STUDIA MATHEMATICA 233 (1) (2016)

Amenability properties of Figà-Talamanca–Herz algebras on inverse semigroups

by

Hasan Pourmahmood-Aghababa (Tabriz and Tehran)

Abstract. This paper continues the joint work with A. R. Medghalchi (2012) and the author's recent work (2015). For an inverse semigroup S, it is shown that $\mathrm{A}_p(S)$ has a bounded approximate identity if and only if $l^1(S)$ is amenable (a generalization of Leptin's theorem) and that $\mathrm{A}(S)$, the Fourier algebra of S, is operator amenable if and only if $l^1(S)$ is amenable (a generalization of Ruan's theorem).

1. Introduction. A discrete semigroup S is called an *inverse semigroup* if for each $s \in S$ there is a unique element $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. An element $e \in S$ is called an *idempotent* if $e^2 = e = e^*$. The set of idempotents is denoted by $E(S)$.

In [MP], we extended the Figà-Talamanca–Herz algebras $\mathrm{A}_p(G)$ introduced by Herz [Her] (on locally compact groups G) to algebras $\mathrm{A}_p(S)$ on inverse semigroups. Then we studied pseudomeasures and pseudofunctions on inverse semigroups in [Pou]. Here we continue this work, investigating some properties of $\mathrm{A}_p(S)$ such as amenability-like properties and existence of a bounded approximate identity. Indeed, after giving some preliminaries in Section 2, we obtain generalizations of Leptin's theorem and Ruan's theorem in Section 3. The Leptin theorem asserts that for a locally compact group G, the Figà-Talamanca–Herz algebra $\mathrm{A}_p(G)$ has a bounded approximate identity if and only if G is amenable; by Johnson's theorem, this is equivalent to the amenability of $L^1(G)$. Also, the celebrated theorem of Ruan states that the operator amenability of the Fourier algebra $\mathrm{A}(G)$ is equivalent to the amenability of G and hence to the amenability of the group algebra $L^1(G)$. We extend these two theorems to (discrete) inverse semigroups. Indeed, we show that $\mathrm{A}_p(S)$ has a bounded approximate identity if and only if the semigroup algebra $l^1(S)$ is amenable if and only if $\mathrm{A}(S)$ is operator amenable.

2010 *Mathematics Subject Classification*: Primary 43A15; Secondary 20M18.

Key words and phrases: Figà-Talamanca–Herz algebras, amenability, semigroup algebras.
Received 14 March 2015; revised 3 April 2016.
Published online 5 May 2016.

DOI: 10.4064/sm8250-4-2016

The amenability of $\mathrm{A}_p(S)$ is also characterized in the sense of amenability of Figà-Talamanca–Herz algebras of maximal subgroups of S. Furthermore, we show that the character space of $\mathrm{A}_p(S)$ is equal to S (extension of [MP, Theorem 6.3]).

2. Preliminaries. Let S be an inverse semigroup and let $l^1(S)$ denote its semigroup algebra. We also consider the restricted semigroup algebra $l^1_r(S) = (l^1(S), \bullet, \sim)$, defined in [AM]. We recall that for $f \in l^1_r(S)$ the functions $\check{f}$ and $\tilde{f}$ are elements of $l^1_r(S)$ defined by $\check{f}(s) = f(s^*)$ and $\tilde{f}(s) = \overline{f(s^*)}$. Also the action $\bullet$ on $l^1_r(S)$ is given by

$$\Big(\sum_{s\in S} a_s\delta_s\Big) \bullet \Big(\sum_{t\in S} b_t\delta_t\Big) = \sum_{s,t\in S,\, tt^*=s^*s} a_s b_t \delta_{st} \quad \Big(\sum_{s\in S} a_s\delta_s, \sum_{t\in S} b_t\delta_t \in l^1_r(S)\Big).$$

It is proved in [AM] that $l^1_r(S)$ is a Banach $*$-algebra with approximate identity.

The linear span of $\{\delta_t : t \in S\}$ is denoted by $F(S)$. In fact, $F(S)$ consists of all finite support functions in $l^1(S)$. Also $F(S)_+$ denotes the space of non-negative functions in $F(S)$.

Let S be an inverse semigroup and let $p, q \in (1,\infty)$ be such that $1/p + 1/q = 1$. The *Figà-Talamanca–Herz algebra* of S, introduced in [MP], is denoted by $\mathrm{A}_p(S)$. It consists of those $\mathbf{u} \in c_0(S)$ such that there are sequences $(f_n)_{n=1}^\infty \subseteq l^p(S)$ and $(g_n)_{n=1}^\infty \subseteq l^q(S)$ with $\sum_{n=1}^\infty \|f_n\|_p\|g_n\|_q < \infty$ and $\mathbf{u} = \sum_{n=1}^\infty f_n \bullet \check{g}_n$. The norm of $\mathbf{u} \in \mathrm{A}_p(S)$ is

$$\|\mathbf{u}\|_{\mathrm{A}_p} = \inf\Big\{\sum_{n=1}^\infty \|f_n\|_p\|g_n\|_q : \mathbf{u} = \sum_{n=1}^\infty f_n \bullet \check{g}_n\Big\}.$$

By [MP, Theorem 3.6], $(\mathrm{A}_p(S), \|\cdot\|_{\mathrm{A}_p})$ is a Banach algebra under pointwise multiplication. The space $\mathrm{A}(S) := \mathrm{A}_2(S)$ is called the *Fourier algebra* of S.

