STUDIA MATHEMATICA 232 (3) (2016)

Dichotomy of global density of Riesz capacity

by

HIROAKI AIKAWA (Sapporo)

Abstract. Let C_α be the Riesz capacity of order α, $0<\alpha<n$, in $\mathbb{R}^n$. We consider the Riesz capacity density

$$\underline{\mathcal{D}}(C_\alpha, E, r) = \inf_{x\in\mathbb{R}^n} \frac{C_\alpha(E\cap B(x,r))}{C_\alpha(B(x,r))}$$

for a Borel set $E\subset\mathbb{R}^n$, where $B(x,r)$ stands for the open ball with center at x and radius r. In case $0<\alpha\le 2$, we show that $\lim_{r\to\infty}\underline{\mathcal{D}}(C_\alpha,E,r)$ is either 0 or 1; the first case occurs if and only if $\underline{\mathcal{D}}(C_\alpha,E,r)$ is identically zero for all $r>0$. Moreover, it is shown that the densities with respect to more general open sets enjoy the same dichotomy. A decay estimate for α-capacitary potentials is also obtained.

1. Introduction. Let φ be a nonnegative set function on $\mathbb{R}^n$, $n\ge 2$, such that:

(i) If $E\subset F$, then $\varphi(E)\le\varphi(F)$.
(ii) If U is a nonempty bounded open set, then $0<\varphi(U)<\infty$.

We denote by $B(x,r)$ the open ball with center at x and radius r. Following the previous paper [3], we consider the lower and upper densities

$$\underline{\mathcal{D}}(\varphi, E, r) = \inf_{x\in\mathbb{R}^n} \frac{\varphi(E\cap B(x,r))}{\varphi(B(x,r))}, \qquad \overline{\mathcal{D}}(\varphi, E, r) = \sup_{x\in\mathbb{R}^n} \frac{\varphi(E\cap B(x,r))}{\varphi(B(x,r))}$$

with respect to φ. (Note that the order of the parameters is changed from [3].) By definition $0\le\underline{\mathcal{D}}(\varphi,E,r)\le\overline{\mathcal{D}}(\varphi,E,r)\le 1$. We note that $\underline{\mathcal{D}}(\varphi,E,r)>0$ means that E is uniformly distributed in $\mathbb{R}^n$ in scale r with respect to φ. We are interested in the limits of $\underline{\mathcal{D}}(\varphi,E,r)$ and $\overline{\mathcal{D}}(\varphi,E,r)$ as $r\to\infty$. Typical examples of φ are the n-dimensional Lebesgue outer measure m and various capacities.

2010 *Mathematics Subject Classification*: Primary 31B15.
Key words and phrases: dichotomy, Riesz capacity, density, fractional Laplacian, α-harmonic.
Received 22 January 2016; revised 12 April 2016.
Published online 2 May 2016.

DOI: 10.4064/sm8511-4-2016

There is a significant difference between the Lebesgue outer measure m and capacities. In [3] we observed that, for each $0 \le c \le 1$, there exists a closed set $E \subset \mathbb{R}^n$ such that

$$\lim_{r\to\infty} \underline{\mathcal{D}}(m,E,r) = \lim_{r\to\infty} \overline{\mathcal{D}}(m,E,r) = c.$$

If φ is a capacity, then the situation is very different. For many capacities φ the limit $\lim_{r\to\infty} \underline{\mathcal{D}}(\varphi,E,r)$ is either 0 or 1. Stegenga [7] first proved this dichotomy for logarithmic capacity in $\mathbb{R}^2$. In [1] and [2] we implicitly observed the same phenomenon for the Newtonian capacity in $\mathbb{R}^n$. In the previous paper [3], we showed the dichotomy for the L^p-capacity

$$\mathrm{Cap}_p(E) = \inf\Big\{ \int_{\mathbb{R}^n} |\nabla u|^p\,dx : u \ge 1 \text{ on } E,\ u \in C_0^\infty(\mathbb{R}^n)\Big\}$$

with $1 < p < n$.

Theorem A. *Let E be a Borel set in $\mathbb{R}^n$. Then $\lim_{r\to\infty} \underline{\mathcal{D}}(\mathrm{Cap}_p,E,r)$ is either 0 or 1; the first case occurs if and only if $\underline{\mathcal{D}}(\mathrm{Cap}_p,E,r)$ is identically zero for all $r>0$.*

In this note we shall study the dichotomy for the Riesz capacity

$$C_\alpha(E) = \inf\{\|\mu\| : U_\alpha^\mu \ge 1 \text{ on } E\}, \qquad U_\alpha^\mu(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n}\,d\mu(y), \tag{1.1}$$

with $0<\alpha<n$ and $\alpha \le 2$. Note that if $\alpha = 2 < n$, then $C_2(E)$ is the Newtonian capacity up to a multiplicative constant. Our main result is the following.

