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RD-INJECTIVITY OF TENSOR PRODUCTS OF MODULES

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Abstract. A classical question due to Yoneda is, "When is the tensor product of any two injective modules injective?" Enochs and Jenda gave a complete and explicit answer to this question in 1991. Since *RD*-injective modules are a generalization of injective modules, it is natural to ask whether the tensor product of any two *RD*-injective modules is *RD*-injective. In this paper we deal with this question.

1. Introduction. The notion of purity has a substantial role in module theory and also in model theory. There are several variants of this notion (see [2], [3], [5], [8], [20], [22] and [23]). For example, the concept of *RD*-purity (or relative divisible-purity), with the related notions of *RD*-injective, *RD*-projective and *RD*-flat module, was introduced by Warfield [22] in 1969. They have been an object of deep study in the past forty years. Apart from the pioneering work of Warfield [22, 23], let us recall here the studies of Facchini [8], Puninski [20, 21], and, in the commutative case, Couchot [5].

Yoneda raised the question of when the tensor product of any two injective modules is injective. Ishikawa [15] proved that for a commutative Noetherian ring R, if the injective envelope $E(R)$ is flat, then the tensor product of any two injective R-modules is injective. Finally, a complete answer to the question of Yoneda was given by Enochs and Jenda [7, Theorem 2.4]. They proved that, over a commutative Noetherian ring R, the injective envelope $E(R)$ is flat if and only if the tensor product of any two injective R-modules is injective. Also, they showed that R is Gorenstein if and only if the torsion product of any two injective R-modules is injective. Pournaki et al. [18] have studied the analogous question for pure-injective modules. Since *RD*-injective modules are a generalization of injective modules, it is natural to ask when the tensor product of two *RD*-injective modules is *RD*-injective. In this paper we deal with this question.

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There are examples in the literature showing that the tensor product of two RD-injective modules need not be RD-injective in general. In this paper, we obtain some conditions which guarantee that it is. In this direction, first we show that if R is a Σ-RD-injective module such that the tensor product of any two RD-injective R-modules is RD-injective, then R is a quasi-Frobenius ring and each direct sum of RD-injective R-modules is RD-injective (Theorem 2.1). As a consequence, R is then a finite product of pure-semisimple rings (i.e., every R-module is a direct sum of finitely generated modules) or finite rings (Corollary 2.2).

In Theorem 2.3, it is shown that if R is a Σ-RD-injective module, then conditions (1)–(4) below are equivalent and imply condition (5), and when R is either a Bézout ring or a non-finite local ring, the following five conditions are equivalent:

(1) R is a pure-semisimple ring;
(2) every R-module is RD-injective;
(3) every R-module is pure-injective;
(4) the tensor product of two pure-injective R-modules is pure-injective; and
(5) the tensor product of two RD-injective R-modules is RD-injective.

We provide an example of an Artinian ring R for which the tensor product of two Artinian R-modules may not be RD-injective (Example 2.6). In this regard, it is shown that if every simple R-module is RD-injective, then the tensor product of any two Artinian R-modules is RD-injective. The converse is also true when R is an Artinian ring (Theorem 2.7). As a consequence, if R is a von Neumann regular ring, then the tensor product of any two Artinian R-modules is injective (Corollary 2.8).

We show that for a semihereditary ring R, the tensor product of a finitely presented R-module and an RD-injective R-module is RD-injective (Proposition 2.9). Moreover, for a semiprime Goldie ring R, the tensor product of a finitely presented torsion-free p-injective R-module and an RD-injective R-module is a torsion-free p-injective pure-injective R-module (Theorem 2.10).

Hattori [13] proved that the tensor product of any two injective R-modules is injective when R is a domain. As an analogue, we show that if R is either a domain or a semiheriditary semiprime Goldie ring, then the tensor product of a finitely presented torsion-free p-injective R-module and an RD-injective R-module is injective (Proposition 2.11). Moreover, a semiprime Goldie ring R is semisimple if and only if the tensor product of an RD-injective R-module and a finitely generated projective R-module is p-injective (Proposition 2.12). Also, over an integral domain R, the tensor product of an FP-injective R-module and an injective flat R-module is FP-injective (Proposition 2.13). Consequently, over an integral domain R, the tensor product of a finitely presented FP-injective R-module and an injective flat R-module is injective (Corollary 2.14). Finally, if every cyclic R-module is RD-injective, then the tensor product of any two finitely generated RD-projective R-modules is RD-injective (Proposition 2.15).