By [MP, Proposition 3.2], for each $1 < p < \infty$, $F(S)$ is a dense subset of $\mathrm{A}_p(S)$. For each $D \subseteq S$ we define

$$\mathrm{A}_p(D) = \{\mathbf{u}|_D : \mathbf{u} \in \mathrm{A}_p(S)\},$$

with the induced quotient norm, i.e.,

$$\|\mathbf{u}|_D\|_{\mathrm{A}_p(D)} = \inf\{\|\mathbf{v}\|_{\mathrm{A}_p} : \mathbf{v} \in \mathrm{A}_p(S),\, \mathbf{v}|_D = \mathbf{u}|_D\} \quad (\leq \|\mathbf{u}\|_{\mathrm{A}_p}).$$

In other words,

$$\|\mathbf{u}|_D\|_{\mathrm{A}_p(D)} = \inf\Big\{\sum_{n=1}^\infty \|f_n\|_p\|g_n\|_q : \mathbf{u} = \sum_{n=1}^\infty f_n \bullet \check{g}_n \text{ on } D\Big\}.$$

Clearly $\mathrm{A}_p(D)$ is a Banach algebra under pointwise multiplication.

We remark that for $f \in l^q(S)$, $g \in l^p(S)$ and $u \in S$, by [MP, (3.3) and (3.4)], we have

$$f \bullet \check{g}(u) = \sum_{t\in S,\, tt^*=u^*u} f(ut)g(t) = \sum_{s\in S,\, ss^*=uu^*} f(s)\check{g}(s^*u). \tag{2.1}$$

For an inverse semigroup S, a linear functional $\mathbf{m} \in l^\infty(S)^*$ is called an *invariant mean* if $\langle \mathbf{1}, \mathbf{m}\rangle = \|\mathbf{m}\| = 1$ and $\mathbf{m}(l_s f) = \mathbf{m}(f)$ for all $f \in l^\infty(S)$ and $s \in S$, where $\mathbf{1}$ denotes the constant unit function on S and $l_s f(t) = f(st)$ for all $t \in S$. The semigroup S is termed *amenable* if there exists an invariant mean on $l^\infty(S)$.

Let A be a Banach algebra and X a Banach A-bimodule. Then X^*, the Banach space dual of X, is also a Banach A-bimodule. A *derivation* from A into X is a bounded linear map satisfying

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in A).$$

For $x \in X$ we denote by ad_x the derivation $\mathrm{ad}_x(a) = a \cdot x - x \cdot a$ for all $a \in A$, which is called an *inner derivation*. We denote by $\mathcal{Z}^1(A, X)$ the space of all derivations from A into X and by $\mathcal{B}^1(A, X)$ the space of all inner derivations from A into X. The *first cohomology group* of A with coefficients in X, denoted by $\mathcal{H}^1(A, X)$, is the quotient space $\mathcal{Z}^1(A, X)/\mathcal{B}^1(A, X)$.

A Banach algebra A is *amenable* if $\mathcal{H}^1(A, X^*) = \{0\}$ for every Banach A-bimodule X, and *contractible* if $\mathcal{H}^1(A, X) = \{0\}$ for each Banach A-bimodule X.

For two Banach spaces X and Y we denote by $X \widehat{\otimes} Y$ their projective tensor product and by $\|\cdot\|_\wedge$ its projective norm. We let $\mathcal{B}(X)$ be the space of all bounded operators on X. Throughout, we use "$\cong$" to denote isometric isomorphism of Banach spaces, algebras, modules, etc. while "$\simeq$" denotes isomorphism with equivalent norms, but not necessarily isometric.

3. Generalizations of Leptin's and Ruan's theorems. Let S be a semigroup. By [How, Chapter 2], there is an equivalence relation $\mathcal{D}$ on S defined by $s\mathcal{D}t$ if and only if there exists $x \in S$ such that

$$Ss \cup \{s\} = Sx \cup \{x\} \quad \text{and} \quad tS \cup \{t\} = xS \cup \{x\}.$$

If S is an inverse semigroup, then by [How, Proposition 5.1.2(4)], $s\mathcal{D}t$ if and only if there exists $x \in S$ such that

$$s^*s = xx^* \quad \text{and} \quad t^*t = x^*x. \tag{3.1}$$

For more details on the relation $\mathcal{D}$ see [MP, Section 4].

Now let S be an inverse semigroup and D a $\mathcal{D}$-class of S. For each $e, f \in E(D)$ we set

$$D_{e,f} = \{s \in D : ss^* = e,\ s^*s = f\}, \qquad D_e = \bigcup_{f\in E(D)} D_{e,f} = \{s \in D : ss^* = e\}.$$

Our first aim is to show that the existence of a bounded approximate identity in $\mathrm{A}_p(S)$ implies finiteness of $E(S)$. We show this via several lemmata. In the lemmata we assume that S is an inverse semigroup, D is a $\mathcal{D}$-class of S and $e, e', f, f' \in E(D)$.