Theorem 1.1. *Let $0<\alpha<n$ and $\alpha\le 2$. Let E be a Borel set in $\mathbb{R}^n$. Then $\lim_{r\to\infty} \underline{\mathcal{D}}(C_\alpha,E,r)$ is either 0 or 1; the first case occurs if and only if $\underline{\mathcal{D}}(C_\alpha,E,r)$ is identically zero for all $r>0$.*

The density over general open sets can be considered. We write $E(x,r) = \{x+ry \in \mathbb{R}^n : y \in E\}$ for $E\subset\mathbb{R}^n$, $r>0$ and $x\in\mathbb{R}^n$, i.e., $E(x,r)$ is E dilated by a factor of r and translated by x. If $x=0$, then we simply write rE for $E(0,r)$. Note that if E is the unit ball $B(0,1)$, then $E(x,r)=B(x,r)$. Let us consider the density over $\Omega(x,r)$, where Ω is an open set satisfying the following condition.

Definition 1.2. Let Ω be an open set. We say that Ω satisfies the *interior corkscrew condition* with $0<\kappa<1$ and $r_0>0$ if

$$\xi\in\partial\Omega \text{ and } 0<r\le r_0 \ \Rightarrow\ B(\xi,r)\cap\Omega \text{ contains a ball of radius } \kappa r.$$

Theorem 1.3. *Let $0<\alpha<n$ and $\alpha\le 2$. Let Ω be a bounded open set satisfying the interior corkscrew condition with $0<\kappa<1$ and $r_0>0$. Let*

E be a Borel set in $\mathbb{R}^n$. For $r > 0$ define

$$\underline{\mathcal{D}}_{\Omega}(C_\alpha, E, r) = \inf_{x\in\mathbb{R}^n} \frac{C_\alpha(E \cap \Omega(x,r))}{C_\alpha(\Omega(x,r))}.$$

Then $\lim_{r\to\infty} \underline{\mathcal{D}}_{\Omega}(C_\alpha, E, r)$ is either 0 or 1; the first case occurs if and only if $\underline{\mathcal{D}}_{\Omega}(C_\alpha, E, r)$ is identically zero for all $r > 0$.

We readily obtain Theorem 1.1 from Theorem 1.3 by letting $\Omega = B(0,1)$. Another typical example of Ω is an open cube. However, Theorem 1.3 treats more general open sets Ω, which may even be disconnected.

There are significant differences between the cases $\alpha = 2$ and $0 < \alpha < 2$. While the process corresponding to the case $\alpha = 2$ is the Brownian motion, the process corresponding to $0 < \alpha < 2$ is a jump process, so that the maximum principle is unavailable. While the classical harmonic measure in an open set D is supported on the boundary ∂D, the α-harmonic measure is supported in the complement of D if $0 < \alpha < 2$. So, the arguments in [1]–[3] do not extend to the case $0 < \alpha < 2$ straightforwardly. We shall get around the difficulty by the technique of Bogdan [4]. Hereafter, we let $0 < \alpha < 2$ since Theorem 1.1, in case $\alpha = 2$, is known from [1] and [2], at least implicitly.

Our argument is based on a decay estimate with respect to α-capacitary potentials, which may be of independent interest. For a set $E \subset \mathbb{R}^n$ we let $\overline{E}$ be the closure of E. It is known that a bounded Borel set E has the α-capacitary potential $U_\alpha^{\mu_E}$ such that $C_\alpha(E) = \|\mu_E\|$ and

$$\begin{aligned} &U_\alpha^{\mu_E} \le 1 \quad \text{in } \mathbb{R}^n,\\ &U_\alpha^{\mu_E} = 1 \quad \text{q.e. on } E, \qquad (1.2)\\ &\operatorname{supp} \mu_E \subset \overline{E}, \end{aligned}$$

where 'q.e.' (quasieverywhere) means the property holds outside a set of C_α-capacity zero. The measure μ_E is called the *capacitary measure* for E. See [6, p. 274]. For simplicity we write U_α^E for $U_\alpha^{\mu_E}$.

DEFINITION 1.4. We say that a closed set F satisfies the *capacity density condition* with $0 < \eta < 1$ and $r_0 > 0$ if

$$\frac{C_\alpha(F \cap B(\xi,r))}{C_\alpha(B(\xi,r))} \ge \eta \quad \text{for } \xi \in F \text{ and } 0 < r \le r_0.$$

It is easy to see that if D satisfies the the interior corkscrew condition with $0 < \kappa < 1$ and $r_0 > 0$, then $\overline{D}$ satisfies the capacity density condition with $0 < \eta = \eta(\kappa, \alpha, n) < 1$. Here $\eta = \eta(\kappa, \alpha, n)$ means that η depends only on κ, α and n. Our decay estimate for an α-capacitary potential is as follows.

