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On some non-linear projections of self-similar sets in $\mathbb{R}^3$

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Abstract. In the last years considerable attention has been paid to orthogonal projections and non-linear images of self-similar sets. In this paper we consider homothetic self-similar sets in $\mathbb{R}^3$, i.e. the generating IFS has the form $\{\lambda_i \underline{x} + \underline{t}_i\}_{i=1}^q$. We show that if the dimension of the set is strictly greater than 1 then the image of the set under some non-linear function to the real line has dimension 1. As an application, we show that the distance set of such a self-similar set has dimension 1. Moreover, the third algebraic product of a self-similar set with itself on the real line has dimension 1 if its dimension is at least 1/3.

1. Introduction and statements of results. We call a non-empty compact set Λ in $\mathbb{R}^d$ *self-similar* if there exists an iterated function system (IFS) Φ of the form

$$\Phi = \{f_i(\underline{x}) = \lambda_i O_i \underline{x} + \underline{t}_i\}_{i=1}^q, \tag{1.1}$$

where $\lambda_i \in (0,1)$, $\underline{t}_i \in \mathbb{R}^d$ and O_i is an orthogonal transformation of $\mathbb{R}^d$ for every $i = 1, \ldots, q$, and Λ is the attractor of Φ, i.e. the unique non-empty compact set $\Lambda = \bigcup_{i=1}^q f_i(\Lambda)$. We call a measure μ *self-similar* if there exists an IFS Φ of the form (1.1) and a probability vector $(p_1, \ldots, p_q)$ such that $\mu = \sum_{i=1}^q p_i (f_i)_* \mu$, where $(f)_*\mu = \mu \circ f^{-1}$.

We denote by $\Pi_{d,k}$ the set of orthogonal projections from $\mathbb{R}^d$ to $\mathbb{R}^k$. The classical results of Marstrand [15] and Kaufman [14] state that for any Borel set $A \subseteq \mathbb{R}^d$ we have $\dim_{\mathrm{H}} \pi A = \min\{k, \dim_{\mathrm{H}} A\}$ for almost every $\pi \in \Pi_{d,k}$, where $\dim_{\mathrm{H}}$ denotes Hausdorff dimension. Denote the packing dimension by $\dim_{\mathrm{P}}$ and the box dimension by $\dim_{\mathrm{B}}$. For the definition and basic properties of Hausdorff, packing and box dimensions we refer to [3].

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Hochman and Shmerkin [13] proved that if the IFS Φ satisfies the *strong separation condition* (SSC), i.e. $f_i(\Lambda) \cap f_j(\Lambda) = \emptyset$ for every $i \neq j$, and the orthogonal transformations of the IFS Φ satisfy the *minimality assumption*, that is, there exists a $\pi \in \Pi_{d,k}$ such that

$$\bigcup_{n=1}^{\infty} \{\pi O_{i_1} \cdots O_{i_n} : 1 \leq i_1, \ldots, i_n \leq q\} \tag{1.2}$$

is dense in $\Pi_{d,k}$, then $\dim_{\mathrm{H}} \pi\Lambda = \min\{k, \dim_{\mathrm{H}} \Lambda\}$ for every $\pi \in \Pi_{d,k}$, and $\dim_{\mathrm{H}} g(\Lambda) = \min\{k, \dim_{\mathrm{H}} \Lambda\}$ for every $g \in C^1(\mathbb{R}^d \to \mathbb{R}^k)$ without singular points. In particular, if the minimality assumption holds then (1.2) holds for all $\pi \in \Pi_{d,k}$. Recently, Farkas [7] generalized this result by omitting the strong separation condition.

Dekking [2], Rams and Simon [19, 20] and Falconer and Jin [5] considered orthogonal projections and non-linear images of random self-similar sets. For more detailed surveys on projections of fractal sets and measures, see [4] or [21].

In this paper, we focus on *homothetic self-similar sets* (HSS sets) in $\mathbb{R}^3$, with $O_i = I$ for every $i = 1, \ldots, q$, where I denotes the identity. Similarly, we consider homothetic self-similar measures (HSS measures).

It is well known that in this case the dimension may drop under some orthogonal projections. However, if Λ is an HSS set with SSC on $\mathbb{R}^2$ then $\dim_{\mathrm{H}} g(\Lambda) = \min\{1, \dim_{\mathrm{H}} \Lambda\}$ for certain C^2 functions $g : \mathbb{R}^2 \to \mathbb{R}$. This result was first published by Bond, Łaba and Zahl [1, Proposition 2.6], but they attribute the proof to Hochman.

Our goal is to generalize this result to HSS sets in $\mathbb{R}^3$ with $\dim_{\mathrm{H}}$ large enough.

We will have a special interest in the radial projection $P_d : \mathbb{R}^d \setminus \{\underline{0}\} \to S^{d-1}$, where S^{d-1} denotes the unit sphere in $\mathbb{R}^d$. Precisely, $P_d(\underline{x}) = \underline{x}/\|\underline{x}\|$. We denote by $\nabla_{\underline{x}} g$ the gradient vector of a function $g : \mathbb{R}^d \to \mathbb{R}$ at a point $\underline{x}$.

Theorem 1.1. *Let Λ be a homothetic self-similar set in $\mathbb{R}^3$ such that $\dim_{\mathrm{H}} \Lambda > 1$ and Λ is not contained in any plane (but may not satisfy SSC). Suppose that $g : \mathbb{R}^3 \to \mathbb{R}$ is a C^1 function on an open set $V \supseteq \Lambda$ such that*

(1) *$\|\nabla_{\underline{x}} g\| \neq 0$ for every $\underline{x} \in \Lambda$.*
(2) *$\|\nabla_{t\cdot\underline{x}} g \times \nabla_{\underline{x}} g\| = 0$ for all $\underline{x} \in V$ and $t \in \mathbb{R}$ such that $t \cdot \underline{x} \in V$.*
(3) *h_g is bi-Lipschitz on $P_3(\Lambda) \subseteq S^2$, where $h_g(\underline{x}) = P_3(\nabla_{t\cdot\underline{x}} g)$ for any $t \in \mathbb{R}$ such that $t \cdot \underline{x} \in V$.*

Then $\dim_{\mathrm{H}} g(\Lambda) = 1$.

We apply Theorem 1.1 in two ways. First, we show a corollary for the distance sets of HSS sets in $\mathbb{R}^3$. Recall that the *distance set* of $A \subset \mathbb{R}^d$ is

$$D(A) = \{\|\underline{x} - \underline{y}\| : \underline{x}, \underline{y} \in A\}. \tag{1.3}$$

For every $\underline{x} \in A$, we define the *pinned distance set* of $A \subset \mathbb{R}^d$ at $\underline{x}$ by

$$D_{\underline{x}}(A) = \{\|\underline{x} - \underline{y}\| : \underline{y} \in A\}. \tag{1.4}$$

Falconer's *distance set conjecture* states that if $\dim_H A > d/2$ then $D(A)$ has positive Lebesgue measure for any measurable $A \subseteq \mathbb{R}^d$. Recently, Orponen [17] showed that for any self-similar set Λ in $\mathbb{R}^2$, if $\mathcal{H}^1(\Lambda) > 0$ then $\dim_H D(\Lambda) = 1$, where $\mathcal{H}^1$ denotes the Hausdorff measure. We improve Orponen's result for HSS sets in $\mathbb{R}^3$ in the following way.

THEOREM 1.2. *Let Λ be an HSS set in $\mathbb{R}^3$ such that* $\dim_H \Lambda > 1$. *Then* $\dim_H D_{\underline{x}}(\Lambda) = 1$ *for every $\underline{x} \in \Lambda$. In particular,* $\dim_H D(\Lambda) = 1$.

As a second application, we consider the algebraic product of a self-similar set on the real line with itself. Let $A, B \subset \mathbb{R}$ and set

$$A \cdot B = \{x \cdot y : x \in A \text{ and } y \in B\}.$$

As a consequence of the result of Bond, Łaba and Zahl [1] we show that for every self-similar set Λ on the real line,

$$\dim_H \Lambda \cdot \Lambda = \min\{2 \dim_H \Lambda, 1\} \tag{1.5}$$

(see Corollary 2.9). We generalize this result to $\Lambda \cdot \Lambda \cdot \Lambda$ in the following way.

THEOREM 1.3. *Let Λ be a self-similar set in $\mathbb{R}$ such that* $\dim_H \Lambda > 1/3$. *Then* $\dim_H \Lambda \cdot \Lambda \cdot \Lambda = 1$.

2. Preliminaries and non-linear projections in $\mathbb{R}^2$. This section is devoted to listing our tools for the proof of Theorem 1.1. The projections in $\mathbb{R}^2$ were studied by several authors, including Hochman [10], Hochman and Shmerkin [13], and Bond, Łaba and Zahl [1]. For the convenience of the reader, we here state the relevant theorems and give short proofs.

First, we introduce some notation. Let Φ be an IFS on $\mathbb{R}^d$ with contracting similitudes of the form (1.1). Denote the attractor of Φ by Λ. Let $\mathcal{S} = \{1, \ldots, q\}$ be the set of symbols and denote the symbolic space by $\Sigma = \mathcal{S}^{\mathbb{N}}$. Let σ be the left-shift operator on Σ. We define the natural projection ρ from Σ to Λ in the usual way, i.e. for any $\mathbf{i} = (i_0, i_1, \ldots) \in \Sigma$,

$$\rho(\mathbf{i}) = \lim_{n\to\infty} f_{i_0} \circ f_{i_1} \circ \cdots \circ f_{i_n}(\underline{0}),$$

where $\underline{0} = (0, \ldots, 0) \in \mathbb{R}^d$. It is easy to see that $\rho(\mathbf{i}) = f_{i_0}(\rho(\sigma\mathbf{i}))$.

