_k^y \in \{-1, +1\}$

be such that $\sigma_k^y h_{Q^k_{\alpha,y}}(y) = |h_{Q^k_{\alpha,y}}(y)|$. Then, for a fixed $y \in X$, $(\sigma_k^y \varepsilon_{Q^k_{\alpha,y}})_k$ is a sequence of random signs. Since the functions h_Q form a martingale difference sequence and by (4.2) we know that $|h_Q|\mu(Q)^{1/2} \lesssim 1$ for every Q, we have

$$\begin{aligned}\Big\|\sum_Q x_Q h_Q\Big\|_p^p &\eqsim \int_X\int_\Omega \Big\|\sum_k \sigma_k^y \varepsilon_{Q^k_{\alpha,y}}(\eta) x_{Q^k_{\alpha,y}} h_{Q^k_{\alpha,y}}(y)\Big\|_E^p \, d\mathbb{P}(\eta)\, d\mu(y)\\ &= \int_X\int_\Omega \Big\|\sum_k \varepsilon_{Q^k_{\alpha,y}}(\eta) x_{Q^k_{\alpha,y}} \frac{|h_{Q^k_{\alpha,y}}(y)|\mu(Q^k_{\alpha,y})^{1/2}}{\mu(Q^k_{\alpha,y})^{1/2}}\Big\|_E^p \, d\mathbb{P}(\eta)\, d\mu(y)\\ &\lesssim \Big\|\sum_Q \varepsilon_Q x_Q \frac{1_Q}{\mu(Q)^{1/2}}\Big\|_{\Omega,p}^p,\end{aligned}$$

by the UMD property of E, Fubini's theorem and Kahane's contraction principle. Write $\sum_Q x_Q h_Q = \sum_{i=1}^N x_i h_{Q_i}$ where the cubes Q_i satisfy $\operatorname{lev}(Q_1) \le \operatorname{lev}(Q_2) \le \cdots \le \operatorname{lev}(Q_N)$. Then, by Lemma 2.10, we have $\mathbb{E}[|h_{Q_i}| \mid \mathscr{F}_i] = 1_Q\langle |h_Q|\rangle_Q$ where $\mathscr{F}_i$ is the σ-algebra generated by $\mathscr{D}^{\operatorname{lev}(Q_i)}$. Thus, since $1/(\mu(Q)^{1/2}\langle |h_Q|\rangle_Q) \eqsim 1$, the previous estimates, Stein's inequality and Kahane's contraction principle (in this order) give

$$\begin{aligned}\Big\|\sum_{i=1}^N x_i h_{Q_i}\Big\|_p^p \eqsim \Big\|\sum_{i=1}^N \varepsilon_i x_i |h_{Q_i}|\Big\|_{\Omega,p}^p &\gtrsim \Big\|\sum_{i=1}^N \varepsilon_i x_i \mathbb{E}[|h_{Q_i}| \mid \mathscr{F}_i]\Big\|_{\Omega,p}^p\\ &= \Big\|\sum_i^N \varepsilon_i x_i \frac{1_{Q_i}}{\mu(Q_i)^{1/2}} \mu(Q_i)^{1/2}\langle |h_{Q_i}|\rangle_{Q_i}\Big\|_{\Omega,p}^p\\ &\gtrsim \Big\|\sum_i^N \varepsilon_i x_i \frac{1_{Q_i}}{\mu(Q_i)^{1/2}}\Big\|_{\Omega,p}^p,\end{aligned}$$

which proves the claim. ■

4.2. Shift operators. Fix $m \ge 1$ and let $\tau\colon \mathscr{D} \to \mathscr{D}$ be an injective function such that

(1) $\tau\mathscr{D}^k \subseteq \mathscr{D}^k$ for every $k \in \mathbb{Z}$;
(2) for every $Q \in \mathscr{D}$ we have $\tau(Q) \subseteq mB_Q$;
(3) the measures of the cubes Q and $\tau(Q)$ are approximately the same:

$$\mu(Q) \eqsim \mu(\tau(Q)). \tag{4.4}$$

Let $\{h_Q\}_{Q\in\mathscr{D}}$ be a system of Haar functions. Then we can define the shift operator $T := T_\tau$ as the linear extension of the operator $\hat{T}$:

$$\hat{T}h_Q = h_{\tau(Q)}.$$

It is easy to see that without condition (4.4) an estimate of the type (1.1) is out of reach for all $p \in (1,\infty)$. More precisely: by property (4.2) we have

$\|h_Q\|_p \eqsim \mu(Q)^{1/p-1/2}$ for every cube Q, and thus without condition (4.4) the estimate cannot hold simultaneously for all $p \in (1,2]$ and all $q \in (2,\infty)$. We note that (4.4) is automatically valid in metric measure spaces that satisfy an Ahlfors-regularity-type condition.

4.3. L^p-boundedness of shift operators. With the help of Proposition 3.1 and Lemma 4.3, we can now prove the following theorem quite easily.