LEMMA 3.1. *$l^p(D_{e,f}) \bullet l^q(D_{e',f'})^\vee$ is a subset of $\mathrm{A}_p(D_{e,e'})$ if $f = f'$, and it is $\{0\}$ otherwise.*

Proof. Let $h \in l^p(D_{e,f})$ and $g \in l^q(D_{e',f'})$. Then

$$h \bullet \check{g}(u) = \sum_{s\in S,\, ss^*=uu^*} h(s)\check{g}(s^*u) = \sum_{s\in D_{e,f},\, u^*s\in D_{e',f'},\, uu^*=e} h(s)g(u^*s).$$

Since

$$e' = (u^*s)(u^*s)^* = u^*eu = u^*(uu^*)u = u^*u,$$

and

$$f' = (u^*s)^*(u^*s) = s^*(uu^*)s = s^*es = s^*(ss^*)s = s^*s = f,$$

we have

$$h \bullet \check{g}(u) = \begin{cases} \displaystyle\sum_{s\in D_{e,f},\, u^*s\in D_{e',f}} h(s)g(u^*s) & \text{if } u \in D_{e,e'} \text{ and } f = f', \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $h \bullet \check{g} \in \mathrm{A}_p(D_{e,e'})$ if $f = f'$ and $h \bullet \check{g} = 0$ otherwise. ■

LEMMA 3.2. *$\mathrm{A}_p(D_{e,e'}) = l^p(D_e) \bullet l^q(D_{e'})^\vee$, that is,*

$$\mathrm{A}_p(D_{e,e'}) = \Big\{\sum_{n=1}^{\infty} f_n \bullet \check{g}_n : f_n \in l^p(D_e),\, g_n \in l^q(D_{e'}),\, \sum_{n=1}^{\infty} \|f_n\|_p\|g_n\|_q < \infty\Big\}.$$

Proof. Using Lemma 3.1 we have

$$\begin{aligned} l^p(D_e) \bullet l^q(D_{e'})^\vee &= \Big(\bigoplus_{f\in E(D)} l^p(D_{e,f})\Big) \bullet \Big(\bigoplus_{f'\in E(D)} l^q(D_{e',f'})\Big)^\vee \\ &= \bigoplus_{f,f'\in E(D)} \big(l^p(D_{e,f}) \bullet l^q(D_{e',f'})^\vee\big) \subseteq \mathrm{A}_p(D_{e,e'}). \end{aligned}$$

Now let $u \in \mathrm{A}_p(D_{e,e'})$ and $\tilde{u} \in \mathrm{A}_p(D)$ be such that $\tilde{u}|_{D_{e,e'}} = u$. Let $\tilde{u} = \sum_{n=1}^{\infty} h_n \bullet \check{g}_n$, where $(h_n) \subseteq l^p(D)$, $(g_n) \subseteq l^q(D)$ and $\sum_{n=1}^{\infty} \|h_n\|_p\|g_n\|_q < \infty$. Since $D = \bigcup_{e\in E(D)} D_e$, for each $n \in \mathbb{N}$ we have $h_n = \sum_{e\in E(D)} h_{n,e}$ and

$g_n = \sum_{e\in E(D)} g_{n,e}$, where $h_{n,e} \in l^p(D_e)$ and $g_{n,e} \in l^q(D_e)$. Hence

$$\begin{aligned} u_1 &= \sum_{n=1}^{\infty} h_n \bullet \check{g}_n = \sum_{n=1}^{\infty}\Big(\sum_{e\in E(D)} h_{n,e}\Big) \bullet \Big(\sum_{e'\in E(D)} \check{g}_{n,e'}\Big) \\ &= \sum_{e,e'\in E(D)} \sum_{n=1}^{\infty} h_{n,e} \bullet \check{g}_{n,e'} \\ &\in \bigoplus_{e,e'\in E(D)} l^p(D_e) \bullet l^q(D_{e'})^{\vee} \subseteq \bigoplus_{e,e'\in E(D)} \mathrm{A}_p(D_{e,e'}), \end{aligned}$$

which means $u = u_1|_{D_{e,e'}} \in l^p(D_e) \bullet l^q(D_{e'})^{\vee}$. ∎

LEMMA 3.3. *For each $u \in \mathrm{A}_p(D_{e,e'})$ there is $\tilde{u} \in \mathrm{A}_p(D)$ such that $\tilde{u} = u$ on $D_{e,e'}$ and $\tilde{u} = 0$ on $D \setminus D_{e,e'}$.*

Proof. Let $u_1 \in \mathrm{A}_p(D)$ be as in the proof of Lemma 3.2. Since

$$u_1 = \sum_{n=1}^{\infty} h_n \bullet \check{g}_n = \sum_{e,e'\in E(D)} \sum_{n=1}^{\infty} h_{n,e} \bullet \check{g}_{n,e'} \in \bigoplus_{e,e'\in E(D)} \mathrm{A}_p(D_{e,e'}),$$

we obtain

$$u = u_1|_{D_{e,e'}} = \sum_{n=1}^{\infty} h_{n,e} \bullet \check{g}_{n,e'} \in \mathrm{A}_p(D_{e,e'}).$$

Let $\tilde{h}_{n,e} \in l^p(D)$ and $\tilde{g}_{n,e'} \in l^q(D)$ be extensions of $h_{n,e}$ and $g_{n,e'}$ to D, respectively, by assuming $\tilde{h}_{n,e} = 0$ on $D \setminus D_e$ and $\tilde{g}_{n,e'} = 0$ on $D \setminus D_{e'}$. Now set $\tilde{u} = \sum_{n=1}^{\infty} \tilde{h}_{n,e} \bullet \check{\tilde{g}}_{n,e'}$. ∎

LEMMA 3.4. *If $\tilde{u} \in \mathrm{A}_p(D)$ is an extension of $u \in \mathrm{A}_p(D_{e,e'})$ with $\tilde{u} = 0$ on $D \setminus D_{e,e'}$, then $\|\tilde{u}\|_{\mathrm{A}_p(D)} = \|u\|_{\mathrm{A}_p(D_{e,e'})}$.*