Let $\underline{p} = (p_1, \ldots, p_q)$ be a probability vector with strictly positive elements. Denote the Bernoulli measure on Σ by $\nu = \underline{p}^{\mathbb{N}}$. Then ν is left-shift invariant and ergodic. Then the measure $\mu = \rho_* \nu = \nu \circ \rho^{-1}$ is the unique self-similar measure with $\operatorname{spt} \mu = \Lambda$ and $\mu = \sum_{i=1}^q p_i (f_i)_* \mu$.

Let $\Sigma^* = \bigcup_{n=0}^\infty \mathcal{S}^n$ denote the set of finite length words in symbols $\mathcal{S}$. For $\bar{\imath} = (i_0, \ldots, i_{n-1}) \in \Sigma^*$, let $|\bar{\imath}|$ be the length of $\bar{\imath}$ and for any $\bar{\imath}, \bar{\jmath} \in \Sigma^*$, let the juxtaposition $\overline{\imath\jmath}$ be the finite length word $(i_0, \ldots, i_{|\bar{\imath}|-1}, j_0, \ldots, j_{|\bar{\jmath}|-1})$.

We write $f_{\overline{\imath}}$ for the composition $f_{i_0} \circ \dots \circ f_{i_{n-1}}$ where $\overline{\imath} = (i_0, \dots, i_{n-1})$. We denote by $\mathrm{Fix}(f_{\overline{\imath}})$ the fixed point of $f_{\overline{\imath}}$. Let $[\overline{\imath}]$ be the cylinder set formed by $\overline{\imath}$,

$$[\overline{\imath}] := \{\mathbf{j} = (j_0, j_1, \dots) \in \Sigma : i_0 = j_0, \dots, i_{|\overline{\imath}|-1} = j_{|\overline{\imath}|-1}\}.$$

Its projection $\Lambda_{\overline{\imath}} = \rho([\overline{\imath}]) = f_{\overline{\imath}}(\Lambda)$ is called a *cylinder set* of Λ. We note that if μ is an HSS measure (or Λ is an HSS set) with IFS $\Phi = \{f_i(\underline{x}) = \lambda_i \underline{x} + \underline{t}_i\}_{i=1}^q$ in $\mathbb{R}^d$ with SSC then for any $\overline{\imath} \in \Sigma^*$ the measure $\mu_{\overline{\imath}} := \mu|_{\Lambda_{\overline{\imath}}}/\mu(\Lambda_{\overline{\imath}})$ (or respectively $\Lambda_{\overline{\imath}}$) is also a self-similar measure (or self-similar set) with IFS $\Phi_{\overline{\imath}} := \{\lambda_i \underline{x} + f_{\overline{\imath}}(t_i)\}_{i=1}^q$. On the other hand, for any $\pi \in \Pi_{d,k}$ the measure $\pi\mu = \mu \circ \pi^{-1}$ is an HSS measure (or respectively $\pi\Lambda$ is an HSS set) as well, with IFS $\pi\Phi := \{\lambda x + \pi(\underline{t}_i)\}_{i=1}^q$. We denote the nth iteration of the IFS by $\Phi^n = \{f_{\overline{\imath}}\}_{\overline{\imath} \in \mathcal{S}^n}$.

Our first approach to the study of homothetic self-similar sets is to find a proper approximating subsystem.

PROPOSITION 2.1. *Let Λ be an HSS set in $\mathbb{R}^d$ with IFS $\Phi = \{f_i(\underline{x}) = \lambda_i \underline{x} + \underline{t}_i\}_{i=1}^q$. For every $\varepsilon > 0$, there exists an IFS Φ' of the form $\{g_j(\underline{x}) = \lambda \underline{x} + \underline{t}'_j\}_{j=1}^{q'}$ with $\lambda \in (0,1)$ such that the attractor Λ' of Φ' satisfies SSC, $\Lambda' \subseteq \Lambda$ and $\dim_{\mathrm{H}} \Lambda' > \dim_{\mathrm{H}} \Lambda - \varepsilon$. Moreover, the functions of Φ' can be written as compositions of functions in Φ.*

We call the attractor and self-similar measures for Φ' homogeneous homothetic self-similar (HHSS).

The proof is analogous to the proof of Peres and Shmerkin [18, Proposition 6], therefore we omit it.

Denote the Hausdorff dimension of a measure μ by $\dim_{\mathrm{H}} \mu$, that is,

$$\dim_{\mathrm{H}} \mu = \inf\{\dim_{\mathrm{H}} A : \mu(A) > 0\}.$$

We define the *upper* and *lower local dimension* of μ at a point $\underline{x}$ in the usual way by

$$\underline{d}_\mu(\underline{x}) = \liminf_{r \to 0+} \frac{\log \mu(B_r(\underline{x}))}{\log r}, \qquad \overline{d}_\mu(\underline{x}) = \limsup_{r \to 0+} \frac{\log \mu(B_r(\underline{x}))}{\log r},$$

where $B_r(\underline{x})$ is the ball with radius r centred at $\underline{x}$. By [6, Theorem 1.2],

$$\dim_{\mathrm{H}} \mu = \mu - \operatorname*{ess\,inf}_{\underline{x}} \underline{d}_\mu(\underline{x}). \tag{2.1}$$

We say that the measure μ is *exact dimensional* if $\underline{d}_\mu(\underline{x}) = \overline{d}_\mu(\underline{x})$ for μ-a.e. $\underline{x}$. By [6, Corollary 2.1], if μ is exact dimensional then

$$\dim_{\mathrm{H}} \mu = \inf\{\dim_{\mathrm{H}} A : \mu(A) = 1\}.$$

LEMMA 2.2. *Let μ and ν be Borel probability measures such that $\mu \ll \nu$ (that is, μ is absolutely continuous with respect to ν) and ν is exact dimensional. Then $\dim_{\mathrm{H}} \mu = \dim_{\mathrm{H}} \nu$.*

Proof. Since $\mu \ll \nu$, for any measurable set A, if $\mu(A) > 0$ then $\nu(A) > 0$. Thus, $\dim_{\mathrm{H}} \mu \geq \dim_{\mathrm{H}} \nu$. On the other hand, since ν is exact dimensional,

$$\begin{aligned}\dim_{\mathrm{H}} \nu = {} & \inf\{\dim_{\mathrm{H}} A : \nu(A) = 1\} = \inf\{\dim_{\mathrm{H}} A : \nu(A^c) = 0\} \\ \geq {} & \inf\{\dim_{\mathrm{H}} A : \mu(A^c) = 0\} = \inf\{\dim_{\mathrm{H}} A : \mu(A) = 1\} \geq \dim_{\mathrm{H}} \mu,\end{aligned}$$

where A^c denotes the complement of A. ∎

Our second approach is to approximate non-linear projections of HSS measures with SSC by orthogonal projections. Let $g : \mathbb{R}^d \to \mathbb{R}$ be a C^1 function. We denote the projection of a Borel measure μ on $\mathbb{R}^d$ by $g_*\mu = \mu \circ g^{-1}$. Let $\nabla_{\underline{x}} g$ be the gradient of g at $\underline{x} = (x_1, \ldots, x_d)$,

$$\nabla_{\underline{x}} g = \begin{pmatrix} g'_{x_1}(\underline{x}) \\ \vdots \\ g'_{x_d}(\underline{x}) \end{pmatrix}.$$

Denote by $\pi_{g,\underline{x}} \in \Pi_{d,1}$ the orthogonal projection from $\mathbb{R}^d$ to the subspace spanned by $\nabla_{\underline{x}} g$, that is, $\pi_{g,\underline{x}}(\underline{y}) = \langle \nabla_{\underline{x}} g, \underline{y}\rangle / \|\nabla_{\underline{x}} g\|$, where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on $\mathbb{R}^d$ and $\|\cdot\|$ is the induced norm. The next theorem is a consequence of the results of Hochman [10].

THEOREM 2.3. *Let μ be an HSS measure with SSC in $\mathbb{R}^d$ and let $g : \mathbb{R}^d \to \mathbb{R}$ be a C^1 function with $\|\nabla_{\underline{x}} g\| \neq 0$ for every $\underline{x} \in \operatorname{spt} \mu$. Then*

$$\dim_{\mathrm{H}} g_*\mu \geq \mu - \operatorname*{ess\,inf}_{\underline{x}} \dim_{\mathrm{H}} \pi_{g,\underline{x}} \mu.$$

Proof. Let μ be an HSS measure with SSC in $\mathbb{R}^d$. Then by [10, Example 4.3] the measure μ is *homogeneous uniformly scaling* [10, Definitions 1.5(3) and 1.35]. Let P be the ergodic fractal distribution generated by μ [10, Definitions 1.2, and 1.5(1), and Proposition 1.36]. For $\pi \in \Pi_{d,1}$, let

$$E_P(\pi) = \int \dim_{\mathrm{H}} \pi\nu \, dP(\nu).$$

Applying [10, Theorem 1.23 and Proposition 1.36] we find that for any C^1 function $g : \mathbb{R}^d \to \mathbb{R}$ with $\|\nabla_{\underline{x}} g\| \neq 0$,

$$\dim_{\mathrm{H}} g_*\mu \geq \mu - \operatorname*{ess\,inf}_{\underline{x}} E_P(\pi_{g,\underline{x}}).$$

By [10, Proposition 1.36] for P-a.e. measure ν there exists a ball B such that $\mu \ll (T_B)_*\nu$, where $T_{B_r(\underline{x})}(\underline{y}) = (\underline{y} - \underline{x})/r$. Hence, $\pi\mu \ll \pi(T_B)_*\nu$ for every $\pi \in \Pi_{d,1}$ and P-a.e. ν. On the other hand, by [10, Theorem 1.22] the measure $\pi\nu$ is exact dimensional for P-a.e. ν. Since T_B is bi-Lipschitz, by Lemma 2.2 we have $\dim_{\mathrm{H}} \pi\mu = \dim_{\mathrm{H}} \pi\nu$ for every $\pi \in \Pi_{d,1}$ and P-a.e. ν, which implies that $E_P(\pi) = \dim_{\mathrm{H}} \pi\mu$. ∎

As a consequence of Theorem 2.3 and [11, Theorem 1.8], we state a modified version of a result of Hochman, published by Bond, Łaba and Zahl [1, Proposition 2.6].