Throughout the paper, R will denote a commutative ring with identity and all modules will be assumed to be unitary. Recall that a ring R is called *von Neumann regular* if every finitely generated ideal of R is generated by an idempotent. A ring R is *local* in case R has a unique maximal ideal. A *semiprime ideal* in a ring R is any ideal of R which is an intersection of prime ideals. A *semiprime ring* is any ring in which 0 is a semiprime ideal. A ring R is said to be *Goldie* if R has finite uniform dimension and satisfies the ascending chain condition on its annihilators. For an R-module M, we denote by $E(M)$ the injective envelope of M.

2. Results. Recall that an exact sequence of left R-modules $0 \to A \to B \to C \to 0$ is *pure exact* if it remains exact when tensoring it with any right R-module. In this case we say that A is a *pure submodule* of B. When $rB \cap A = rA$ for every $r \in R$, we say that A is an *RD-submodule* of B (relatively divisible) and that the sequence is *RD-exact*.

An R-module M is said to be *pure-injective* (resp. *RD-injective*) if M has the injective property with respect to each pure exact sequence (resp. RD-exact sequence). An R-module M is *pure-projective* (resp. *RD-projective*) if M has the projective property with respect to each pure exact sequence (resp. RD-exact sequence). Also, a left R-module M is *Σ-pure-injective* (resp. *Σ-RD-injective*) if $M^{(I)}$ is pure-injective (resp. RD-injective) for each index set I. *Pure-essential extension*, *RD-essential extension*, *pure-injective envelope*, and *RD-injective envelope* are defined as in Warfield [22]. For every R-module M, we denote its pure-injective envelope (resp. RD-injective envelope) by $PE(M)$ (resp. $RDE(M)$).

A left R-module A is said to be *RD-coflat* if every RD-exact sequence $0 \to A \to B \to C \to 0$ of left R-modules is pure exact, a notion defined by Couchot [5]. Thus every RD-injective left R-module is RD-coflat.

Recall that a ring R is called *quasi-Frobenius* if R is Artinian and self-injective. Also, a ring R is *perfect* in case R satisfies the descending chain condition on principal ideals.

THEOREM 2.1. *If R is a Σ-RD-injective module such that the tensor product of any two RD-injective R-modules is RD-injective, then R is a quasi-Frobenius ring and each direct sum of RD-injective R-modules is RD-injective.*

Proof. Assume R is a Σ-RD-injective module and M an RD-injective R-module. Thus $R^{(I)}$ is an RD-injective R-module for each index set I.

Also, since the tensor product of any two *RD*-injective *R*-modules is *RD*-injective, $R^{(I)} \otimes_R M$ is *RD*-injective. Thus $M^{(I)}$ is an *RD*-injective *R*-module, since $R^{(I)} \otimes_R M \cong M^{(I)}$. Therefore, every *RD*-injective *R*-module is Σ-*RD*-injective. This implies that every *RD*-injective *R*-module is Σ-pure-injective, since every *RD*-injective is pure injective.

Now, let N be an *RD*-coflat *R*-module. Thus $RDE(N)$ is Σ-pure-injective. Also, N is a pure submodule of the Σ-pure-injective *R*-module $RDE(N)$. Thus, by [14, Corollary 8], N is a direct summand of $RDE(N)$, and so N is *RD*-injective. Hence every *RD*-coflat *R*-module is *RD*-injective, and so the *RD*-injective *R*-modules and the *RD*-coflat *R*-modules coincide. Therefore, each direct sum of *RD*-injective *R*-modules is *RD*-injective by [5, Proposition I.3].

So, if $\{E_i\}_{i\in I}$ is a family of injective *R*-modules, then $\bigoplus_{i\in I} E_i$ is *RD*-injective. Also, clearly $\bigoplus_{i\in I} E_i$ is *RD*-coflat. It follows that $\bigoplus_{i\in I} E_i$ is injective, since it is a pure submodule of $\prod_{i\in I} E_i$. Therefore, R is Noetherian by Bass's Theorem. Moreover, we know that any Σ-pure-injective ring is semiprimary (i.e., $R/J(R)$ is semisimple and $J(R)$ is nilpotent). Thus R is also perfect, and so it is an Artinian ring. Therefore, [5, Proposition I.2] allows us to conclude. ∎

Recall that a ring R is called *pure-semisimple* if every *R*-module is a direct sum of finitely generated modules, or equivalently, if every *R*-module is pure-injective.