Proof. If $\tilde{u} = \sum_{n=1}^{\infty} h_n \bullet \check{g}_n$ with $\sum_{n=1}^{\infty} \|h_n\|_p \|g_n\|_q < \infty$, then as in the proof of Lemma 3.3,

$$\tilde{u} = \sum_{n=1}^{\infty} h_n \bullet \check{g}_n = \sum_{e,e'\in E(D)} \sum_{n=1}^{\infty} h_{n,e} \bullet \check{g}_{n,e'} \in \bigoplus_{e,e'\in E(D)} \mathrm{A}_p(D_{e,e'}).$$

On the other hand, the condition $\tilde{u}|_{D\setminus D_{e,e'}} = 0$ implies $\tilde{u} = \sum_{n=1}^{\infty} \tilde{h}_{n,e} \bullet \check{\tilde{g}}_{n,e'}$. Now using the inequality $\sum_{n=1}^{\infty} \|\tilde{h}_{n,e}\|_p \|\tilde{g}_{n,e'}\|_q \le \sum_{n=1}^{\infty} \|h_n\|_p \|g_n\|_q$ we have

$$\begin{aligned} \|\tilde{u}\|_{\mathrm{A}_p(D)} &= \inf\Big\{\sum_{n=1}^{\infty} \|h_n\|_p \|g_n\|_q : \tilde{u} = \sum_{n=1}^{\infty} h_n \bullet \check{g}_n\Big\} \\ &= \inf\Big\{\sum_{n=1}^{\infty} \|\tilde{h}_{n,e}\|_p \|\tilde{g}_{n,e'}\|_q : \tilde{u} = \sum_{n=1}^{\infty} \tilde{h}_{n,e} \bullet \check{\tilde{g}}_{n,e'}\Big\} \end{aligned}$$

$$= \inf\Big\{\sum_{n=1}^{\infty} \|h_{n,e}\|_p \|g_{n,e'}\|_q : u = \sum_{n=1}^{\infty} h_{n,e} \bullet \check{g}_{n,e'}\Big\}$$
$$= \|u\|_{\mathrm{A}_p(D_{e,e'})}. \ \blacksquare$$

LEMMA 3.5. *There is a norm decreasing epimorphism*

$$\varphi : \mathrm{A}_p(D) \to l^1\text{-}\bigoplus_{e\in E(D)} \mathrm{A}_p(D_{e,e}).$$

Proof. For each $u \in \mathrm{A}_p(D)$ define $\varphi(u) = (u_{e,e})$, where $u_{e,e} = u|_{D_{e,e}}$. Clearly φ is linear. We show that it is onto. Let

$$u = (u_e) \in l^1\text{-}\bigoplus_{e\in E(D)} \mathrm{A}_p(D_{e,e}).$$

Using Lemma 3.4, let $\tilde{u}_e \in \mathrm{A}_p(D)$ be an extension of u_e on D with $\tilde{u}_e = 0$ on $D \setminus D_{e,e}$ and $\|\tilde{u}_e\|_{\mathrm{A}_p(D)} = \|u_e\|_{\mathrm{A}_p(D_{e,e})}$. Set $\tilde{u} = \sum_{e\in E(D)} \tilde{u}_e$. Since $\mathrm{A}_p(D)$ is a Banach space and

$$\sum_{e\in E(D)} \|\tilde{u}_e\|_{\mathrm{A}_p(D)} = \sum_{e\in E(D)} \|u_e\|_{\mathrm{A}_p(D_{e,e})} = \|u\| < \infty,$$

the series $\sum_{e\in E(D)} \tilde{u}_e$ converges in $\mathrm{A}_p(D)$, that is, $\tilde{u} \in \mathrm{A}_p(D)$. Now it is clear that $\varphi(\tilde{u}) = u$.

In order to see that φ is norm decreasing, let $u \in \mathrm{A}_p(D)$. For $e, e' \in E(D)$ let $u_{e,e'} = u|_{D_{e,e'}}$. If $\tilde{u}_{e,e'} \in \mathrm{A}_p(D)$ is an extension of $u_{e,e'}$ with $\tilde{u}_{e,e'} = 0$ on $D \setminus D_{e,e'}$, then clearly $u = \sum_{e,e'\in E(D)} \tilde{u}_{e,e'}$. Now let $u = \sum_{n=1}^{\infty} h_n \bullet \check{g}_n$, where $(h_n) \subseteq l^p(D)$, $(g_n) \subseteq l^q(D)$ and $\sum_{n=1}^{\infty} \|h_n\|_p \|g_n\|_q < \infty$. Then by Lemma 3.1, we have

$$u = \sum_{n=1}^{\infty} h_n \bullet \check{g}_n = \sum_{n=1}^{\infty} \sum_{e,e',f\in E(D)} h_{n,e,f} \bullet \check{g}_{n,e',f}$$
$$= \sum_{e,e'\in E(D)} \Big(\sum_{f\in E(D)} \sum_{n=1}^{\infty} h_{n,e,f} \bullet \check{g}_{n,e',f}\Big) \in \bigoplus_{e,e'\in E(D)} \mathrm{A}_p(D_{e,e'}),$$

and

$$u_{e,e'} = \sum_{f\in E(D)} \sum_{n=1}^{\infty} h_{n,e,f} \bullet \check{g}_{n,e',f},$$

where $h_n = \sum_{e,f\in E(D)} h_{n,e,f}$, $g_n = \sum_{e,f\in E(D)} g_{n,e,f}$, $h_{n,e,f} \in l^p(D_{e,f})$ and $g_{n,e,f} \in l^q(D_{e,f})$. So $u_{e,e} = \sum_{f\in E(D)} \sum_{n=1}^{\infty} h_{n,e,f} \bullet \check{g}_{n,e,f}$ and by Hölder's inequality,