Proposition 2.4. *Let μ be an HHSS measure with SSC in $\mathbb{R}^2$ such that* $\operatorname{spt}\mu$ *is not contained in any line. Suppose that $g : \mathbb{R}^2 \to \mathbb{R}$ is a C^2 map such that $\|\nabla_{\underline{x}} g\| \neq 0$ and*

$$\left\|\begin{pmatrix} g''_{xx}(\underline{x})g'_y(\underline{x}) - g''_{xy}(\underline{x})g'_x(\underline{x}) \\ g''_{xy}(\underline{x})g'_y(\underline{x}) - g''_{yy}(\underline{x})g'_x(\underline{x}) \end{pmatrix}\right\| \neq 0$$

for every $\underline{x} \in \operatorname{spt}\mu$. Then

$$\dim_{\mathrm{H}} g_*\mu = \min\{1, \dim_{\mathrm{H}}\mu\}.$$

To prove the proposition, we need a technical lemma.

Lemma 2.5. *Let μ be an HHSS measure with SSC in $\mathbb{R}^2$ such that* $\operatorname{spt}\mu$ *is not contained in any line. Then there exists a constant $c > 0$ that* $\dim_{\mathrm{H}}\pi\mu \geq c > 0$ *for every $\pi \in \Pi_{2,1}$.*

Proof. Let $\{f_i(\underline{x}) = \lambda\underline{x} + \underline{t}_i\}_{i=1}^q$ be the IFS corresponding to μ and $\underline{p} = (p_1, \ldots, p_q)$ the corresponding probability vector.

Since $\operatorname{spt}\mu$ is not contained in any line, there exist three fixed points of the functions, say f_1, f_2 and f_3, that form a triangle. Denote the sides of the triangle by a, b and c. Let $\kappa = \inf_{\pi\in\Pi_{2,1}} \max\{|\pi a|, |\pi b|, |\pi c|\} > 0$ and let $N = \lceil \log\kappa/(4|\operatorname{spt}\mu|)/\log\lambda\rceil$, where $|\cdot|$ denotes diameter. Let $x \in \pi\operatorname{spt}\mu$ and

$$z_n(x) := \sum_{\substack{\bar{\imath}\in\mathcal{S}^{nN} \\ B_{(\kappa/4)\lambda^{nN}}(x)\cap\pi\Lambda_{\bar{\imath}}\neq\emptyset}} \nu([\bar{\imath}]).$$

It is easy to see by the definition of N and κ that there exists an $\bar{\imath} \in \Sigma^*$ with $|\bar{\imath}| = N$ such that $B_{(\kappa/4)\lambda^N}(x) \cap \pi\Lambda_{\bar{\imath}} = \emptyset$. Thus, $z_1(x) \leq 1 - p_{\min}^N$, where $p_{\min} = \min\{p_1, \ldots, p_q\}$. On the other hand, for every $\bar{\imath} \in \Sigma^*$ with $|\bar{\imath}| = nN$ and $B_{(\kappa/4)\lambda^{nN}}(x) \cap \pi\Lambda_{\bar{\imath}} \neq \emptyset$ there exists a $\bar{\jmath} \in \Sigma$ with $|\bar{\jmath}| = N$ such that $B_{(\kappa/4)\lambda^{(n+1)N}}(x) \cap \pi\Lambda_{\bar{\imath}\bar{\jmath}} = \emptyset$. Thus,

$$\sum_{\substack{\bar{\jmath}\in\mathcal{S}^{N} \\ B_{(\kappa/4)\lambda^{(n+1)N}}(x)\cap\pi\Lambda_{\bar{\imath}\bar{\jmath}}\neq\emptyset}} \nu([\bar{\jmath}]) \leq 1 - p_{\min}^N. \tag{2.2}$$

Now we prove by induction that $z_n(x) \leq (1 - p_{\min}^N)^n$. For $n = 1$ this has already been shown. Assume that it holds for n. Then by (2.2),

$$\begin{aligned} z_{n+1}(x) &= \sum_{\substack{\bar{\imath}\in\mathcal{S}^{(n+1)N} \\ B_{(\kappa/4)\lambda^{(n+1)N}}(x)\cap\pi\Lambda_{\bar{\imath}}\neq\emptyset}} \nu([\bar{\imath}]) \\ &= \sum_{\substack{\bar{\imath}\in\mathcal{S}^{nN} \\ B_{(\kappa/4)\lambda^{nN}}(x)\cap\pi\Lambda_{\bar{\imath}}\neq\emptyset}} \sum_{\substack{\bar{\jmath}\in\mathcal{S}^{N} \\ B_{(\kappa/4)\lambda^{(n+1)N}}(x)\cap\pi\Lambda_{\bar{\imath}\bar{\jmath}}\neq\emptyset}} \nu([\bar{\imath}\bar{\jmath}]) \end{aligned}$$

$$= \sum_{\substack{\overline{\imath}\in\mathcal{S}^{nN} \\ B_{(\kappa/4)\lambda^{nN}}(x)\cap\pi\Lambda_{\overline{\imath}}\neq\emptyset}} \nu([\overline{\imath}]) \sum_{\substack{\overline{\jmath}\in\mathcal{S}^{N} \\ B_{(\kappa/4)\lambda^{(n+1)N}}(x)\cap\pi\Lambda_{\overline{\imath}\overline{\jmath}}\neq\emptyset}} \nu([\overline{\jmath}])$$
$$\leq (1-p_{\min}^N)z_n(x) \leq (1-p_{\min}^N)^{n+1}.$$

Hence, for any $x \in \operatorname{spt}\mu$,

$$\liminf_{r\to 0+} \frac{\log \mu(B_r(x))}{\log r} = \liminf_{n\to\infty} \frac{\log \mu(B_{\frac{\kappa}{4}\lambda^{nN}}(x))}{nN\log\lambda}$$
$$\geq \liminf_{n\to\infty} \frac{\log z_n(x)}{nN\log\lambda} = \frac{\log(1-p_{\min}^N)}{N\log\lambda} > 0.$$

By (2.1) this implies that $\dim_{\mathrm{H}} \pi\mu \geq \frac{\log(1-p_{\min}^N)}{N\log\lambda} > 0$ for all $\pi \in \Pi_{2,1}$. ∎

Proof of Proposition 2.4. Since $\dim_{\mathrm{H}} g_*\mu \leq \min\{1, \dim_{\mathrm{H}}\mu\}$, it is enough to show the lower bound. By Theorem 2.3 we have

$$\dim_{\mathrm{H}} g_*\mu \geq \mu - \operatorname*{ess\,inf}_{\underline{x}} \dim_{\mathrm{H}} \pi_{g,\underline{x}}\mu.$$

Thus, it is enough to show that

$$\dim_{\mathrm{H}} \pi_{g,\underline{x}}\mu = \min\{1, \dim_{\mathrm{H}}\mu\} \quad \text{for } \mu\text{-a.e. } \underline{x}. \tag{2.3}$$

If μ is an HHSS measure with IFS $\{\lambda\underline{x} + \underline{t}_i\}_{i=1}^q$ then for any $\pi \in \Pi_{2,1}$ the measure $\pi\mu$ is HHSS as well, with IFS $\{\lambda x + \pi(\underline{t}_i)\}_{i=1}^q$. By using the parametrization $\pi_\theta(\underline{x}) = \langle(\cos\theta, \sin\theta), \underline{x}\rangle$ and [11, Theorem 1.8], it follows that

$$\dim_{\mathrm{P}}\bigl\{\theta \in [0,\pi) : \dim_{\mathrm{H}} \pi_\theta\mu < \min\{1, \dim_{\mathrm{H}}\mu\}\bigr\} = 0.$$

Hence, to verify (2.3) it is enough to show that

$$\dim_{\mathrm{H}} f_*\mu > 0,$$

where $f(\underline{x}) = \arctan(g'_{x_1}(\underline{x})/g'_{x_2}(\underline{x}))$. By our assumption $\|\nabla_{\underline{x}} f\| \neq 0$ for every $\underline{x} \in \operatorname{spt}\mu$. By applying Theorem 2.3 and Lemma 2.5, we get

$$\dim_{\mathrm{H}} f_*\mu \geq \mu - \operatorname*{ess\,inf}_{\underline{x}} \dim_{\mathrm{H}} \pi_{f,\underline{x}}\mu \geq \inf_{\pi\in\Pi_{2,1}} \dim_{\mathrm{H}} \pi\mu \geq c > 0. \ ∎$$

As a consequence of Proposition 2.4, we state an analogue of [1, Proposition 2.5] for measures, which plays an important role for the further studies.