Theorem 1.5. *Let D be a bounded open set whose closure is included in a ball B and let $F = \overline{B} \setminus D$. If F satisfies the capacity density condition with $0 < \eta < 1$ and $r_0 > 0$, then there exist positive constants $\beta = \beta(\eta, \alpha, n)$ and $A = A(\eta, \alpha, n, r_0)$ such that*

$$1 - U_\alpha^F(x) \le A\delta_D(x)^\beta \quad \text{for } x \in D.$$

Here $\delta_D(x) = \operatorname{dist}(x, \partial D)$.

We use the following notation. The symbol A stands for a positive constant whose exact value is unimportant and may change from one occurrence to the next. If necessary, we use $A_0, A_1, \dots,$ to specify them. We say that positive quantities f and g are *comparable* and write $f \approx g$ if $A^{-1} \le f/g \le A$ with some constant $A \ge 1$. The constant A is referred to as the *constant of comparison*. We have to pay attention to the dependency of the constant of comparison.

2. Preliminaries. In view of the definition (1.1), we have the following lemma.

Lemma 2.1. *Let $E \subset \mathbb{R}^n$ and $\kappa > 0$. If there is a measure μ such that $U_\alpha^\mu \ge \kappa$ on E, then $C_\alpha(E) \le \|\mu\|/\kappa$.*

The α-capacity of a ball is well-known. Recall that $\overline{E}$ stands for the closure of E. In particular, $\overline{B}(x, r)$ is the closed ball with center at x and radius r.

Lemma 2.2 ([6, p. 163]). *There exists a positive constant $A_0 = A_0(\alpha, n)$ such that $C_\alpha(B(x, r)) = C_\alpha(\overline{B}(x, r)) = A_0 r^{n-\alpha}$.*

Let us give some general observations for α-harmonic measure, α-reduced function, α-Green function and so on. Let D be an open set. We denote by $\omega_D^x(E)$ the α-harmonic measure on D of $E \subset \mathbb{R}^n \setminus D$ evaluated at $x \in D$. In case x is understood from the context, we suppress the superscript x and simply write $\omega_D(E)$. Note that, in case $0 < \alpha < 2$, the support of $\omega_D(E)$ is not concentrated on the boundary ∂D. In fact, the α-harmonic measure on the ball $B(0, r)$ is explicitly given by

$$\omega_{B(0,r)}^x(E) = A \int_E \left(\frac{r^2 - |x|^2}{|y|^2 - r^2} \right)^{\alpha/2} \frac{dy}{|x - y|^n} \qquad \text{for } x \in B(0, r)$$

with $A = A(\alpha, n)$ (see e.g. [6, p. 265] or [4, (2.6)]). In particular, if $\rho \ge 2r$, then

$$\begin{aligned} \omega_{B(0,r)}(B(0, \rho)^c) &\le A \int_{|y| \ge \rho} \left(\frac{r^2}{|y|^2 - (|y|/2)^2} \right)^{\alpha/2} \frac{dy}{(|y| - |y|/2)^n} \\ &\le A_1 \left(\frac{r}{\rho} \right)^\alpha \qquad \text{on } B(0, r) \end{aligned}$$

with $A_1 = A_1(\alpha, n) > 1$. By translation we have

$$\omega_{B(x,r)}(B(x,\rho)^c) \le A_1 \left(\frac{r}{\rho}\right)^{\alpha} \quad \text{on } B(x,r). \tag{2.1}$$

For $E \subset D$ we define ${}^D\mathbf{R}_1^E(x)$ by

$$\inf\{u(x) : u \ge 0 \text{ in } \mathbb{R}^n,\ u \text{ is } \alpha\text{-superharmonic in } D,\ u \ge 1 \text{ on } E\},$$

and let ${}^D\widehat{\mathbf{R}}_1^E$ be the lower semicontinuous regularization of ${}^D\mathbf{R}_1^E$. Observe that if E is a compact set in D, then

$$\omega_{D\setminus E}(D^c) = 1 - {}^D\widehat{\mathbf{R}}_1^E \quad \text{on } D \setminus E. \tag{2.2}$$

We extend the α-harmonic measure to arbitrary sets $E \subset D$ by the right hand side of (2.2). We denote by $G_D(x,y)$ the α-Green function for D. We have

$${}^D\widehat{\mathbf{R}}_1^E(x) = \int G_D(x,y)\, d\tilde{\mu}_E(y),$$

where $\tilde{\mu}_E$ is the capacitary measure for E in D (see [5]). Since $G_D(x,y) \le |x-y|^{\alpha-n}$, it follows that $U_\alpha^{\tilde{\mu}_E} \ge G_D\tilde{\mu}_E \ge 1$ on E outside a set of null C_α-capacity. Hence

$$C_\alpha(E) \le \|\tilde{\mu}_E\| \tag{2.3}$$

by Lemma 2.1. We have an estimate of α-harmonic measure in terms of C_α-capacity.