$$\begin{aligned}\sum_{e\in E(D)} \|u_{e,e}\|_{\mathrm{A}_p(D_{e,e})} &\le \sum_{n=1}^{\infty} \sum_{e,f\in E(D)} \|h_{n,e,f}\|_p \|g_{n,e,f}\|_q \\ &\le \sum_{n=1}^{\infty} \Big(\sum_{e,f\in E(D)} \|h_{n,e,f}\|_p^p \Big)^{1/p} \Big(\sum_{e,f\in E(D)} \|g_{n,e,f}\|_q^q \Big)^{1/q} \\ &= \sum_{n=1}^{\infty} \|h_n\|_p \|g_n\|_q,\end{aligned}$$

which implies

$$\|\varphi(u)\| = \sum_{e\in E(D)} \|u_{e,e}\|_{\mathrm{A}_p(D_{e,e})} \le \|u\|_{\mathrm{A}_p(D)}. \ \blacksquare$$

LEMMA 3.6. *If* $\mathrm{A}_p(D)$ *has a bounded approximate identity, then* $E(D)$ *is finite.*

Proof. If $\mathrm{A}_p(D)$ has a bounded approximate identity, by Lemma 3.5 so does $l^1\text{-}\bigoplus_{e\in E(D)} \mathrm{A}_p(D_{e,e})$. Now it follows from [MP, Lemma 6.4] that $E(D)$ must be finite. ■

THEOREM 3.7. *Let* S *be an inverse semigroup and* $1 < p < \infty$. *Then* $E(S)$ *is finite provided that* $\mathrm{A}_p(S)$ *has a bounded approximate identity.*

Proof. Let $\{D_\lambda : \lambda \in \Lambda\}$ be the family of $\mathcal{D}$-classes of S indexed by some set Λ. Then by [MP, equation (4.2)],

$$\mathrm{A}_p(S) \cong l^1\text{-}\bigoplus_{\lambda\in\Lambda} \mathrm{A}_p(D_\lambda),$$

where the right hand side is a commutative Banach algebra with componentwise product. It follows from [MP, Lemma 6.4] that Λ is finite and each $\mathrm{A}_p(D_\lambda)$ has a bounded approximate identity. Now by Lemma 3.6, $E(S) = \bigcup_{\lambda\in\Lambda} E(D_\lambda)$ is finite. ■

Let G be a group with identity e, and let I be a non-empty set. Then the *Brandt inverse semigroup* corresponding to G and I, denoted by $\mathcal{M}^0(G,I)$, is the collection of all $I \times I$ matrices $(g)_{ij}$ with $g \in G$ in the (i,j) entry and zero elsewhere and the $I \times I$ zero matrix 0. Multiplication in $\mathcal{M}^0(G,I)$ is given by the formula

$$(g)_{ij}(h)_{kl} = \begin{cases} (gh)_{il} & \text{if } j = k \\ 0 & \text{if } j \ne k \end{cases} \qquad (g,h \in G,\ i,j,k,l \in I),$$

and $(g)_{ij}{}^* = (g^{-1})_{ji}$ and $0^* = 0$. The set of all idempotents is

$$E(\mathcal{M}^0(G,I)) = \{(e)_{ii} : i \in I\} \cup \{0\}.$$

The element $(e)_{ij}$ will be denoted by E_{ij}. If I has finitely many elements, say $|I| = n$, $\mathcal{M}^0(G,n)$ will be used instead of $\mathcal{M}^0(G,I)$. We remark that $\mathcal{M}^0(G,I) \setminus \{0\}$ is a $\mathcal{D}$-class in $\mathcal{M}^0(G,I)$.

Let S be a semigroup. A *principal series* of ideals for S is a chain

$$S = S_1 \supsetneqq S_2 \supsetneqq \cdots \supsetneqq S_m \supsetneqq S_{m+1} = \emptyset, \tag{3.2}$$

where $S_1, \dots, S_m$ are ideals in S and there is no ideal of S strictly between S_j and S_{j+1} for every $1 \le j \le m$.

Later on we will use the following lemma [DN, p. 315].

Lemma 3.8. *Let S be an inverse semigroup with finitely many idempotent elements. Then S has a principal series as in* (3.2). *Moreover, for every $k = 1, \dots, m$ there is a natural number n_k and a group G_k such that $S_k/S_{k+1} = \mathcal{M}^0(G_k, n_k)$. Also, the maximal subgroups of S (up to isomorphism) are precisely G_k for $k = 1, \dots, m$.*

Theorem 3.9. *Let $S = \mathcal{M}^0(G,I)$ be a Brandt inverse semigroup. Then*

(i) $\mathrm{A}_p(S)$ *has a bounded approximate identity if and only if I is finite and G is amenable.*
(ii) $\mathrm{A}_p(S)$ *is amenable if and only if I is finite and* $\mathrm{A}_p(G)$ *is amenable.*

Proof. (i) If $\mathrm{A}_p(S)$ has a bounded approximate identity, then in view of Theorem 3.7, I is finite. If we set $G_{ij} = GE_{ij}$ for each $i, j \in I$, then $\{G_{ij} : i, j \in I\} \cup \{0\}$ is a partition of S. Since the operation in $\mathrm{A}_p(S)$ is pointwise, by the open mapping theorem,

$$\mathrm{A}_p(S) \simeq \Big(l^1\text{-} \bigoplus_{i,j\in I} \mathrm{A}_p(G_{ij})\Big) \oplus_1 \mathbb{C}\delta_0,$$

and also $\mathrm{A}_p(G_{ij}) = \mathrm{A}_p(G)$. By [MP, Lemma 6.4], $\mathrm{A}_p(G)$ has a bounded approximate identity, and thus, by Leptin's theorem, G is amenable.