From Theorem 2.1, [5, Theorem II.1], and [12, Theorem 4.3], we have:

COROLLARY 2.2. *If R is a Σ-RD-injective module such that the tensor product of any two RD-injective R-modules is RD-injective, then R is a finite product of pure-semisimple rings or finite rings.*

Recall that a ring R is *Bézout* if each of its finitely generated ideals is principal.

THEOREM 2.3. *Let R be Σ-RD-injective as a module over itself. Consider the following conditions:*

(1) *R is a pure-semisimple ring;*
(2) *every R-module is RD-injective;*
(3) *every R-module is pure-injective;*
(4) *the tensor product of two pure-injective R-modules is pure-injective;*
(5) *the tensor product of two RD-injective R-modules is RD-injective.*

Then:

(a) *Conditions* (1)–(4) *are equivalent and imply condition* (5).
(b) *When R is either a Bézout ring or a non-finite local ring, the five conditions are equivalent.*

Proof. (1)⇒(2). Assume that R is a pure-semisimple ring. Thus by [1, Theorem 3.1] and [12, Theorem 4.3], R is an Artinian principal ideal ring. So by [1, Corollary 3.3], every R-module is a direct sum of cyclic R-modules, and so by [22, Corollary 1] every R-module is RD-projective, since R is a principal ideal ring. This implies that every RD-exact sequence splits. Therefore, every R-module is RD-injective.

(2)⇒(3) is clear, since every RD-injective R-module is pure-injective.

(3)⇒(4) and (2)⇒(5) are clear.

(4)⇒(1). Assume that the tensor product of two pure-injective R-modules is pure-injective. Then R is also a Σ-pure-injective module, since R is Σ-RD-injective. Therefore, [18, Theorem 2.5] allows us to conclude.

(5)⇒(1). First assume that R is a Bézout ring and the tensor product of any two RD-injective R-modules is RD-injective. Then by Theorem 2.1, R is an Artinian ring. Thus R is a principal ideal ring, since R is Bézout. Therefore, by [1, Corollary 3.3], R is a pure-semisimple ring.

Now, assume that R is a non-finite local ring and the tensor product of any two RD-injective R-modules is RD-injective. Then Corollary 2.2 allows us to conclude. ■

Recall that a ring R is a *valuation ring* if it is uniserial as an R-module. Also, an *RD-ring* is a ring over which purity and RD-purity coincide. In [22, Corollary 5] and [23, Theorem 3] Warfield proved that the class of commutative RD-rings is exactly the class of *Prüfer rings* (i.e., localizations at maximal ideals are valuation rings). See [21] for more details on RD-rings.

EXAMPLE 2.4. (1) By Huisgen-Zimmermann [14, Observation 3(4) and Theorem 6], we know that every Artinian module over a commutative ring is Σ-pure-injective. Thus every commutative Artinian RD-ring is Σ-RD-injective. This implies that every commutative Artinian principal ideal ring is Σ-RD-injective (see [21, Proposition 6.5]).

(2) By [19, Example 4.4.14], every local RD-ring with $J(R)^2 = 0$ is Σ-RD-injective. In particular, every uniserial ring with $J(R)^2 = 0$ is Σ-RD-injective (see [21, Remark 2.7]).

REMARK 2.5. Warfield proved in [23, Theorem 1] that every finitely presented module over a valuation ring is a finite direct sum of cyclically presented modules. Thus purity and RD-purity over a valuation ring coincide. Couchot showed in [6, Theorem 12] that if R is a valuation ring, then $RDE(R) \otimes_R M$ is RD-injective for every finitely generated R-module M. Also, Warfield showed in [22, Theorem 6] that if S is a maximal immediate extension of a valuation ring R and M is a finitely generated R-module, then $M \otimes_R S$ is RD-injective. Therefore, the tensor product of two not necessarily RD-injective modules can be RD-injective. Also, by [18, Proposition 2.1], we know that for a ring R the tensor product of a finitely presented R-module and a pure-injective R-module is a pure-injective R-module. Thus, if R is an RD-ring, then the tensor product of a finitely presented R-module and an RD-injective R-module is an RD-injective R-module.