LEMMA 2.3. *There exists a positive constant $A_2 = A_2(\alpha, n) < 1$ with the following property: if $E \subset B(x,r)$ satisfies $C_\alpha(E) \ge \eta C_\alpha(B(x,r))$, then*

$$\omega_{B(x,2r)\setminus E}(B(x,2r)^c) \le 1 - A_2\eta \quad \text{on } \overline{B}(x,r). \tag{2.4}$$

Proof. It is easy to see that

$$G_{B(x,2r)}(y,z) \ge A|y-z|^{\alpha-n} \quad \text{for } y,z \in \overline{B}(x,r),$$

where $0 < A = A(\alpha,n) < 1$. Let $\tilde{\mu}_E$ be the α-capacitary measure of E with respect to $B(x,2r)$. Since $\operatorname{supp}\tilde{\mu}_E \subset \overline{B}(x,r)$, it follows that if $y \in \overline{B}(x,r)$, then

$$\begin{aligned}{}^{B(x,2r)}\widehat{\mathbf{R}}_1^E(y) &= G_{B(x,2r)}\tilde{\mu}_E(y) \ge A\int |y-z|^{\alpha-n}\, d\tilde{\mu}_E(z) \ge A\int (2r)^{\alpha-n}\, d\tilde{\mu}_E(z) \\ &\ge A(2r)^{\alpha-n}C_\alpha(E) \ge A(2r)^{\alpha-n}\eta C_\alpha(B(x,r)) = A2^{\alpha-n}A_0\eta\end{aligned}$$

by assumption and by (2.3) and Lemma 2.2. Hence (2.4) with $A_2 = A2^{\alpha-n}A_0$ follows from (2.2). ∎

3. Decay estimate for α-capacitary potentials. The main aim of this section is to prove Theorem 1.5. The following lemma provides a substantial estimate.

Lemma 3.1. *Let A_1 and A_2 be as in (2.1) and Lemma 2.3, respectively. Let $0 < \eta < 1$ and take τ such that $1 - A_2\eta < \tau < 1$. Choose $M > 4$ such that*

$$1 - A_2\eta + \frac{2^{\alpha+1}A_1}{\tau M^\alpha} \le \tau \quad \textit{and} \quad \tau M^\alpha \ge 2. \tag{3.1}$$

Let F be a compact set and let $\xi \in F$. Let $\rho > 0$ and let k be a positive integer. If

$$\frac{C_\alpha(F \cap B(\xi, r))}{C_\alpha(B(\xi, r))} \ge \eta \quad \textit{for } \rho \le r \le M^k\rho, \tag{3.2}$$

then

$$1 - U_\alpha^F \le \tau^k \quad \textit{on } \overline{B}(\xi, \rho). \tag{3.3}$$

Proof. We prove the lemma by induction on k. Observe that

$$1 - U_\alpha^F(x) \le \int_{B(\xi,2\rho)^c} (1 - U_\alpha^F(y))\, \omega^x_{B(\xi,2\rho)\setminus F}(dy) \quad \text{for } x \in B(\xi, 2\rho). \tag{3.4}$$

First, let $k = 1$. Decompose $B(\xi, 2\rho)^c$ into $B(\xi, M\rho)\setminus B(\xi, 2\rho)$ and $B(\xi, M\rho)^c$. Then (3.4) gives

$$\begin{aligned} 1 - U_\alpha^F(x) &\le \omega^x_{B(\xi,2\rho)\setminus F}(B(\xi, M\rho) \setminus B(\xi, 2\rho)) + \omega^x_{B(\xi,2\rho)\setminus F}(B(\xi, M\rho)^c) \\ &\le \omega^x_{B(\xi,2\rho)\setminus F}(B(\xi, 2\rho)^c) + \omega^x_{B(\xi,2\rho)}(B(\xi, M\rho)^c) \\ &\le 1 - A_2\eta + A_1\Big(\frac{2}{M}\Big)^\alpha < \tau \end{aligned}$$

by Lemma 2.3 and (2.1), and by (3.1). Thus (3.3) with $k = 1$ holds.

Second, let $k \ge 2$ and suppose the lemma holds up to $k-1$. For simplicity we let $a_j = \sup_{\overline{B}(\xi,M^j\rho)}(1 - U_\alpha^F)$ for $0 \le j \le k$. By induction hypothesis we have $a_j \le \tau^{k-j}$ for $1 \le j \le k$. (Apply the lemma with $M^j\rho$ in place of ρ.) We have to show that $a_0 \le \tau^k$. Decompose $B(\xi, 2\rho)^c$ into

$$[B(\xi, M\rho) \setminus B(\xi, 2\rho)] \cup \bigcup_{j=2}^k [B(\xi, M^j\rho) \setminus B(\xi, M^{j-1}\rho)] \cup B(\xi, M^k\rho)^c.$$