COROLLARY 2.6. *If μ is an HHSS measure with SSC in $\mathbb{R}^2$ such that $\underline{0} \notin \operatorname{spt}\mu$ and $\operatorname{spt}\mu$ is not contained in any line, then*

$$\dim_{\mathrm{H}} (P_2)_*\mu = \min\{1, \dim_{\mathrm{H}}\mu\}. \tag{2.4}$$

Proof. Since μ can be written as a convex combination of self-similar measures restricted to cylinder sets, we have

$$\dim_{\mathrm{H}} (P_2)_*\mu = \min_{\overline{\imath}\in\mathcal{S}^n} \dim_{\mathrm{H}} (P_2)_*\mu_{\overline{\imath}}$$

for every $n \geq 1$. Thus it is enough to show that for sufficiently large $n \geq 1$, (2.4) holds for any $\bar{\imath} \in \mathcal{S}^n$. By choosing n sufficiently large and by applying a rotation, we may assume that $\operatorname{spt} \mu_{\bar{\imath}}$ is contained in the upper half-plane and separated away from the x-axis.

Since $x \mapsto (x, \sqrt{1-x^2})$ is bi-Lipschitz for every $x \in (-1+\varepsilon, 1+\varepsilon)$, it is enough to show that the map $g : (x, y) \mapsto x/\sqrt{x^2+y^2}$ satisfies $\dim_{\mathrm{H}} g_*\mu_{\bar{\imath}} = \min\{1, \dim_{\mathrm{H}} \mu_{\bar{\imath}}\}$. Indeed, g satisfies the assumptions of Proposition 2.4. ∎

As another consequence of Proposition 2.4 we state the following theorem for general self-similar sets in $\mathbb{R}^2$.

THEOREM 2.7. *Let Λ be an arbitrary self-similar set in $\mathbb{R}^2$ not contained in any line. Suppose that $g : \mathbb{R}^2 \to \mathbb{R}$ is a C^2 map such that $\|\nabla_{\underline{x}} g\| \neq 0$ and*

$$\left\| \begin{pmatrix} g''_{xx}(\underline{x})g'_y(\underline{x}) - g''_{xy}(\underline{x})g'_x(\underline{x}) \\ g''_{xy}(\underline{x})g'_y(\underline{x}) - g''_{yy}(\underline{x})g'_x(\underline{x}) \end{pmatrix} \right\| \neq 0$$

for every $\underline{x} \in \Lambda$. Then

$$\dim_{\mathrm{H}} g(\Lambda) = \min\{1, \dim_{\mathrm{H}} \Lambda\}.$$

Proof. By [17, Lemma 3.4], for every $\varepsilon > 0$ there exists a self-similar set $\Lambda' \subseteq \Lambda$ not contained in any line such that $\dim_{\mathrm{H}} \Lambda' \geq \dim_{\mathrm{H}} \Lambda - \varepsilon$ and the attractor of the corresponding IFS Φ' satisfies SSC. If one of the functions of Φ' contains an irrational rotation then by [13, Corollary 1.7],

$$\dim_{\mathrm{H}} g(\Lambda) \geq \dim_{\mathrm{H}} g(\Lambda') = \min\{1, \dim_{\mathrm{H}} \Lambda'\} \geq \min\{1, \dim_{\mathrm{H}} \Lambda\} - \varepsilon.$$

If none of the functions of Φ' contains an irrational rotation then by [17, Lemma 4.2] there exists a self-similar set $\Lambda'' \subseteq \Lambda$ such that $\dim_{\mathrm{H}} \Lambda'' \geq \dim_{\mathrm{H}} \Lambda - 2\varepsilon$ and the similitudes of the corresponding IFS Φ'' do not contain any rotation or reflection, i.e. Λ'' is an HSS set with SSC. By Proposition 2.1, there exists an HHSS set Λ''' with SSC such that $\Lambda''' \subseteq \Lambda$ and $\dim_{\mathrm{H}} \Lambda''' \geq \dim_{\mathrm{H}} \Lambda - 3\varepsilon$.

Let μ be the natural self-similar measure on Λ''', that is, μ is the equidistributed self-similar measure on the cylinder sets. Hence, $\dim_{\mathrm{H}} \mu = \dim_{\mathrm{H}} \Lambda'''$. By Proposition 2.4,

$$\begin{aligned} \dim_{\mathrm{H}} g(\Lambda) \geq \dim_{\mathrm{H}} g_*\mu &= \min\{1, \dim_{\mathrm{H}} \mu\} \\ &= \min\{1, \dim_{\mathrm{H}} \Lambda'''\} \geq \min\{1, \dim_{\mathrm{H}} \Lambda\} - 3\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the statement of the theorem is proven. ∎

As a corollary, one can prove a weaker version of Falconer's distance set conjecture in $\mathbb{R}^2$. This is just a little stronger than Orponen's result [17, Theorem 1.2], since we only assume that $\dim_{\mathrm{H}} \Lambda \geq 1$ and we do not need that $\mathcal{H}^1(\Lambda) > 0$.

COROLLARY 2.8. *Let Λ be a self-similar set in $\mathbb{R}^2$ with $\dim_{\mathrm{H}} \Lambda \geq 1$. Then*

$$\dim_{\mathrm{H}} D(\Lambda) = 1,$$

where $D(\Lambda)$ denotes the distance set of Λ defined in (1.3).

Proof. If Λ is contained in a line then $\dim_{\mathrm{H}} D(\Lambda) = \dim_{\mathrm{H}} \Lambda$. So, we may assume that Λ is not contained in any line. Let $\underline{a} \in \Lambda$ and let $\Lambda_{\bar{\imath}}$ be a cylinder set such that $\mathrm{dist}(\underline{a}, \Lambda_{\bar{\imath}}) > 0$. Then $D_{\underline{a}}(\underline{x}) = \|\underline{x} - \underline{a}\|$ satisfies the conditions of Theorem 2.7 with self-similar set $\Lambda_{\bar{\imath}}$. Thus,

$$\begin{aligned}\dim_{\mathrm{H}} D(\Lambda) &\geq \dim_{\mathrm{H}} D_{\underline{a}}(\Lambda) \geq \dim_{\mathrm{H}} D_{\underline{a}}(\Lambda_{\bar{\imath}}) \\ &= \min\{1, \dim_{\mathrm{H}} \Lambda_{\bar{\imath}}\} = \min\{1, \dim_{\mathrm{H}} \Lambda\} = 1. \ \blacksquare\end{aligned}$$

Another corollary of Theorem 2.7 is (1.5).

COROLLARY 2.9. *Let Λ be a self-similar set in $\mathbb{R}$. Then*

$$\dim_{\mathrm{H}} \Lambda \cdot \Lambda = \min\{2 \dim_{\mathrm{H}} \Lambda, 1\}.$$

Proof. Without loss of generality, we may assume that Λ is not a singleton. Then there exists a cylinder set $\Lambda_{\bar{\imath}}$ of Λ such that every element in $\Lambda_{\bar{\imath}}$ is either strictly positive or strictly negative.

By [18, Proposition 6], for every $\varepsilon > 0$ there exists a self-similar set $\Lambda' \subseteq \Lambda_{\bar{\imath}}$ such that $\dim_{\mathrm{H}} \Lambda' \geq \dim_{\mathrm{H}} \Lambda - \varepsilon$ and it is the attractor of an IFS Φ satisfying SSC and having the form

$$\Phi = \{f_i(x) = \lambda x + t_i\}_{i=1}^q.$$

Then $\Lambda' \times \Lambda'$ is a self-similar set with SSC in $\mathbb{R}^2$ with IFS

$$\Phi' = \{h_i(\underline{x}) = \lambda \underline{x} + (t_i, t_j)\}_{i,j=1}^q.$$

Let $g(x, y) = xy$. Then $\|\nabla_{\underline{x}} g\| = \sqrt{y^2 + x^2} \neq 0$ and

$$\left\| \begin{pmatrix} g''_{xx}(\underline{x}) g'_y(\underline{x}) - g''_{xy}(\underline{x}) g'_x(\underline{x}) \\ g''_{xy}(\underline{x}) g'_y(\underline{x}) - g''_{yy}(\underline{x}) g'_x(\underline{x}) \end{pmatrix} \right\| = \sqrt{x^2 + y^2} \neq 0$$

for any $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Thus by Theorem 2.7,

$$\begin{aligned}\dim_{\mathrm{H}} \Lambda \cdot \Lambda &\geq \dim_{\mathrm{H}} \Lambda' \cdot \Lambda' = \dim_{\mathrm{H}} g(\Lambda' \times \Lambda') \\ &= \min\{1, \dim_{\mathrm{H}} \Lambda' \times \Lambda'\} \geq \min\{1, 2 \dim_{\mathrm{H}} \Lambda\} - 2\varepsilon,\end{aligned}$$

where we have used the fact that $\dim_{\mathrm{H}} \Lambda' \times \Lambda' = 2 \dim_{\mathrm{H}} \Lambda'$ (see [3, Corollary 7.4]). Since $\varepsilon > 0$ was arbitrary, the proof is complete. ∎

Similarly to the case of Proposition 2.4, to prove our main Theorem 1.1, we need an upper bound for the exceptional directions for orthogonal projections in $\Pi_{3,1}$. For a vector $\underline{n} \in S^{d-1}$ let $\pi_{\underline{n}} \in \Pi_{d,1}$ be the orthogonal projection to the subspace generated by $\underline{n}$, i.e. $\pi_{\underline{n}}(\underline{x}) = \langle \underline{x}, \underline{n} \rangle$.