Conversely, if I is finite and G is amenable, then by Leptin's theorem $\mathrm{A}_p(G_{ij}) = \mathrm{A}_p(G)$ has a bounded approximate identity, and so does $\mathrm{A}_p(S) \simeq (l^1\text{-} \bigoplus_{i,j\in I} \mathrm{A}_p(G_{ij})) \oplus_1 \mathbb{C}\delta_0$.

(ii) Since every amenable Banach algebra has a bounded approximate identity, this is a direct consequence of part (i) and the fact that $\mathrm{A}_p(S) \simeq (l^1\text{-} \bigoplus_{i,j\in I} \mathrm{A}_p(G_{ij})) \oplus_1 \mathbb{C}\delta_0$. ∎

Let A be a Banach algebra with a bounded approximate identity (e_α), and let I be a closed ideal of A. Since $\|a + I\|_{A/I} \le \|a\|$ for each $a \in A$, $(e_\alpha + I)$ becomes a bounded approximate identity for A/I.

Lemma 3.10. *Let $S = \mathcal{M}^0(G,I)$ be a Brandt inverse semigroup. Then* $\mathrm{A}_p(S)$ *has a bounded approximate identity if and only if* $\mathrm{A}_p(S)/\mathbb{C}\delta_0$ *has a bounded approximate identity.*

Proof. Since S is a Brandt semigroup, it has two $\mathcal{D}$-classes, $S\setminus\{0\}$ and $\{0\}$. Therefore, by [MP, equation (4.2)], we have $\mathrm{A}_p(S) = \mathrm{A}_p(S\setminus\{0\}) \oplus_1 \mathbb{C}\delta_0$. Hence, if $\mathrm{A}_p(S)/\mathbb{C}\delta_0$ admits a bounded approximate identity, then so does $\mathrm{A}_p(S\setminus\{0\}) \cong \mathrm{A}_p(S)/\mathbb{C}\delta_0$. Denote by (e_α) the bounded approximate identity for $\mathrm{A}_p(S\setminus\{0\})$. Then it can be easily checked that $(e_\alpha + \delta_0)$ is a bounded approximate identity for $\mathrm{A}_p(S)$.

The converse is clear as we mentioned before the lemma. ∎

In the following theorem we characterize the existence of a bounded approximate identity in $\mathrm{A}_p(S)$.

THEOREM 3.11. *Let S be an inverse semigroup and $1 < p < \infty$. Then $\mathrm{A}_p(S)$ has a bounded approximate identity if and only if $E(S)$ is finite and each maximal subgroup of S is amenable.*

Proof. Suppose $\mathrm{A}_p(S)$ has a bounded approximate identity. Then, by Theorem 3.7, $E(S)$ is finite. Now by Lemma 3.8, S has a principal series

$$S = S_1 \supsetneqq S_2 \supsetneqq \cdots \supsetneqq S_m \supsetneqq S_{m+1} = \emptyset \tag{3.3}$$

such that for each $k = 1,\ldots,m$, we have $S_k/S_{k+1} = \mathcal{M}^0(G_k, n_k)$ for some group G_k and $n_k \in \mathbb{N}$. Let (e_α) be a bounded approximate identity for $\mathrm{A}_p(S)$. Since multiplication in $\mathrm{A}_p(S)$ is pointwise, $(e_\alpha|_{S_k})$ forms a bounded approximate identity for $\mathrm{A}_p(S_k)$ for $k = 1,\ldots,m$. As $S_k/S_{k+1} = \mathcal{M}^0(G_k, n_k)$ has exactly two $\mathcal{D}$-classes, $S_k \setminus S_{k+1} = \mathcal{M}^0(G_k, n_k) \setminus \{0\}$ and $\{0\}$, by [MP, equation (4.2)] we have $\mathrm{A}_p(S_k) \cong \mathrm{A}_p(S_{k+1}) \oplus_1 \mathrm{A}_p(S_k \setminus S_{k+1})$. Therefore,

$$\begin{aligned} \mathrm{A}_p(S_k)/\mathrm{A}_p(S_{k+1}) &\cong \mathrm{A}_p(S_k \setminus S_{k+1}) \cong \mathrm{A}_p(S_k/S_{k+1})/\mathbb{C}\delta_0 \\ &\cong \mathrm{A}_p(\mathcal{M}^0(G_k, n_k))/\mathbb{C}\delta_0. \end{aligned} \tag{3.4}$$

This means that $\mathrm{A}_p(S_k)/\mathrm{A}_p(S_{k+1})$ (and so, by Lemma 3.10, $\mathrm{A}_p(\mathcal{M}^0(G_k, n_k))$) has a bounded approximate identity for each $k = 1,\ldots,m$. Now it follows from the above discussion and Theorem 3.9 that each G_k is amenable.