We see from (3.4) that if $x \in \overline{B}(\xi, \rho)$, then $1 - U_\alpha^F(x)$ is bounded by

$$\begin{aligned} a_1\omega^x_{B(\xi,2\rho)\setminus F}(B(\xi, M\rho) \setminus B(\xi, 2\rho)) + \sum_{j=2}^k a_j\omega^x_{B(\xi,2\rho)}(B(\xi, M^j\rho)\setminus B(\xi, M^{j-1}\rho)) \\ + \omega^x_{B(\xi,2\rho)}(B(\xi, M^k\rho)^c). \end{aligned}$$

Use the induction hypothesis and take the supremum over $x \in \overline{B}(\xi, \rho)$. By

Lemma 2.3, (2.1) and (3.1) we obtain

$$\begin{aligned}a_0 &\le \tau^{k-1}(1-A_2\eta)+\sum_{j=2}^{k}\tau^{k-j}A_1\left(\frac{2}{M^{j-1}}\right)^{\alpha}+A_1\left(\frac{2}{M^{k}}\right)^{\alpha}\\ &=\tau^{k-1}\Big\{1-A_2\eta+2^{\alpha}A_1\sum_{j=2}^{k}(\tau M^{\alpha})^{1-j}+2^{\alpha}A_1\tau(\tau M^{\alpha})^{-k}\Big\}\\ &\le \tau^{k-1}\Big\{1-A_2\eta+2^{\alpha}A_1\frac{2}{\tau M^{\alpha}}\Big\}\le \tau^{k},\end{aligned}$$

which completes the induction. ∎

Proof of Theorem 1.5. Let $M > 4$ be as in Lemma 3.1. Let $x \in D$. Without loss of generality we may assume that $0 < \delta_D(x) < M^{-1}r_0$. Let k be the positive integer such that $M^{-k-1}r_0 \le \delta_D(x) < M^{-k}r_0$. We find a point $\xi \in \partial D$ such that $|x-\xi| < M^{-k}r_0$, i.e., $x \in B(\xi, M^{-k}r_0)$. Invoke Lemma 3.1 with $\rho = M^{-k}r_0$. We have

$$1-U_{\alpha}^{F}(x)\le\tau^{k}=\left(\frac{1}{M}\right)^{-(\log\tau/\log M)k}\le\left(\frac{M}{r_0}\delta_D(x)\right)^{-\log\tau/\log M},$$

since $M^{-k-1}r_0 \le \delta_D(x)$. Hence we have the required estimate with $\beta = -\log\tau/\log M$. ∎

Theorem 1.5 gives the following decay estimate of more familiar form.

COROLLARY 3.2. *Let D be a bounded open set and let K be a compact subset of D. If D^c satisfies the capacity density condition with $0 < \eta < 1$ and $r_0 > 0$, then*

$${}^{D}\widehat{\mathbf{R}}_1^{K}(x)\le A\delta_D(x)^{\beta}\quad \textit{for } x\in D$$

with $\beta = \beta(\eta,\alpha,n) > 0$ and $A = A(\eta, r_0, K, D, \alpha, n) > 0$.

Proof. Take an open ball B containing $\overline{D}$ and set $F = \overline{B}\setminus D$. Observe that $0 < c = \inf_K(1-U_{\alpha}^{F}) \le 1$. Define a nonnegative function u by $u = {}^{D}\widehat{\mathbf{R}}_1^{K}$ on D and $u = 0$ on $\mathbb{R}^n \setminus D$. Since u is regular α-harmonic in $D\setminus K$, and $u \le (1-U_{\alpha}^{F})/c$ on $\mathbb{R}^n\setminus(D\setminus K)$, it follows that $u \le (1-U_{\alpha}^{F})/c$ in $D\setminus K$. Hence Theorem 1.5 gives the required estimate. ∎

4. Approximation of α-capacity. In this section we give a uniform approximation of α-capacity from the inside. The decay estimate (Lemma 3.1) in the previous section plays an important role. Let us begin with an approximation from the outside. For $\varepsilon > 0$ the closed ε-neighborhood of K is denoted by

$$K[\varepsilon]=\{x\in\mathbb{R}^n:\operatorname{dist}(x,K)\le\varepsilon\}.$$

Let us estimate $C_\alpha(K[\varepsilon])$ in case K is the closure of a bounded open set D satisfying a condition slightly weaker than the interior corkscrew condition.

LEMMA 4.1. *Let D be a bounded open set. Suppose there exist $\kappa > 0$ and $0 < r_1 < r_2/2$ such that*

$$\text{(4.1)} \qquad \xi \in \partial D \text{ and } r_1 \le r \le r_2 \Rightarrow D \cap B(\xi, r) \text{ contains a ball of radius } \kappa r.$$