THEOREM 4.5. *Let $p \in (1,\infty)$ and $f \in L^p(X;E)$. Then*

$$\|Tf\|_p \leq C(\log(2m)+1)^\alpha \|f\|_p$$

where $C = C(p,X,E,\alpha)$, $\alpha = 1/\min\{t_E,p\} - 1/\max\{q_E,p\} < 1$, and t_E and q_E are the type and cotype of the space E.

Proof. By the properties of the Haar functions and Proposition 3.1, we may assume that

$$f = \sum_{i=1}^{L} \sum_{j=0}^{4T-1} \sum_{\omega=1}^{K} \sum_{Q \in \mathscr{D}_{i,j,\omega}} x_Q h_Q$$

where $x_Q \neq 0$ only for finitely many Q. Thus, for simplicity, we denote $f = \sum_{i,j,\omega} \sum_{k=1}^{n} x_k h_{Q_k}$ where $\operatorname{lev}(Q_1) \leq \cdots \leq \operatorname{lev}(Q_n)$.

For every $k = 1,\ldots,n$, let $\mathscr{F}_k$ be the σ-algebra generated by

$$F_k := \Big(\mathscr{D}(\omega)^{\operatorname{lev}(Q_k)-3} \setminus \bigcup_{\substack{l=1,\ldots,n\\ \operatorname{lev}(Q_l)=\operatorname{lev}(Q_k)}} \{P_{Q_l}, P_{\tau(Q_l)}\}\Big) \cup \bigcup_{\substack{l=1,\ldots,n\\ \operatorname{lev}(Q_l)=\operatorname{lev}(Q_k)}} \{P^{Q_l}\},$$

where $P^{Q_l} := P_{Q_l} \cup P_{\tau(Q_l)}$. Notice that if $\operatorname{lev}(Q_{k_1}) = \operatorname{lev}(Q_{k_2})$, then $F_{k_1} = F_{k_2}$. By (3.3) we know that F_k is a partition of the space X, and by (3.4) the sequence $(\mathscr{F}_k)$ is nested. Thus, for every $k = 1,\ldots,n$,

$$\mathbb{E}[1_{Q_k}|\mathscr{F}_k] \overset{2.10}{=} 1_{P^{Q_k}} \langle 1_{Q_k} \rangle_{P^{Q_k}} \overset{(4.4)}{\eqsim} 1_{P^{Q_k}} \frac{\mu(Q_k)}{\mu(P_{Q_k})} \eqsim 1_{P^{Q_k}}. \tag{4.6}$$

In particular,

$$\begin{aligned}
\Big\|\sum_k x_k h_{\tau(Q_k)}\Big\|_p &\overset{4.3}{\eqsim} \Big\|\sum_k \varepsilon_k \frac{x_k}{\mu(\tau(Q_k))^{1/2}} 1_{\tau(Q_k)}\Big\|_{\Omega,p} \\
&\overset{\substack{2.15\\(4.4)}}{\lesssim} \Big\|\sum_k \varepsilon_k \frac{x_k}{\mu(Q_k)^{1/2}} 1_{P^{Q_k}}\Big\|_{\Omega,p} \\
&\overset{(4.6)}{\eqsim} \Big\|\sum_k \varepsilon_k \frac{x_k}{\mu(Q_k)^{1/2}} \mathbb{E}[1_{Q_k}|\mathscr{F}_k]\Big\|_{\Omega,p} \\
&\overset{2.14}{\lesssim} \Big\|\sum_k \varepsilon_k \frac{x_{Q_k}}{\mu(Q_k)^{1/2}} 1_{Q_k}\Big\|_{\Omega,p}.
\end{aligned}$$

Hence, since by Section 2.4.1 the space $L^p(X;E)$ has a non-trivial type $t>1$ and a non-trivial cotype $q<\infty$, we have

$$\begin{aligned}\Big\|\sum_{i,j,\omega}\sum_k x_k h_{\tau(Q_k)}\Big\|_p &\lesssim \bigg(\sum_{i,j,\omega}\Big\|\sum_k \varepsilon_k \frac{x_{Q_k}}{\mu(Q_k)^{1/2}}1_{Q_k}\Big\|_{\Omega,p}^t\bigg)^{1/t}\\ &\le (4TKL)^{1/t-1/q}\bigg(\sum_{\omega,i,j}\Big\|\sum_k \varepsilon_k \frac{x_{Q_k}}{\mu(Q_k)^{1/2}}1_{Q_k}\Big\|_{\Omega,p}^q\bigg)^{1/q}\\ &\lesssim T^{1/t-1/q}\Big\|\sum_{\omega,i,j}\sum_k \varepsilon_k \frac{x_{Q_k}}{\mu(Q_k)^{1/2}}1_{Q_k}\Big\|_{\Omega,p}\\ &\overset{4.3}{\lesssim} (\log(2m)+1)^{1/t-1/q}\|f\|_p\end{aligned}$$

by Hölder's inequality. ■

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