Proposition 2.10. *Let μ be an HHSS measure in $\mathbb{R}^3$ with SSC. Then*

$$\dim_{\mathrm{P}}\left\{\underline{n}\in S^2 : \dim_{\mathrm{H}}\pi_{\underline{n}}\mu<\min\{1,\dim_{\mathrm{H}}\mu\}\right\}\le 1. \tag{2.5}$$

Proposition 2.10 follows from Hochman's [12, Theorem 1.10].

Finally, we state the dimension conservation phenomena for HSS measures, first showed by Furstenberg [9] and generalized by Falconer and Jin [5].

Theorem 2.11. *Let μ be an HSS measure with SSC in $\mathbb{R}^d$ and let $\pi\in\Pi_{d,k}$. Then*

$$\dim_{\mathrm{H}}\pi\mu+\dim_{\mathrm{H}}\mu_{\pi^{-1}(\underline{x})}=\dim_{\mathrm{H}}\mu \quad \text{for } \pi\mu\text{-a.e. } \underline{x}\in\mathbb{R}^k, \tag{2.6}$$

where $\mu_{\pi^{-1}(\underline{x})}$ denotes the conditional measure of μ on the fibre $\pi^{-1}(\underline{x})$. Moreover,

$$\pi\mapsto\dim_{\mathrm{H}}\pi\mu \text{ is lower semicontinuous.} \tag{2.7}$$

For the proof we refer to Hochman [10, Theorem 1.37].

3. Radial projection in $\mathbb{R}^3$. The critical point of our study is the examination of the radial projection. Unfortunately, we cannot prove the analogue of Corollary 2.6 in general. However, we are able to show that if an HSS set has dimension strictly larger than 1 then there exists an HHSS measure whose support is contained in the HSS set, and whose radial projection has dimension strictly larger than 1.

Theorem 3.1. *Let Λ be an HSS set in $\mathbb{R}^3$ such that* $\dim_{\mathrm{H}}\Lambda>1$ *and Λ is not contained in any plane. Then there exists an HHSS measure μ such that* $\operatorname{spt}\mu\subseteq\Lambda$ *and* $\dim_{\mathrm{H}}\mu\ge\dim_{\mathrm{H}}(P_3)_*\mu>1$.

Let $C_{\alpha,\underline{v}}(\underline{x})$ denote the closed double cone with vertex $\underline{x}\in\mathbb{R}^3$, angle α, and axis $\underline{v}\in\mathbb{R}^3$ with $\|\underline{v}\|=1$. That is,

$$C_{\alpha,\underline{v}}(\underline{x})=\{\underline{y}\in\mathbb{R}^3 : |\langle\underline{x}-\underline{y},\underline{v}\rangle|\ge|\cos(\alpha)|\,\|\underline{x}-\underline{y}\|\}.$$

In other words, the angle between $\underline{x}-\underline{y}$ and $\underline{v}$ is less than or equal to α. First, we show the following lemma.

Lemma 3.2. *Let Λ be an HHSS set in $\mathbb{R}^3$, not contained in any plane. Then for every vector $\underline{v}\in\mathbb{R}^3$ with $\|\underline{v}\|=1$ and $\underline{x}\in\Lambda$ there exists $\pi/2>\alpha>0$ such that for every $r>0$,*

$$\operatorname{int}(B_r(\underline{x})\cap C_{\alpha,\underline{v}}(\underline{x}))\cap\Lambda\ne\emptyset,$$

where int *denotes interior.*

Proof. Assume for contradiction that there exist $\underline{v}\in\mathbb{R}^3$ with $\|\underline{v}\|=1$ and $\underline{x}\in\Lambda$ such that for every $\pi/2>\alpha>0$ there exists an $r=r(\alpha)>0$ such that

$$\operatorname{int}(B_r(\underline{x})\cap C_{\alpha,\underline{v}}(\underline{x}))\cap\Lambda=\emptyset.$$

Let $\Phi = \{f_i(\underline{x}) = \lambda\underline{x} + \underline{t}_i\}_{i=1}^q$ be the corresponding IFS and let $n(r) = \min\{n : \lambda^n < r\}$.

For $\pi/2 > \alpha > 0$, if $\underline{x} \in \Lambda_{\bar\imath}$ with $|\bar\imath| \geq n(r(\alpha))$ then $\Lambda_{\bar\imath} \subseteq \mathrm{int}(B_{r(\alpha)}(\underline{x})) \cap \Lambda$. Thus by our assumption $\Lambda_{\bar\imath} \cap \mathrm{int}(C_{\alpha,\underline{v}}(\underline{x})) = \emptyset$. Since Φ does not contain any orthogonal transformation,

$$\Lambda \cap \mathrm{int}(C_{\alpha,\underline{v}}(f_{\bar\imath}^{-1}(\underline{x}))) = f_{\bar\imath}^{-1}(\Lambda_{\bar\imath} \cap \mathrm{int}(C_{\alpha,\underline{v}}(\underline{x}))) = \emptyset.$$

Thus, for every $\pi/2 > \alpha > 0$ there exists $N \geq 1$ such that for every $\bar\imath$ with $\underline{x} = \rho(\mathbf{i}) = f_{\bar\imath}(\rho(\sigma^{|\bar\imath|}\mathbf{i}))$ and $|\bar\imath| \geq N$,

$$\Lambda \cap \mathrm{int}(C_{\alpha,\underline{v}}(f_{\bar\imath}^{-1}(\underline{x}))) = \emptyset.$$

Let $\underline{y}$ be a density point of the sequence $\{f_{\bar\imath}^{-1}(\underline{x})\}$. Since Λ is compact, we have $\underline{y} \in \Lambda$ and $\Lambda \cap \mathrm{int} C_{\alpha,\underline{v}}(\underline{y}) = \emptyset$. But α was arbitrary, so Λ must be contained in a plane with normal vector $\underline{v}$ and containing $\underline{y}$, a contradiction. ■

Denote by V_π the subspace to which $\pi \in \Pi_{3,2}$ projects, and denote by $\underline{n}_\pi \in \mathbb{R}^3$ the normal vector to V_π with $\|\underline{n}_\pi\| = 1$. For a projection $\pi \in \Pi_{3,2}$, let

$$\sin(\varepsilon_\pi) = \inf_{\underline{x} \neq \underline{y} \in \Lambda} \frac{\|\underline{n}_\pi \times (\underline{x} - \underline{y})\|}{\|\underline{x} - \underline{y}\|}. \tag{3.1}$$

Since Λ is compact, if $\pi\Lambda$ satisfies the SSC then $\sin(\varepsilon_\pi) > 0$. Let

$$\Gamma_\pi = \left\{(\underline{x}, \underline{y}) \in \Lambda \times \Lambda : \underline{x} \neq \underline{y} \ \&\ \sin(\varepsilon_\pi) = \frac{\|\underline{n}_\pi \times (\underline{x} - \underline{y})\|}{\|\underline{x} - \underline{y}\|}\right\}.$$

Since $\Phi = \{f_i\}_{i=1}^q$ is orthogonal transformation free,

$$\sin(\varepsilon_\pi) = \min_{i \neq j} \inf_{\underline{x} \in f_i(\Lambda),\, \underline{y} \in f_j(\Lambda)} \frac{\|\underline{n}_\pi \times (\underline{x} - \underline{y})\|}{\|\underline{x} - \underline{y}\|}.$$

Moreover, by compactness, there are $i \neq j$, $\underline{x}_i \in f_i(\Lambda)$, $\underline{y}_j \in f_j(\Lambda)$ such that $(\underline{x}_i, \underline{y}_j) \in \Gamma_\pi \neq \emptyset$. By definition

$$\mathrm{int}(C_{\varepsilon_\pi,\underline{n}_\pi}(\underline{x})) \cap \Lambda = \emptyset \quad \text{for every } \underline{x} \in \Lambda. \tag{3.2}$$

Lemma 3.3. *Let Λ be an HHSS set not contained in any plane and suppose that $\pi\Lambda$ satisfies SSC. If $(\underline{x}, \underline{y}), (\underline{x}, \underline{z}) \in \Gamma_\pi$ then $(\underline{y}, \underline{z}) \notin \Gamma_\pi$. Thus, if $(\underline{x}, \underline{y}) \in \Gamma_\pi$ then $l_{\underline{x},\underline{y}} \cap \Lambda \setminus \{\underline{x}, \underline{y}\} = \emptyset$, where $l_{\underline{x},\underline{y}}$ is the line through $\underline{x}$ and $\underline{y}$.*

Proof. Suppose that $(\underline{x}, \underline{y}), (\underline{x}, \underline{z}), (\underline{y}, \underline{z}) \in \Gamma_\pi$. It is easy to see that $\underline{x}, \underline{y}, \underline{z}$ must be contained in one line. Indeed, $\underline{z}$ must be a common element of the boundary of the cones $C_{\varepsilon_\pi,\underline{n}_\pi}(\underline{x})$ and $C_{\varepsilon_\pi,\underline{n}_\pi}(\underline{y})$ (see Figure 1). Without loss of generality, assume that $\underline{z}$ is between $\underline{x}$ and $\underline{y}$. Let V be the common tangent plane of $C_{\varepsilon_\pi,\underline{n}_\pi}(\underline{x})$ and $C_{\varepsilon_\pi,\underline{n}_\pi}(\underline{y})$, and let $\underline{v}$ be its normal vector. By Lemma 3.2, there exists $\pi/2 > \alpha > 0$ such that $\mathrm{int}(B_r(\underline{z}) \cap C_{\alpha,\underline{v}}(\underline{z})) \cap \Lambda$