Then there exist positive constants $\beta = \beta(\kappa, \alpha, n)$, $A_3 = A_3(\kappa, \alpha, n)$ and $A_4 = A_4(\kappa, \alpha, n)$ such that

$$\frac{C_\alpha(K[2r_1])}{C_\alpha(K)} \le \frac{1}{1 - A_3(2r_1/r_2)^\beta} \quad \text{with } K = \overline{D},$$

provided $2r_1/r_2 \le A_4$. In particular, if D satisfies the interior corkscrew condition, then, for each $b > 1$, there exists $\varepsilon = \varepsilon(b, \kappa, \rho_0, \alpha, n) > 0$ such that

$$\frac{C_\alpha(K[\varepsilon])}{C_\alpha(K)} \le b.$$

Proof. Let $K = \overline{D}$. First we claim that (4.1) implies

$$\text{(4.2)} \qquad \xi \in K \text{ and } 2r_1 \le r \le r_2 \Rightarrow D \cap B(\xi, r) \text{ contains a ball of radius } \kappa r/2.$$

We need to show the claim only for $\xi \in D$. In this case we find $\xi^* \in \partial D$ such that $|\xi - \xi^*| = \delta_D(\xi)$. If $0 < r \le 2\delta_D(\xi)$, then $D \cap B(\xi, r) \supset B(\xi, r/2)$; if $r \ge 2\delta_D(\xi)$, then $D \cap B(\xi, r) \supset D \cap B(\xi^*, r/2)$, which contains a ball of radius $\kappa r/2$ by (4.1). Thus the claim is proved.

In view of (4.2) and Lemma 2.2, we find $0 < \eta = \eta(\kappa, \alpha, n) < 1$ such that

$$\frac{C_\alpha(K \cap B(\xi, r))}{C_\alpha(B(\xi, r))} \ge \eta \quad \text{for } 2r_1 \le r \le r_2$$

whenever $\xi \in K$. Let $0 < \tau < 1$ and $M > 4$ be as in Lemma 3.1 and let $\beta = -\log \tau / \log M > 0$. Let k be the integer such that $2r_1 M^k \le r_2 < 2r_1 M^{k+1}$. Observe that $\tau^{k+1} < (2r_1/r_2)^\beta \le \tau^k$. If $2r_1/r_2 \le M^{-1}$, then $k \ge 1$, so that Lemma 3.1 yields

$$1 - U_\alpha^K \le \tau^k < \tau^{-1}(2r_1/r_2)^\beta \quad \text{on } \overline{B}(\xi, 2r_1).$$

Since $\xi \in K$ is arbitrary, the same inequality holds on $K[2r_1]$. Hence Lemma 2.1 gives the required estimate with $A_3 = \tau^{-1}$ and $A_4 = M^{-1}$. ∎

For a bounded open set D and $\varepsilon > 0$ we write $D_\varepsilon = \{x \in D : \delta_D(x) > \varepsilon\}$. We know the following geometric properties of D_ε.

LEMMA 4.2 ([3, Lemma 5.3]). *Suppose a bounded open set D satisfies the interior corkscrew condition with $0 < \kappa < 1$ and $r_0 > 0$. If $0 < \varepsilon < \kappa r_0/2$, then*

(i) $\overline{D} \subset \{y : \operatorname{dist}(y, D_\varepsilon) \le (1 + 2/\kappa)\varepsilon\}$;
(ii) *if* $\xi \in \partial D_\varepsilon$ *and* $2\varepsilon/\kappa \le r \le r_0$, *then* $B(\xi, r) \cap D_\varepsilon$ *contains a ball of radius* $\frac{1}{2}\kappa(1-\kappa)r$.

By translation and dilation and by Lemma 4.2 we have the following approximation of α-capacity from the inside as a corollary to Lemma 4.1.

THEOREM 4.3. *Suppose a bounded open set* Ω *satisfies the interior corkscrew condition with* $0 < \kappa < 1$ *and* $r_0 > 0$. *Then there exist positive constants* $\beta = \beta(\kappa, \alpha, n)$, $A_3 = A_3(\kappa, \alpha, n)$ *and* $A_4 = A_4(\kappa, \alpha, n)$ *such that if* $0 < \varepsilon \le A_4\kappa r_0/4$, *then*

$$\frac{C_\alpha(\overline{\Omega}(x,\rho))}{C_\alpha(\overline{\Omega}_\varepsilon(x,\rho))} \le \frac{1}{1 - A_3[4\varepsilon/(\kappa r_0)]^\beta} \quad \textit{uniformly for } x \in \mathbb{R}^n \textit{ and } \rho > 0.$$

Proof. Let $x \in \mathbb{R}^n$ and $\rho > 0$. By translation and dilation and by Lemma 4.2 we have

(i) $\overline{\Omega}(x,\rho) \subset \{y : \operatorname{dist}(y, \Omega_\varepsilon(x,\rho)) \le (1 + 2/\kappa)\varepsilon\rho\}$;
(ii) if $\xi \in \partial\Omega_\varepsilon(x,\rho)$ and $2\varepsilon\rho/\kappa \le r \le r_0\rho$, then $\Omega_\varepsilon(x,\rho) \cap B(\xi, r)$ contains a ball of radius $\frac{1}{2}\kappa(1-\kappa)r$.