Conversely, if $E(S)$ is finite, then S has a principal series as in (3.3). Since G_k is amenable, by Theorem 3.9, $\mathrm{A}_p(\mathcal{M}^0(G_k, n_k))$ has a bounded approximate identity for each k. Therefore, $\mathrm{A}_p(S_k \setminus S_{k+1}) \cong \mathrm{A}_p(\mathcal{M}^0(G_k, n_k))/\mathbb{C}\delta_0$ has a bounded approximate identity. Now using the isomorphisms $\mathrm{A}_p(S_k) \cong \mathrm{A}_p(S_{k+1}) \oplus \mathrm{A}_p(S_k \setminus S_{k+1})$ for $k = 1,\ldots,m$, we obtain

$$\mathrm{A}_p(S) \cong \bigoplus_{k=1}^{m} \mathrm{A}_p(S_k \setminus S_{k+1}), \tag{3.5}$$

which shows that $\mathrm{A}_p(S)$ has a bounded approximate identity. ∎

Using the above theorem and [Pat, Theorem A.0.3], we get one of the main results of this paper:

Corollary 3.12 (A generalization of Leptin's theorem). *Let S be an inverse semigroup and $1 < p < \infty$. Then $\mathrm{A}_p(S)$ has a bounded approximate identity if and only if $l^1(S)$ is amenable.*

Another application of Theorem 3.11 gives a characterization of amenability of $\mathrm{A}_p(S)$ which was left open in [MP].

Theorem 3.13. *Let S be an inverse semigroup and $1 < p < \infty$. Then $\mathrm{A}_p(S)$ is amenable if and only if $E(S)$ is finite and $\mathrm{A}_p(G)$ is amenable for each maximal subgroup G of S.*

Proof. If $\mathrm{A}_p(S)$ is amenable, then by Theorem 3.7, $E(S)$ is finite. Thus S has a principal series as in (3.3), and hence by (3.4) and (3.5),

$$\mathrm{A}_p(S) \cong \bigoplus_{k=1}^{m} \mathrm{A}_p(\mathcal{M}^0(G_k, n_k))/\mathbb{C}\delta_0. \tag{3.6}$$

It follows from Theorem 3.9 and [Run, Proposition 2.3.1 and Theorem 2.3.10] that $\mathrm{A}_p(G_k)$ is amenable for each k.

Conversely, consider the principal series (3.3). According to Theorem 3.9, for each k, amenability of $\mathrm{A}_p(G_k)$ implies amenability of $\mathrm{A}_p(\mathcal{M}^0(G_k, n_k))$, hence of $\mathrm{A}_p(\mathcal{M}^0(G_k, n_k))/\mathbb{C}\delta_0$. Now the amenability of $\mathrm{A}_p(S)$ follows from the isomorphism (3.6). ∎

Let $\mathcal{C}$ be the bicyclic semigroup with generators p, q. Then $E(\mathcal{C}) = \{p^n q^n : n \in \mathbb{N}\}$. Therefore, $\mathrm{A}_p(\mathcal{C})$ has no bounded approximate identity and consequently is not amenable.

We remark that if $E(S)$ is finite and each maximal subgroup of S is almost abelian, then $\mathrm{A}_p(S)$ is amenable (see [LNR, Remark 2]). For $p = 2$ we have a better characterization:

Corollary 3.14. *Let S be an inverse semigroup. Then $\mathrm{A}(S)$ is amenable if and only if $E(S)$ is finite and each maximal subgroup of S is almost abelian.*

Proof. This is immediate by Theorem 3.13 and [FR, Theorem 2.3]. ∎

As in [MP], we define the *restricted left regular representation* $\pi : S \to \mathcal{B}(l^2(S))$ by

$$\pi(s)(\delta_t) = \begin{cases} \delta_{st} & \text{if } tt^* = s^*s, \\ 0 & \text{otherwise.} \end{cases}$$

As usual, setting $\pi(\delta_s) = \pi(s)$ extends π to a representation of the Banach algebra $l_r^1(S)$, called the *left regular representation*. Therefore, $\pi : l_r^1(S) \to \mathcal{B}(l^2(S))$ is defined by $\pi(f)(g) = f \bullet g$.

As in [MP, Definition 2.6], the ultra-weak closure of $\pi(l_r^1(S))$ in $\mathcal{B}(l^2(S))$ is denoted by $\mathrm{VN}(S)$. Using [Mur, Theorem 4.2.4], one can easily see that $\mathrm{VN}(S)$ is a weakly closed $*$-subalgebra of $\mathcal{B}(l^2(S))$ containing the identity operator, hence a von Neumann algebra by [Mur, Theorem 4.2.5]. We call $\mathrm{VN}(S)$ the *von Neumann algebra* of S. By [MP, Theorem 3.7], $\mathrm{VN}(S)$ is the dual of $\mathrm{A}(S)$ and the duality is given by

$$\langle \mathbf{u}, T\rangle = \sum_{n=1}^{\infty} \langle f_n, T(g_n)\rangle \qquad \Big(\mathbf{u} = \sum_{n=1}^{\infty} f_n \bullet \check{g}_n \in \mathrm{A}(S),\ T \in \mathrm{VN}(S)\Big).$$

Since $\mathrm{A}(S)^* = \mathrm{VN}(S)$ is a von Neumann algebra, $\mathrm{A}(S)$ has a canonical operator space structure and so we can discuss its operator amenability. We refer the reader to [ER] for a complete account of the theory of operator spaces.

THEOREM 3.15 (A generalization of Ruan's theorem). *Let S be an inverse semigroup. Then $\mathrm{A}(S)$ is operator amenable if and only if $l^1(S)$ is amenable.*

Proof. If $\mathrm{A}(S)$ is operator amenable, then it has a bounded approximate identity [ER, Proposition 16.1.1], and so by Theorem 3.11, $E(S)$ is finite and each maximal subgroup of S is amenable. Therefore, $l^1(S)$ is amenable by [Pat, Theorem A.0.3].