By [18, Theorem 2.6], we know that the tensor product of two Artinian R-modules is always pure-injective. But mind the following example:

EXAMPLE 2.6. Assume that K is a field and $R := K[x, y]/\langle x^2, y^2, xy\rangle$. Hence R is an Artinian ring. By [20, Corollary 4], ${}_RR$ is not RD-injective. Thus the R-module $R \otimes_R R$ is not RD-injective. Consequently, the tensor product of two Artinian R-modules is not RD-injective, in general.

Next, we obtain a condition for the RD-injectivity of the tensor product of any two Artinian R-modules.

THEOREM 2.7. *If every simple R-module is RD-injective, then the tensor product of any two Artinian R-modules is RD-injective. The converse is also true when R is an Artinian ring.*

Proof. Assume that every simple R-module is RD-injective. If M and N are two Artinian R-modules, then by [9, Proposition 6.1], $M \otimes_R N$ is an R-module of finite length. Also, by [5, Theorem IV.1], every finite length R-module is RD-injective, and so $M \otimes_R N$ is RD-injective. The converse is straightforward. ∎

COROLLARY 2.8. *If R is a von Neumann regular ring, then the tensor product of any two Artinian R-modules is injective.*

Proof. Assume that R is a von Neumann regular ring. Hence by [16, Corollary 3.73], all simple R-modules are injective. Thus by Theorem 2.7, the tensor product of any two Artinian R-modules is RD-injective. Also, since R is von Neumann regular, every RD-injective R-module is injective. Therefore, the tensor product of any two Artinian R-modules is injective. ∎

Recall that a ring R is said to be *semihereditary* if every finitely generated ideal of R is projective.

PROPOSITION 2.9. *If R is a semihereditary ring, then the tensor product of a finitely presented R-module and an RD-injective R-module is RD-injective.*

Proof. Assume that R is a semihereditary ring and $\mathcal{M}$ is a maximal ideal of R. Since R is semihereditary, so is the localization $R_{\mathcal{M}}$. Therefore, every finitely generated ideal of $R_{\mathcal{M}}$ is a projective $R_{\mathcal{M}}$-module, and so every finitely generated ideal of $R_{\mathcal{M}}$ is a free $R_{\mathcal{M}}$-module, since $R_{\mathcal{M}}$ is local. It follows that $R_{\mathcal{M}}$ is a domain, and so $R_{\mathcal{M}}$ is a Prüfer domain. Thus by [22, Corollary 5] and [23, Theorem 3], R is an RD-ring. Therefore, Remark 2.5 allows us to conclude. ∎

Assume that R is a semiprime Goldie ring and M is an R-module. Then

$$T(M) := \{m \in M \mid rm = 0 \text{ for some regular } r \in R\}$$

is a submodule of M. An R-module M is *torsion* if $T(M) = M$, and *torsion-free* if $T(M) = 0$.

Recall that an R-module M is said to be *p-injective* (or *divisible*) if every R-homomorphism $f : I \to M$ extends to $g : R \to M$ for each principal ideal I of R, or equivalently, if every system of equations $rx = m \in M$ $(r \in R)$ is solvable in M (see [16, Proposition 3.17]).

Theorem 2.10. *Let R be a semiprime Goldie ring. Then the tensor product of a finitely presented torsion-free p-injective R-module and an RD-injective R-module is a torsion-free p-injective pure-injective R-module.*

Proof. Assume that M is a finitely presented torsion-free p-injective R-module and N an RD-injective R-module. Since N is RD-injective, by [4, Theorem 3.7], there is a family $\{K_\lambda\}_{\lambda\in\Lambda}$ of R-algebras which are cyclically presented as R-modules, such that N is isomorphic to a direct summand of $\prod_{\lambda\in\Lambda} \mathrm{Hom}_R(K_\lambda, E)$ where E is an injective cogenerator of R (notice that the notion of RD-injectivity coincides with the $(1,1)$-pure injectivity of [4]). Set $K = \bigoplus_{\lambda\in\Lambda} K_\lambda$. Then

$$\prod_{\lambda\in\Lambda} \mathrm{Hom}_R(K_\lambda, E) \cong \mathrm{Hom}_R(K, E).$$