Let β, A_3 and A_4 be as in Lemma 4.1 with $\frac{1}{2}\kappa(1-\kappa)$ in place of κ. Let us apply Lemma 4.1 to $D = \Omega_\varepsilon(x,\rho)$. Since $(1 + 2/\kappa)\varepsilon\rho \le 4\varepsilon\rho/\kappa$, it follows from (i) that $\overline{\Omega}(x,\rho) \subset \overline{D}[4\varepsilon\rho/\kappa]$, so that from (ii),

$$\frac{C_\alpha(\overline{\Omega}(x,\rho))}{C_\alpha(\overline{\Omega}_\varepsilon(x,\rho))} \le \frac{C_\alpha(\overline{D}[4\varepsilon\rho/\kappa])}{C_\alpha(\overline{D})} \le \frac{1}{1 - A_3[(4\varepsilon\rho/\kappa)/(r_0\rho)]^\beta} = \frac{1}{1 - A_3[4\varepsilon/(\kappa r_0)]^\beta},$$

provided $4\varepsilon/(\kappa r_0) \le A_4$. ∎

5. Proof of Theorem 1.3. The following lemma is a crucial step of the proof of Theorem 1.3.

LEMMA 5.1. *Let* $A_1 > 1$ *and* $0 < A_2 < 1$ *be as in* (2.1) *and Lemma* 2.3, *respectively. Suppose* $E \subset \mathbb{R}^n$ *satisfies*

$$\frac{C_\alpha(E \cap B(x,R))}{C_\alpha(B(x,R))} \ge \eta \quad \textit{for every } x \in \mathbb{R}^n \tag{5.1}$$

with $R > 0$ *and* $0 < \eta < 1$. *If* $M > 4$ *and*

$$A_1\left(\frac{2}{M}\right)^\alpha \le 1 \quad \textit{and} \quad 1 - A_2\eta + M^{-\alpha} < 1, \tag{5.2}$$

then, for every open set D *and* $k \ge 1$,

$$1 - U_\alpha^{E\cap D} \le (1 - A_2\eta + M^{-\alpha})^{k-1} \quad \textit{on } D_k$$

whenever $D_k = \{x \in D : \delta_D(x) \geq (M + \cdots + M^k)R\}$ *is nonempty. Moreover, if* $C_\alpha(D_k) > 0$, *then*

$$\frac{C_\alpha(E \cap D)}{C_\alpha(D_k)} \geq 1 - (1 - A_2\eta + M^{-\alpha})^{k-1}.$$

Proof. For simplicity we let $F = E \cap D$ and $b_k = \sup_{D_k}(1 - U_\alpha^F)$. By definition $0 \leq b_k \leq 1$. So, there is nothing to prove for $k = 1$. Let $k \geq 2$ and $x \in D_k$. Observe that

$$1 - U_\alpha^F \leq \int\limits_{B(x,2R)^c} (1 - U_\alpha^F(y))\, \omega_{B(x,2R)\setminus F}(dy) \quad \text{on } B(x, 2R).$$

We estimate the right hand side by decomposing $B(x, 2R)^c$. Since $B(x, 2R) \subset B(x, M^k R) \subset D_{k-1}$, it follows from (2.4) and (2.1) with $r = 2R$ and $\rho = M^k R$ that $1 - U_\alpha^F(x)$ is bounded by

$$\begin{aligned} b_{k-1}\omega^x_{B(x,2R)\setminus F}(B(x, M^k R) \setminus B(x, 2R)) + \omega^x_{B(x,2R)\setminus F}(B(x, M^k R)^c) \\ \leq b_{k-1}\omega^x_{B(x,2R)\setminus F}(B(x, 2R)^c) + \omega^x_{B(x,2R)}(B(x, M^k R)^c) \\ \leq b_{k-1}(1 - A_2\eta) + A_1\left(\frac{2}{M^k}\right)^\alpha. \end{aligned}$$

Since $x \in D_k$ is arbitrary, we have

$$b_k \leq b_{k-1}(1 - A_2\eta) + A_1\left(\frac{2}{M^k}\right)^\alpha.$$

Let us consider the sequence $\{c_k\}$ defined by $c_1 = 1$ and

$$c_k = c_{k-1}(1 - A_2\eta) + A_1\left(\frac{2}{M^k}\right)^\alpha \quad \text{for } k \geq 2.$$

Obviously $b_k \leq c_k$. By the above identity with $k - 1$ in place of k, we have

$$c_{k-1} \geq A_1\left(\frac{2}{M^{k-1}}\right)^\alpha = A_1\left(\frac{2}{M^k}\right)^\alpha \cdot M^\alpha$$

for $k \geq 3$. The same inequality holds for $k = 2$ by (5.2), so that

$$c_k \leq c_{k-1}(1 - A_2\eta) + M^{-\alpha}c_{k-1} = (1 - A_2\eta + M^{-\alpha})c_{k-1} \quad \text{for } k \geq 2.$$