Conversely, let $l^1(S)$ be amenable. Then, again by [Pat, Theorem A.0.3], $E(S)$ is finite and each maximal subgroup of S is amenable. Consider the principal series (3.3) for S. Using Ruan's theorem [ER, Theorem 16.2.1], we see that $\mathrm{A}(G_k)$ is operator amenable for each k. Now the operator space version of Theorem 3.9(ii) together with (3.6) gives the operator amenability of $\mathrm{A}(S)$. Note that (3.6) is a complete isometric isomorphism for $p = 2$. ■

In the rest of this section we calculate the character space of $\mathrm{A}_p(S)$.

For each $s \in S$, define the evaluation functional $\varphi_s : \mathrm{A}_p(S) \to \mathbb{C}$, $\varphi_s(\mathbf{u}) = \mathbf{u}(s)$. It is easily seen that φ_s is a character of $\mathrm{A}_p(S)$ (i.e. a multiplicative bounded linear functional on $\mathrm{A}_p(S)$). In the following theorem we show that these are the only characters of $\mathrm{A}_p(S)$, which can also be considered as a generalization of [MP, Theorem 6.3].

THEOREM 3.16. *Let S be an inverse semigroup and $1 < p < \infty$. Then $\Phi_{\mathrm{A}_p(S)} = S$, where $\Phi_{\mathrm{A}_p(S)}$ denotes the space of all non-zero characters of $\mathrm{A}_p(S)$.*

Proof. Let $\varphi \in \Phi_{\mathrm{A}_p(S)}$. Then there is $0 \neq \mathbf{u} \in \mathrm{A}_p(S)$ such that $\varphi(\mathbf{u}) \neq 0$. Since $\mathbf{u} = \sum_{s\in S} \mathbf{u}(s)\delta_s$, there is $s \in S$ with $\varphi(\delta_s) \neq 0$. Take an arbitrary element $t \in S$ with $t \neq s$. Since $\delta_t\delta_s = 0$, we have $\varphi(\delta_t)\varphi(\delta_s) = \varphi(\delta_t\delta_s) = 0$, which implies $\varphi(\delta_t) = 0$. Therefore, $\varphi = \varphi_s$. ■

This theorem shows that the Gelfand transform $\mathcal{G} : \mathrm{A}_p(S) \to c_0(S)$ is the inclusion map, and so we have:

COROLLARY 3.17. *$\mathrm{A}_p(S)$ is a semisimple Banach algebra.*

Singer and Wermer [SW] have shown that $\mathcal{H}^1(A, A) = 0$ for any commutative semisimple Banach algebra A. This yields

COROLLARY 3.18. *Let S be an inverse semigroup and $1 < p < \infty$. Then* $\mathcal{H}^1(\mathrm{A}_p(S), \mathrm{A}_p(S)) = 0$.

Acknowledgements. This research was supported in part by a grant from IPM (No. 92470041). The author would like to thank the referee for his/her useful comments, and specially for pointing out an error in the first draft and for suggesting the proof of Lemma 3.10.

References

[AM] M. Amini and A. Medghalchi, *Restricted algebras on inverse semigroups I. Representation theory*, Math. Nachr. 279 (2006), 1739–1748.

[DN] J. Duncan and I. Namioka, *Amenability of inverse semigroups and their semigroup algebras*, Proc. Roy. Soc. Edinburg Sec. A 80 (1978), 309–321.

[ER] E. G. Effros and Z. J. Ruan, *Operator Spaces*, Oxford Univ. Press, 2000.

[FR] B. E. Forrest and V. Runde, *Amenability and weak amenability of the Fourier algebra*, Math. Z. 250, (2005), 731–744.

[Her] C. Herz, *The theory of p-spaces with applications to convolution operators*, Trans. Amer. Math. Soc. 154 (1971), 69–82.

[How] J. M. Howie, *Fundamentals of Semigroup Theory*, London Math. Soc. Monogr. 12, Clarendon Press, Oxford, 1995.

[LNR] A. Lambert, M. Neufang and V. Runde, *Operator space structure and amenability for Figà-Talamanca–Herz algebras*, J. Funct. Anal. 211 (2004), 245–269.

[MP] A. R. Medghalchi and H. Pourmahmood-Aghababa, *Figà-Talamanca–Herz algebras for restricted inverse semigroups and Clifford semigroups*, J. Math. Anal. Appl. 395 (2012), 473–485.

[Mur] G. J. Murphy, *C^*-algebras and Operator Theory*, Academic Press, San Diego, CA, 1990.

[Pat] A. T. Paterson, *Groupoids, Inverse Semigroups, and their Operator Algebras*, Progr. Math. 170, Birkhäuser, Boston, 1999.

[Pou] H. Pourmahmood-Aghababa, *Pseudomeasures and pseudofunctions on inverse semigroups*, Semigroup Forum 90 (2015), 632–647.

[Run] V. Runde, *Lectures on Amenability*, Lecture Notes in Math. 1774, Springer, Berlin, 2002.

[SW] I. M. Singer and J. Wermer, *Derivations on commutative normed algebras*, Math. Ann. 129 (1955), 260–264.

Hasan Pourmahmood-Aghababa
Department of Mathematics, University of Tabriz
Tabriz, Iran
and
School of Mathematics, Institute for Research in Fundamental Sciences (IPM)
P.O. Box 19395-5746, Tehran, Iran
E-mail: h_p_aghababa@tabrizu.ac.ir
h_pourmahmood@yahoo.com