Thus N is a direct summand of $\mathrm{Hom}_R(K, E)$ where K is the direct sum of a family of cyclically presented R-modules and E is an injective R-module. Therefore, $M \otimes_R N$ is a direct summand of $M \otimes_R \mathrm{Hom}_R(K, E)$. Also, since M is finitely presented,

$$M \otimes_R \mathrm{Hom}_R(K, E) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(M, K), E).$$

We claim that the R-module $\mathrm{Hom}_R(\mathrm{Hom}_R(M, K), E)$ is torsion-free p-injective. To prove the claim we show that over a semiprime Goldie ring R, if A is a torsion-free p-injective R-module, then $\mathrm{Hom}_R(A, B)$ is a torsion-free p-injective R-module for each R-module B. First, assume that A is a p-injective R-module and $rf = 0$ for some $f \in \mathrm{Hom}_R(A, B)$ and regular element $r \in R$. Thus $rf(a) = 0$ and so $f(ra) = 0$ for all $a \in A$. Also, there exists $a' \in A$ such that $ra' = a$, since A is p-injective. Therefore, $f(a) = f(ra') = 0$ and so $f = 0$. Hence, $\mathrm{Hom}_R(A, B)$ is a torsion-free R-module.

Now, assume that A is torsion-free p-injective. We show that $\mathrm{Hom}_R(A, B)$ is a p-injective R-module. Suppose that $a \in A$, $s \in R$ and r is a regular element of R. Since A is p-injective we have $sa = r(sa)'$ where $sa, (sa)' \in A$ and also $sa = s(a'r) = (sa')r$ where $a, a' \in A$. This implies that $(sa)' = sa'$, since A is torsion-free. Also, for $a_1, a_2 \in A$, since A is p-injective we have $r(a_1 + a_2)' = (a_1 + a_2)$ where $(a_1 + a_2)', (a_1 + a_2) \in A$, and also $(a_1 + a_2) =$

$ra_1'+ra_2' = r(a_1'+a_2')$ where $a_1', a_2' \in A$. This implies that $(a_1+a_2)' = a_1'+a_2'$, since A is torsion-free.

Now, assume that $f \in \mathrm{Hom}_R(A, B)$ and r is a regular element of R. Define $g : A \to B$ by $g(a) = f(a')$ where $ra' = a$ and $a' \in A$. Since A is torsion-free, g is well-defined. Also, for each $a_1, a_2 \in A$,

$$g(a_1 + a_2) = f((a_1 + a_2)') = f(a_1' + a_2') = f(a_1') + f(a_2') = g(a_1) + g(a_2),$$

and for each $r \in R$ and $a \in A$,

$$g(ra) = f((ra)') = f(ra') = rf(a') = rg(a).$$

Thus $g \in \mathrm{Hom}_R(A, B)$ and so $\mathrm{Hom}_R(A, B)$ is a p-injective R-module.

Therefore, $\mathrm{Hom}_R(\mathrm{Hom}_R(M, K), E)$ is torsion-free p-injective, since M is torsion-free p-injective. This implies that $M \otimes_R N$ is torsion-free p-injective. Also, by [18, Proposition 2.1], $M \otimes_R N$ is pure-injective, since N is a pure-injective R-module. ∎

Hattori [13] proved that the tensor product of any two injective R-modules is injective when R is a domain. The following proposition is an analogue of this result.

Proposition 2.11. *Let R be either a domain or a semihereditary semiprime Goldie ring. Then the tensor product of a finitely presented torsion-free p-injective R-module and an RD-injective R-module is injective.*

Proof. Assume that M is a finitely presented torsion-free p-injective R-module and N an *RD*-injective R-module.

First assume that R is a domain. Thus by Theorem 2.10, $M \otimes_R N$ is a torsion-free p-injective R-module, since every domain is semiprime Goldie. Also, by [16, Proposition 3.25], over a domain R, a torsion-free R-module is injective if and only if it is p-injective. Thus $M \otimes_R N$ is an injective R-module.