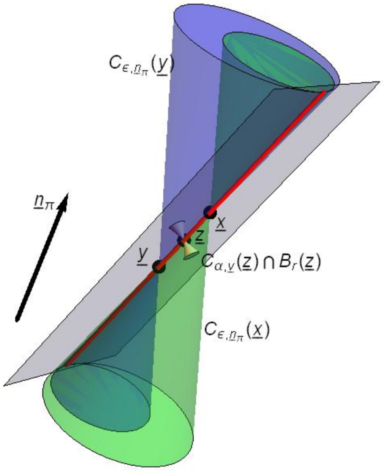

Fig. 1. Cones and points for $(\underline{x}, \underline{y}), (\underline{x}, \underline{z}), (\underline{y}, \underline{z}) \in \Gamma_\pi$

$\neq \emptyset$ for every $r > 0$. Let $r > 0$ be so small that $\mathrm{int}(B_r(\underline{z}) \cap C_{\alpha,\underline{v}}(\underline{z})) \subseteq \mathrm{int}(C_{\varepsilon_\pi,\underline{n}_\pi}(\underline{x}) \cap C_{\varepsilon_\pi,\underline{n}_\pi}(\underline{y}))$. Then

$$\emptyset \neq \mathrm{int}(B_r(\underline{z}) \cap C_{\alpha,\underline{v}}(\underline{z})) \cap \Lambda \subseteq \mathrm{int}(C_{\varepsilon_\pi,\underline{n}_\pi}(\underline{x})) \cap \Lambda.$$

But by (3.2), $\mathrm{int}(C_{\varepsilon_\pi,\underline{n}_\pi}(\underline{x})) \cap \Lambda = \emptyset$, a contradiction. ∎

Proposition 3.4. *Let Λ be an HSS set in $\mathbb{R}^3$ with $\dim_{\mathrm{H}} \Lambda > 1$ and not contained in any plane. Then there exists an orthogonal projection $\pi \in \Pi_{3,2}$ and a self-similar measure μ such that $\mathrm{spt}\,\mu \subseteq \Lambda$, $\mathrm{spt}\,\mu$ is not contained in any plane, $\pi\mu$ satisfies SSC (but not with respect to the IFS generating μ), and $\dim_{\mathrm{H}} \mu > \dim_{\mathrm{H}} \pi\mu > 1$.*

Proof. By Marstrand's projection theorem [16, Corollaries 9.4 and 9.8] there exists a $\pi_1 \in \Pi_{3,2}$ such that $\dim_{\mathrm{H}} \pi_1\Lambda = \min\{2, \dim_{\mathrm{H}} \Lambda\}$. The set $\pi_1\Lambda$ is an HSS set in $\mathbb{R}^2$. By Proposition 2.1, there exists an HHSS set Λ^1 and an IFS $\Phi_1 = \{f_i(\underline{x}) = \lambda\underline{x} + \underline{t}_i\}_{i=1}^q$ with SSC such that $\Lambda^1 \subseteq \Lambda$, $\pi_1\Lambda^1$ satisfies SSC and $2 > \dim_{\mathrm{H}} \Lambda^1 = \dim_{\mathrm{H}} \pi_1\Lambda^1 > 1$.

Let ε_{π_1} be defined in (3.1). By compactness, there are $\underline{x}_i \in f_i(\Lambda^1)$ and $\underline{y}_j \in f_j(\Lambda^1)$ such that $i \neq j$ and $(\underline{x}_i, \underline{y}_j) \in \Gamma_{\pi_1}$. Fix such $i \neq j$ and $\underline{x}_i, \underline{y}_j$. Denote by π_2 the projection onto the subspace with normal vector $(\underline{x}_i - \underline{y}_j)/\|\underline{x}_i - \underline{y}_j\|$. Then by Lemma 3.3, the projection π_2 is 2-to-1 on Λ^1. Thus, by [8, Corollary 4.16], $\dim_{\mathrm{H}} \Lambda^1 = \dim_{\mathrm{H}} \pi_2\Lambda^1$, but clearly SSC does not hold.

Let $\delta > 0$ be so small that $\dim_{\mathrm{H}} \Lambda^1 - 3\delta > 1$. Fix $\bar{\imath}, \bar{\jmath} \in \Sigma^*$ such that $|\bar{\imath}| = |\bar{\jmath}|$, $\underline{x}_i \in f_{\bar{\imath}}(\Lambda^1)$, $\underline{y}_j \in f_{\bar{\jmath}}(\Lambda^1)$, and $m := |\bar{\imath}| = |\bar{\jmath}|$ is so large that the attractor $\widetilde{\Lambda}$ of the IFS $\widetilde{\Phi} := \{f_{i_0} \circ \cdots \circ f_{i_{m-1}}\}_{i_0,\ldots,i_{m-1}=1}^q \setminus \{f_{\bar{\imath}}, f_{\bar{\jmath}}\}$ satisfies

$\dim_{\mathrm{H}} \widetilde{\Lambda} \geq \dim_{\mathrm{H}} \Lambda^1 - \delta$. Since π_2 is still at most 2-to-1 on the smaller set $\widetilde{\Lambda}$, we have $\dim_{\mathrm{H}} \widetilde{\Lambda} = \dim_{\mathrm{H}} \pi_2 \widetilde{\Lambda}$. Observe that $\pi_2 \underline{x}_i = \pi_2 \underline{y}_j \notin \pi_2 \widetilde{\Lambda}$.

Let $\widetilde{\mu}$ be the natural HSS measure on $\widetilde{\Lambda}$. By Theorem 2.11 (see (2.7)) the function $\pi \mapsto \dim_{\mathrm{H}} \pi \widetilde{\mu}$ is lower semicontinuous at π_2. Hence, $\pi \mapsto \dim_{\mathrm{H}} \pi \widetilde{\Lambda}$ is lower semicontinuous at π_2. Let $\beta > 0$ be small enough that for every projection $\pi \in \Pi_{3,2}$ with $\|\underline{n}_\pi \times \underline{n}_{\pi_2}\| < |\sin(\beta)|$, we have $\pi \underline{x}_i, \pi \underline{y}_j \notin \pi \widetilde{\Lambda}$ and

$$\dim_{\mathrm{H}} \pi \widetilde{\Lambda} \geq \dim_{\mathrm{H}} \pi_2 \widetilde{\Lambda} - \delta = \dim_{\mathrm{H}} \widetilde{\Lambda} - \delta.$$

Since the fixed points of the iterates of the functions are dense in $\widetilde{\Lambda}$, by compactness we may find $\hbar_1, \hbar_2 \in \Sigma^*$ such that $\pi(\mathrm{Fix}(f_{\bar{\imath}\hbar_1})), \pi(\mathrm{Fix}(f_{\bar{\jmath}\hbar_2})) \notin \pi \widetilde{\Lambda}$ for every projection $\pi \in \Pi_{3,2}$ with $\|\underline{n}_\pi \times \underline{n}_{\pi_2}\| < |\sin(\beta)|$ and

$$\frac{\|(\mathrm{Fix}(f_{\bar{\imath}\hbar_1}) - \mathrm{Fix}(f_{\bar{\jmath}\hbar_2})) \times (\underline{x}_i - \underline{y}_j)\|}{\|\mathrm{Fix}(f_{\bar{\imath}\hbar_1}) - \mathrm{Fix}(f_{\bar{\jmath}\hbar_2})\| \|\underline{x}_i - \underline{y}_j\|} < |\sin(\beta)|.$$

Denote by π' the projection onto the subspace with normal vector $(\mathrm{Fix}(f_{\bar{\imath}\hbar_1}) - \mathrm{Fix}(f_{\bar{\jmath}\hbar_2}))/\|\mathrm{Fix}(f_{\bar{\imath}\hbar_1}) - \mathrm{Fix}(f_{\bar{\jmath}\hbar_2})\|$. Applying Proposition 2.1 for $\pi' \widetilde{\Phi}$, we see that there exist an HHSS set $\widetilde{\Lambda}^1$ and an IFS $\widetilde{\Phi}_1$ with SSC such that $\widetilde{\Lambda}^1 \subseteq \widetilde{\Lambda}$, $\pi' \widetilde{\Lambda}^1$ satisfies SSC and $\dim_{\mathrm{H}} \widetilde{\Lambda}^1 = \dim_{\mathrm{H}} \pi' \widetilde{\Lambda}^1 > \dim_{\mathrm{H}} \pi' \widetilde{\Lambda} - \delta$.

We claim that there exist $m, k \geq 1$ such that the IFS

$$(\pi' \widetilde{\Phi}_1)^m \cup \Big\{ \overbrace{\pi' f_{\bar{\imath}\hbar_1} \circ \cdots \circ \pi' f_{\bar{\imath}\hbar_1}}^{k} =: \pi' f^k_{\bar{\imath}\hbar_1} \Big\}$$

satisfies SSC and is homogeneous.