Hence $b_k \leq c_k \leq (1 - A_2\eta + M^{-\alpha})^{k-1}$, which gives the first assertion. The second assertion readily follows from Lemma 2.1. ∎

Proof of Theorem 1.3. It is sufficient to show that if $\underline{\mathcal{D}}_\Omega(C_\alpha, E, r) > 0$ for some $r > 0$, then $\lim_{r\to\infty} \underline{\mathcal{D}}_\Omega(C_\alpha, E, r) = 1$. We find $z \in \Omega$ and $0 < R_1 < R_2$ such that $B(z, R_1) \subset \Omega \subset B(z, R_2)$. Observe that $B(x + rz, R_1 r) \subset$

$\Omega(x,r) \subset B(x+rz, R_2 r)$, so that

$$\frac{C_\alpha(E\cap\Omega(x,r))}{C_\alpha(\Omega(x,r))} \le \frac{C_\alpha(E\cap B(x+rz,R_2r))}{C_\alpha(B(x+rz,R_1r))} = \left(\frac{R_2}{R_1}\right)^{n-\alpha}\frac{C_\alpha(E\cap B(x+rz,R_2r))}{C_\alpha(B(x+rz,R_2r))}.$$

Hence $\underline{\mathcal{D}}_\Omega(C_\alpha,E,r) > 0$ implies (5.1) with some $0<\eta<1$ and $R = R_2 r > 0$.

Let $0<c<1$. In view of Theorem 4.3, we find $\varepsilon>0$ so small that

$$\frac{C_\alpha(\Omega(x,\rho))}{C_\alpha(\overline{\Omega}_\varepsilon(x,\rho))} \le \frac{1}{\sqrt{c}} \quad \text{for every } x\in\mathbb{R}^n \text{ and } \rho>0. \tag{5.3}$$

Let $M>4$ satisfy (5.2) and let k be so large that

$$1-(1-A_2\eta+M^{-\alpha})^{k-1} \ge \sqrt{c}.$$

Observe that $\operatorname{dist}(\Omega_\varepsilon(x,\rho),\partial\Omega(x,\rho)) = \rho\operatorname{dist}(\Omega_\varepsilon,\partial\Omega) = \varepsilon\rho$. If

$$\rho \ge \varepsilon^{-1}(M+\cdots+M^k)R,$$

then

$$\overline{\Omega}_\varepsilon(x,\rho) \subset \{y\in\Omega(x,\rho) : \delta_{\Omega(x,\rho)}(y) \ge (M+\cdots+M^k)R\},$$

so that Lemma 5.1 with $D=\Omega(x,\rho)$ yields

$$\frac{C_\alpha(E\cap\Omega(x,\rho))}{C_\alpha(\overline{\Omega}_\varepsilon(x,\rho))} \ge 1-(1-A_2\eta+M^{-\alpha})^{k-1} \ge \sqrt{c}.$$

This, together with (5.3), implies that if $\rho\ge\varepsilon^{-1}(M+\cdots+M^k)R$, then

$$\frac{C_\alpha(E\cap\Omega(x,\rho))}{C_\alpha(\Omega(x,\rho))} \ge c \quad \text{for every } x\in\mathbb{R}^n,$$

so that $\underline{\mathcal{D}}_\Omega(C_\alpha,E,\rho)\ge c$. Since $0<c<1$ is arbitrary, we have

$$\lim_{r\to\infty}\underline{\mathcal{D}}_\Omega(C_\alpha,E,r)=1,$$

as required. ■

Acknowledgements. The author thanks the referee for careful reading and useful suggestions.

This work was supported in part by JSPS KAKENHI Grant Numbers 25287015 and 25610017.

References

[1] H. Aikawa, *Norm estimate of Green operator, perturbation of Green function and integrability of superharmonic functions*, Math. Ann. 312 (1998), 289–318.

[2] H. Aikawa, *Boundary Harnack principle and Martin boundary for a uniform domain*, J. Math. Soc. Japan 53 (2001), 119–145.

[3] H. Aikawa and T. Itoh, *Dichotomy of global capacity density*, Proc. Amer. Math. Soc. 143 (2015), 5381–5393.
[4] K. Bogdan, *The boundary Harnack principle for the fractional Laplacian*, Studia Math. 123 (1997), 43–80.
[5] P. Kim, R. Song, and Z. Vondraček, *Minimal thinness with respect to symmetric Lévy processes*, Trans. Amer. Math. Soc., to appear; arXiv:1405.0297 (2014).
[6] N. S. Landkof, *Foundations of Modern Potential Theory*, Grundlehren Math. Wiss. 180, Springer, New York, 1972,
[7] D. A. Stegenga, *A geometric condition which implies BMOA*, Michigan Math. J. 27 (1980), 247–252.

Hiroaki Aikawa
Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan
E-mail: aik@math.sci.hokudai.ac.jp