Now if R is a semihereditary semiprime Goldie ring, then by Proposition 2.9 and Theorem 2.10, $M \otimes_R N$ is a torsion-free p-injective *RD*-injective R-module. We also have $r(M \otimes_R N) = rE(M \otimes_R N) \cap (M \otimes_R N)$ for each $r \in R$, since $M \otimes_R N$ is p-injective. Thus, by [22, Proposition 2], the exact sequence

$$0 \to M \otimes_R N \hookrightarrow E(M \otimes_R N) \to E(M \otimes_R N)/(M \otimes_R N) \to 0$$

is *RD*-exact, and so it splits, since $M \otimes_R N$ is *RD*-injective. Therefore, $M \otimes_R N$ is an injective R-module. ∎

Proposition 2.12. *Let R be a semiprime Goldie ring. Then the following statements are equivalent:*

(1) *R is a semisimple ring;*
(2) *R is a p-injective module;*

(3) *the tensor product of an RD-injective R-module and a finitely generated projective R-module is p-injective.*

Proof. (1)⇒(2) is clear.

(2)⇒(3). Assume that R is a p-injective module, and M is a finitely generated projective R-module and N an RD-injective R-module. Thus M is finitely presented, since M is finitely generated projective. Also, M is torsion-free p-injective, since R is p-injective and M is a direct summand of a free R-module. Therefore, by Theorem 2.10, $M \otimes_R N$ is a torsion-free p-injective pure-injective R-module.

(3)⇒(1). Assume that N is an RD-injective R-module. Thus by hypothesis, $R \otimes_R N$ is a p-injective R-module, and so N is p-injective, since $R \otimes_R N \cong N$. Thus every RD-injective R-module is p-injective, and so similarly to the proof of Proposition 2.11, this implies that every RD-injective R-module is injective. Now, by [5, Proposition I.1], for each $r \in R$, $\mathrm{Hom}(R/Rr, \mathbb{Q}/\mathbb{Z})$ is an RD-injective R-module, and so it is injective. This implies that R/Rr is a flat R-module for each $r \in R$. Thus the exact sequence

$$0 \to Rr \hookrightarrow R \to R/Rr \to 0$$

is pure exact for each $r \in R$ [16, Theorem 4.85], and so it splits, since R/Rr is a pure-projective R-module. So, every principal ideal of R is a direct summand of R and hence R is a von Neumann regular ring. So, R is a semisimple ring, since a von Neumann regular ring which is a Goldie ring is semisimple (see [10, Theorem 1.17] and [11, Lemma 6.12]). ■

Recall that an R-module A is *FP-injective* (or *absolutely pure*) if it is pure in every R-module that contains it, or equivalently, if $\mathrm{Ext}^1_R(M, A) = 0$ for all finitely presented R-modules M.

Proposition 2.13. *If R is an integral domain, then the tensor product of an FP-injective R-module and an injective flat R-module is FP-injective.*

Proof. Assume that R is an integral domain and A is an FP-injective R-module and B is an injective flat R-module. Thus the exact sequence

$$0 \to A \hookrightarrow E(A) \to E(A)/A \to 0$$

is pure. So, the exact sequence

$$0 \to A \otimes_R B \to E(A) \otimes_R B \to E(A)/A \otimes_R B \to 0$$

is pure, since B is flat. Also, by Hattori's result [13], $E(A) \otimes_R B$ is an injective R-module. This implies that $A \otimes_R B$ is a pure submodule of $E(A \otimes_R B)$. We know that an R-module K is FP-injective if and only if K is pure in its injective envelope. Therefore, $A \otimes_R B$ is FP-injective. ■

Corollary 2.14. *If R is an integral domain, then the tensor product of a finitely presented FP-injective R-module and an injective flat R-module is injective.*

Proof. This follows from Proposition 2.13, [18, Proposition 2.1], and the fact that every FP-injective pure-injective module is injective. ∎

Warfield [22, Corollary 1] proved that an R-module M is RD-projective if and only if it is a direct summand of a direct sum of cyclically presented R-modules. Facchini et al. [8, Theorem 4.6] proved that every right RD-projective module over a one-sided perfect ring is a direct sum of finitely presented cyclic modules.

We conclude this paper with the following result.

PROPOSITION 2.15. *If every cyclic R-module is RD-injective, then the tensor product of two finitely generated RD-projective R-modules is RD-injective.*

Proof. Assume that every cyclic R-module is RD-injective and M and N are two finitely generated RD-projective R-modules. Thus by [17, Theorem 2.1], R is a perfect ring. Hence by [8, Theorem 4.6], M and N are finite direct sums of finitely presented cyclic R-modules. So, the R-module $M \otimes_R N$ is a finite direct sum of cyclic modules, and so by hypothesis it is an RD-injective R-module. ∎

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