Indeed, since $\pi' \widetilde{\Phi}_1$ satisfies SSC and is homogeneous, the same is true of $(\pi' \widetilde{\Phi}_1)^m$ for every $m \geq 1$. By Proposition 2.1, the contraction ratio of $\widetilde{\Phi}_1$ is λ^l for some $l \geq 1$. On the other hand, the contraction ratio of $f_{\bar{\imath}\hbar_1}$ is $\lambda^{|\bar{\imath}\hbar_1|}$. Now, let us fix the ratio $k/m = l/|\bar{\imath}\hbar_1|$. Since $\pi'(\mathrm{Fix}(f_{\bar{\imath}\hbar_1})), \pi'(\mathrm{Fix}(f_{\bar{\jmath}\hbar_2})) \notin \pi' \widetilde{\Lambda}$, we see that by choosing k sufficiently large, the SSC holds.

Let $\Phi' := (\widetilde{\Phi}_1)^m \cup \{f^k_{\bar{\imath}\hbar_1}, f^k_{\bar{\imath}\hbar_2}\}$. Observe that $\pi'(\mathrm{Fix}(f_{\bar{\imath}\hbar_1})) = \pi'(\mathrm{Fix}(f_{\bar{\jmath}\hbar_2}))$. Thus, $\pi' f_{\bar{\imath}\hbar_1} \equiv \pi' f_{\bar{\imath}\hbar_2}$, i.e. there are exact overlaps. Hence, $\pi' \Phi' = (\pi' \widetilde{\Phi}_1)^m \cup \{\pi' f^k_{\bar{\imath}\hbar_1}\}$, and therefore it satisfies SSC.

Let Λ' be the attractor of Φ'. Then

$$\begin{aligned}\dim_{\mathrm{H}} \pi' \Lambda' &\geq \dim_{\mathrm{H}} \pi' \widetilde{\Lambda}^1 \geq \dim_{\mathrm{H}} \pi' \widetilde{\Lambda} - \delta \\ &\geq \dim_{\mathrm{H}} \pi_2 \widetilde{\Lambda} - 2\delta = \dim_{\mathrm{H}} \widetilde{\Lambda} - 2\delta \geq \dim_{\mathrm{H}} \Lambda^1 - 3\delta > 1.\end{aligned}$$

Let μ' be the HHSS measure on Λ' with weights $1/(\sharp(\widetilde{\Phi}_1)^m + 1)$ for the functions in $(\widetilde{\Phi}_1)^m$ and weights $1/(2(\sharp(\widetilde{\Phi}_1)^m + 1))$ for $f^k_{\bar{\imath}\hbar_1}, f^k_{\bar{\imath}\hbar_2}$. Thus, $\pi' \mu'$ is the natural self-similar measure on $\pi' \Lambda'$, and therefore $1 < \dim_{\mathrm{H}} \pi' \Lambda' = \dim_{\mathrm{H}} \pi' \mu'$. Because of the exact overlap and the fact that $\operatorname{spt} \pi' \mu' = \pi' \Lambda'$ cannot be contained in a line, $\operatorname{spt} \mu'$ cannot be contained in a plane. The exact overlap and $\dim_H \mu' \leq \dim_H \Lambda^1 < 2$ imply $\dim_H \mu' > \dim_H \pi'\mu'$, which was to be proven. ∎

By changing coordinates, without loss of generality we may assume that the projection in Proposition 3.4 is the coordinate projection $\pi : (x, y, z) \mapsto (x, y)$. Moreover, since the measure μ in Proposition 3.4 cannot be contained in any plane, we may assume that $\operatorname{spt} \mu$ is supported on an octant, separated away from the z-axis by restricting μ to a cylinder set.

Denote by γ the projection along geodesics on S^2 to S^1. We note that γ is well defined except at the poles. On the other hand, $\gamma \circ P_3 = P_2 \circ \pi$.

Let $\nu := (P_3)_*\mu$. Thus, $\gamma_*\nu = (P_2)_*\pi\mu$. For convenience, we use the cylindrical coordinates in $\mathbb{R}^3$ and the radial coordinates on $\mathbb{R}^2$. That is, for $\mathbb{R}^3 \ni \underline{x} = (r, \varphi, z)$, $\pi(\underline{x}) = (r, \varphi)$, $P_2(\pi(\underline{x})) = \varphi = \gamma(P_3(\underline{x}))$. Denote the conditional measure of μ on $\pi^{-1}(r, \varphi)$ by $\mu_{\pi^{-1}(r,\varphi)}$, the conditional measure of $\pi\mu$ on $P_2^{-1}(\varphi)$ by $\pi\mu_{P_2^{-1}(\varphi)}$, and the conditional measure of ν on $\gamma^{-1}(\varphi)$ by $\nu_{\gamma^{-1}(\varphi)}$ (see Figure 2).

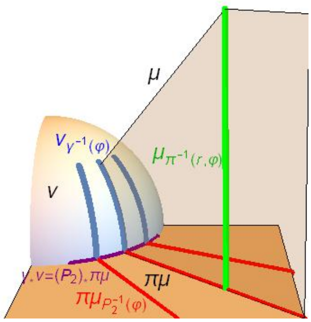

Fig. 2. The conditional and projected measures along P_2, P_3 and γ

Lemma 3.5. *For $\gamma_*\nu$-almost every $\varphi \in S^1$,*

$$\dim_H \nu_{\gamma^{-1}(\varphi)} \geq \dim_H \mu - \dim_H \pi\mu > 0.$$

Proof. By definition of conditional measures, $\nu = \int \nu_{\gamma^{-1}(\varphi)}\, d\gamma_*\nu(\varphi)$. On the other hand, $\pi\mu = \int \pi\mu_{P_2^{-1}(\varphi)}\, d\gamma_*\nu(\varphi)$, and thus $\mu = \int \mu_{\pi^{-1}(r,\varphi)}\, d\pi\mu(r, \varphi)$ $= \iint \mu_{\pi^{-1}(r,\varphi)}\, d\pi\mu_{P_2^{-1}(\varphi)}(r)\, d\gamma_*\nu(\varphi)$. Hence,

$$\nu = (P_3)_*\mu = \iint (P_3)_*\mu_{\pi^{-1}(r,\varphi)}\, d\pi\mu_{P_2^{-1}(\varphi)}(r)\, d\gamma_*\nu(\varphi).$$

Since the conditional measures are uniquely defined up to a measure zero set,

$$\nu_{\gamma^{-1}(\varphi)} = \int (P_3)_*\mu_{\pi^{-1}(r,\varphi)}\, d\pi\mu_{P_2^{-1}(\varphi)}(r) \quad \text{for } \gamma_*\nu\text{-almost every } \varphi. \tag{3.3}$$

Observe that for any compact line segment $I \subset \mathbb{R}^3$ which is not contained in any 1-dimensional subspace of $\mathbb{R}^3$ the map $P_3 : I \to S^2$ is bi-Lipschitz. Hence, by Theorem 2.11 and Proposition 3.4,

$$\begin{aligned}\dim_{\mathrm{H}} (P_3)_*\mu_{\pi^{-1}(r,\varphi)} &= \dim_{\mathrm{H}} \mu_{\pi^{-1}(r,\varphi)} \\ &= \dim_{\mathrm{H}} \mu - \dim_{\mathrm{H}} \pi\mu > 0 \quad \text{for } \pi\mu\text{-a.e. } (r,\varphi).\end{aligned}$$

By the definition of Hausdorff dimension, let $A_{\varphi,n}$ be the set such that $\nu_{\gamma^{-1}(\varphi)}(A_{\varphi,n}) > 0$ and $\dim_{\mathrm{H}} \nu_{\gamma^{-1}(\varphi)} \geq \dim_{\mathrm{H}} A_{\varphi,n} - 1/n$. Thus, by (3.3), for $\gamma_*\nu$-a.e. φ there exists a set $B_{\varphi,n}$ such that $\pi\mu_{P_2^{-1}(\varphi)}(B_{\varphi,n}) > 0$ and for $\pi\mu_{P_2^{-1}(\varphi)}$-a.e. $r \in B_{\varphi,n}$,

$$(P_3)_*\mu_{\pi^{-1}(r,\varphi)}(A_{\varphi,n}) > 0.$$

Hence,

$$\begin{aligned}\dim_{\mathrm{H}} \nu_{\gamma^{-1}(\varphi)} + 1/n \geq \dim_{\mathrm{H}} A_{\varphi,n} &\geq \dim_{\mathrm{H}} (P_3)_*\mu_{\pi^{-1}(r,\varphi)} \\ &= \dim_{\mathrm{H}} \mu - \dim_{\mathrm{H}} \pi\mu > 0 \quad \text{for } \gamma_*\nu\text{-a.e. } \varphi.\end{aligned}$$

Since n was arbitrary, the proof is complete. ∎

Proof of Theorem 3.1. Let μ and π be as in Proposition 3.4. Since $\pi\mu$ is an HHSS measure satisfying SSC, Corollary 2.6 yields

$$\dim_{\mathrm{H}} \gamma_*\nu = \dim_{\mathrm{H}} (P_2)_*\pi\mu = \min\{1, \dim_{\mathrm{H}} \pi\mu\} = 1.$$

By Lemma 3.5,

$$\dim_{\mathrm{H}} \nu_{\gamma^{-1}(\varphi)} \geq \dim_{\mathrm{H}} \mu - \dim_{\mathrm{H}} \pi\mu > 0.$$

Thus, by [10, Lemma 6.13],

$$\dim_{\mathrm{H}} (P_3)_*\mu = \dim_{\mathrm{H}} \nu \geq \dim_{\mathrm{H}} \gamma_*\nu + \dim_{\mathrm{H}} \nu_{\gamma^{-1}(\varphi)} > 1. \ ∎$$

4. Proofs of the main theorems

Proof of Theorem 1.1. Let Λ be an HSS set in $\mathbb{R}^3$ not contained in any plane and with $\dim_{\mathrm{H}} \Lambda > 1$. Moreover, let $g : \mathbb{R}^3 \to \mathbb{R}$ be a C^1 function satisfying assumptions (1)–(3). Since Λ is compact, there exists an open neighbourhood of Λ on which $\|\nabla_{\underline{x}} g\| > 0$. By considering a sufficiently small cylinder of Λ, we may assume that there exists a ball B such that $\Lambda \subseteq B$ and $\|\nabla_{\underline{x}} g\| > 0$ for every $\underline{x} \in B$. Let $f : \underline{x} \in B \mapsto \nabla_{\underline{x}} g / \|\nabla_{\underline{x}} g\|$. By assumption (2), $f(\underline{x}) = f(t \cdot \underline{x})$ for every $t \in \mathbb{R}$ such that $t \cdot \underline{x} \in B$. Thus, by assumption (3), for any HSS measure μ with $\operatorname{spt} \mu \subseteq \Lambda$,

$$\dim_{\mathrm{H}} f_*\mu = \dim_{\mathrm{H}} (P_3)_*\mu.$$

It is enough to show the lower bound. Let μ be the HHSS measure as in Theorem 3.1. Then by Theorem 2.3,

$$\dim_{\mathrm{H}} g(\Lambda) \geq \dim_{\mathrm{H}} g_*\mu \geq \mu - \operatorname*{ess\,inf}_{\underline{x}} \dim_{\mathrm{H}} \pi_{g,\underline{x}}\mu,$$

where we recall that $\pi_{g,\underline{x}}(\underline{y}) = \langle \nabla_{\underline{x}} g, \underline{y}\rangle / \|\nabla_{\underline{x}} g\|$. By Proposition 2.10,

$$\dim_{\mathrm{H}}\{\underline{n} \in S^2 : \dim_{\mathrm{H}} \pi_{\underline{n}}\mu < 1\} \leq 1.$$

But by Theorem 3.1, $\dim_{\mathrm{H}} f_*\mu = \dim_{\mathrm{H}} (P_3)_*\mu > 1$, thus

$$f_*\mu(\{\underline{n} \in S^2 : \dim_{\mathrm{H}} \pi_{\underline{n}}\mu < 1\}) = 0,$$

and therefore $\mu - \operatorname{ess\,inf}_{\underline{x}} \dim_{\mathrm{H}} \pi_{g,\underline{x}}\mu = 1$. ∎

Proof of Theorem 1.2. If Λ is contained in a plane then we refer to Corollary 2.8 or [17, Theorem 1.2]. So we may assume that Λ is not contained in any plane.

By shifting Λ we may assume that $\underline{0} \in \Lambda$. Let $\Lambda_{\bar{\imath}}$ be a cylinder set such that $\mathrm{dist}(\underline{0}, \Lambda_{\bar{\imath}}) > 0$. Then $g(\underline{x}) := \|\underline{x}\|$ satisfies the conditions of Theorem 1.1 with self-similar set $\Lambda_{\bar{\imath}}$. Thus,

$$\begin{aligned}\dim_{\mathrm{H}} D_{\underline{0}}(\Lambda) &\geq \dim_{\mathrm{H}} g(\Lambda) \geq \dim_{\mathrm{H}} g(\Lambda_{\bar{\imath}}) \\ &= \min\{1, \dim_{\mathrm{H}} \Lambda_{\bar{\imath}}\} = \min\{1, \dim_{\mathrm{H}} \Lambda\} = 1. \ \blacksquare\end{aligned}$$

Proof of Theorem 1.3. Let Λ correspond to an IFS $\{\lambda_i x + t_i\}_{i=1}^{q}$, where $\lambda_i \in (-1,1)$. By [18, Proposition 6] there exists a self-similar set $\Lambda' \subseteq \Lambda$ in $\mathbb{R}$ such that $\dim_{\mathrm{H}} \Lambda' > 1/3$ with IFS $\{\lambda x + t_i'\}_{i=1}^{q'}$, where $\lambda \in (0,1)$. Since Λ' is not a singleton, there exists a cylinder set $\Lambda_{\bar{\imath}}'$ with each element either strictly positive or strictly negative.

It is easy to see that $\Lambda_{\bar{\imath}}' \times \Lambda_{\bar{\imath}}' \times \Lambda_{\bar{\imath}}'$ is an HSS set in $\mathbb{R}^3$ separated away from the planes determined by the axes. Thus it is contained in one of the octants. Moreover, $\dim_{\mathrm{H}} \Lambda_{\bar{\imath}}' \times \Lambda_{\bar{\imath}}' \times \Lambda_{\bar{\imath}}' > 1$.

Let $g(x,y,z) = xyz$. Then

$$\nabla_{\underline{x}} g = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}.$$

It is easy to see that $\nabla_{\underline{x}} g$ satisfies assumptions (1) and (2) of Theorem 1.1 on $\Lambda_{\bar{\imath}}' \times \Lambda_{\bar{\imath}}' \times \Lambda_{\bar{\imath}}'$.

To show that g satisfies (3), observe that there exists an open, simply connected set V in S^2 such that $P_3(\Lambda_{\bar{\imath}}') \subset V$ and V is uniformly separated away from the planes $x = 0$, $y = 0$, $z = 0$. Since $\underline{x} \mapsto \nabla_{\underline{x}} g$ is one-to-one on every open octant and $\det(H_{\underline{x}} g) = 2xyz \neq 0$ for any $(x,y,z) \in V$, where $H_{\underline{x}} g$ denotes the Hesse matrix of g, we see that $\underline{x} \mapsto \nabla_{\underline{x}} g$ is a diffeomorphism between V and its image $\nabla_V g$. Now, let $(\varphi, \theta) \mapsto \underline{x}(\varphi, \theta)$ be the natural parametrization of V. Thus,

$$\left\langle \frac{\partial}{\partial\varphi}\nabla_{\underline{x}} g \times \frac{\partial}{\partial\theta}\nabla_{\underline{x}} g, \nabla_{\underline{x}} g \right\rangle = \left\langle \det(H_{\underline{x}} g)((H_{\underline{x}} g)^T)^{-1} \frac{\partial \underline{x}}{\partial\varphi} \times \frac{\partial \underline{x}}{\partial\theta}, \nabla_{\underline{x}} g \right\rangle$$
$$= \left\langle \frac{\partial \underline{x}}{\partial\varphi} \times \frac{\partial \underline{x}}{\partial\theta}, \det(H_{\underline{x}} g)((H_{\underline{x}} g))^{-1}\nabla_{\underline{x}} g \right\rangle = \frac{1}{2}\det(H_{\underline{x}} g)\left\langle \frac{\partial \underline{x}}{\partial\varphi} \times \frac{\partial \underline{x}}{\partial\theta}, \underline{x} \right\rangle \neq 0$$

for every $\underline{x} \in V$. Hence, the normal vector of S^2 at $\nabla_{\underline{x}} g/\|\nabla_{\underline{x}} g\|$ is uniformly transversal to the normal vector of $\nabla_V g$ at $\nabla_{\underline{x}} g$. Thus, P_3 is a diffeomorphism between $\nabla_V g$ and $P_3(\nabla_V g)$, and therefore h_g is bi-Lipschitz.

Thus, by Theorem 1.1,

$$\dim_{\mathrm{H}} \Lambda\cdot\Lambda\cdot\Lambda \geq \dim_{\mathrm{H}} \Lambda'_{\bar\imath}\cdot\Lambda'_{\bar\imath}\cdot\Lambda'_{\bar\imath} = \dim_{\mathrm{H}} g(\Lambda'_{\bar\imath}\times\Lambda'_{\bar\imath}\times\Lambda'_{\bar\imath}) = 1. \ \blacksquare$$

Remark. Unfortunately, our method does not allow us to prove similar statements if $\dim_{\mathrm{H}} \Lambda \leq 1$. The method depends on the dimension of the exceptional directions of orthogonal projections from $\mathbb{R}^3$ to $\mathbb{R}$. By using Hochman's Theorem 2.3, we get

$$\dim_{\mathrm{H}} g_*\mu \geq \mu - \operatorname*{ess\,inf}_{\underline{x}} \dim_{\mathrm{H}} \pi_{g,\underline{x}}\mu.$$

On the other hand, in the case of self-similar sets,

$$\dim_{\mathrm{H}}\{\pi \in \Pi_{3,1} : \dim_{\mathrm{H}} \pi\Lambda < \min\{1, \dim_{\mathrm{H}} \Lambda\}\} \leq 1$$

(see Theorem 2.10). Hence, to prove that the dimension does not drop, it is enough to show that $\dim_{\mathrm{H}} f_*\mu > 1$, where $f : \underline{x} \mapsto P_3(\nabla_{\underline{x}} g)$. However, this is not possible if $\dim_{\mathrm{H}} \Lambda \leq 1$ and in particular if $\dim_{\mathrm{H}} \mu \leq 1$.

Remark. Conditions (2) and (3) in Theorem 1.1 imply that we have to check only that $\dim_{\mathrm{H}} (P_3)_*\mu > 1$. These conditions seem rather technical, and we conjecture that they can be replaced by some more natural condition.

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