

A functorial control of integral torsion in homology

by

Antoine Touzé (Lille)

Abstract. We study the integral torsion of the values of strict polynomial functors defined over the integers. We interpret some classical homological invariants as values of strict polynomial functors, and thus obtain estimates of the integral torsion of these invariants, including cancellation results.

1. Introduction. The aim of this article is to provide an explanation for some phenomena regarding the integral torsion of some homological invariants.

An emblematic example is the stable homology of the Eilenberg–Mac Lane spaces $K(\pi, n)$ associated to an abelian group π . Cartan [6, Exposé 11] computed the (unstable) integral homology of these spaces, and deduced from it their stable homology (the i th stable homology group $\mathrm{HZ}_i(H\pi)$ equals the unstable homology group $\mathrm{H}_{i+n}(K(\pi, n), \mathbb{Z})$ if $i < n$). He observed *a posteriori* the striking fact that the torsion part of the graded abelian group $\mathrm{HZ}_*(H\pi)$ contains only elements of order p for prime integers p , whereas the unstable homology groups contain complicated torsion. For example, the graded abelian group $\mathrm{H}_*(K(\mathbb{Z}, 4), \mathbb{Z})$ contains elements of order p^r for all $r \geq 0$ and for all primes p .

Dold and Puppe [12, Section 10] gave a conceptual proof for this phenomenon, which does not require computing $\mathrm{HZ}_*(H\pi)$. Such a result is useful because if we know *a priori* that the stable homology has only prime torsion then we can recover it (by the universal coefficient theorem) from the stable homology with prime field coefficients $\mathrm{HF}_{p_*}(H\pi)$, whose computation is significantly easier (see e.g. [2]). The argument of Dold and Puppe has been adapted to other contexts later, for example in [17], but its range of application is quite limited.

2010 *Mathematics Subject Classification*: 18A22, 20G05, 18G15, 18G55.

Key words and phrases: polynomial functors, strict polynomial functors, integral torsion.

Received 24 August 2015; revised 1 April 2016.

Published online 14 December 2016.

This article describes a method to obtain information on integral torsion, but with a wider range of application than Dold and Puppe’s argument. The method has two steps:

- (1) In Theorem 1.2, we describe the integral torsion of the values of strict polynomial functors in the sense of Friedlander and Suslin [18, 29].
- (2) Then we give a (nonexhaustive) list of homological invariants which can be interpreted as a value of a strict polynomial functor. In particular we obtain information regarding their integral torsion.

One of the origins of strict polynomial functors is the study of representations of algebraic group schemes and Schur algebras. Algebraic group schemes and their representations do not play any role in this article. However, Theorem 1.2 describing the integral torsion of the values of strict polynomial functors actually yields a new result on representations of Schur algebras. Since those representations are not the focus of this article, we give this result in the Appendix.

We now describe the first step of our method. Given a commutative ring R , we denote by \mathbf{P}_R the category of finitely generated projective R -modules. As recalled in detail in Section 2, strict polynomial functors can be thought of as functors $F : \mathbf{P}_R \rightarrow R\text{-Mod}$, equipped with an additional ‘strict polynomial structure’. There are two nonnegative integers attached to each strict polynomial functor, which are usually easy to determine:

- The weight ⁽¹⁾ of the functor F , determined by its strict polynomial structure. The homogeneous strict polynomial functors of weight s form an abelian category $\mathcal{P}_{s,R}$.
- The degree $\deg(F)$ in the sense of Eilenberg and Mac Lane [15]. This notion of degree is defined for all functors $F : \mathbf{P}_R \rightarrow R\text{-Mod}$, i.e. independently of the strict polynomial structure.

For example, the d th tensor $\otimes^d : M \mapsto M^{\otimes d}$ is canonically a strict polynomial functor of weight d , and its degree is also d . In general, the degree of a strict polynomial functor does not exceed its weight, but the inequality may be strict. The main idea of the first step of the method is that the difference between the weight of F and its degree gives information regarding the integral torsion of the values of F . For example, in Section 3 we prove:

PROPOSITION 1.1. *Let R be a commutative ring, let s be a positive integer, and let $F \in \mathcal{P}_{s,R}$. If $\deg(F) < s$, then the values of F , considered as abelian groups, are torsion groups.*

⁽¹⁾ What we call the ‘weight’ of a strict polynomial functor is called ‘degree’ in [18, 29]. However, we shall not follow the terminology of [18, 29] to prevent confusion with the notion of degree in the sense of Eilenberg and Mac Lane [15].

We will therefore restrict our attention to functors with torsion values. For such functors we have a decomposition in $\mathcal{P}_{s,R}$,

$$F \simeq \bigoplus_{p \text{ prime}} (p)F,$$

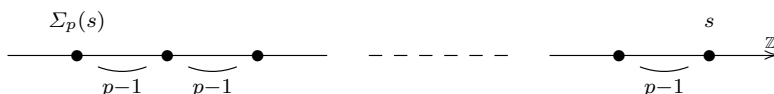
where $(p)F$ denotes the p -primary part of F , that is, for all $M \in \mathcal{P}_R$,

$$(p)F(M) = \{x \in F(M) \mid \exists r \in \mathbb{N} \ p^r x = 0\}.$$

Since $\deg((p)F) \leq \deg(F)$, we can restrict our attention further to functors with values in p -primary abelian groups, which we call *p-primary functors*. Given a positive integer s , we denote by $\Sigma_p(s)$ the sum of the digits in the p -adic expansion of s , and by $\mathcal{I}(p, s)$ the ‘interval’ of integers

$$\mathcal{I}(p, s) = \{n \in \mathbb{N} \mid n \equiv s \pmod{p-1} \text{ and } \Sigma_p(s) \leq n \leq s\},$$

which can be pictured as follows (note that $\Sigma_p(s)$ is the smallest element of $\mathcal{I}(p, s)$):



We say that the torsion of a strict polynomial functor F is *bounded by an integer n* if for all finitely generated projective R -modules M and all $x \in F(M)$ we have $nx = 0$ (in particular, the torsion of F is bounded by 1 if and only if $F = 0$). In Section 3, we prove the following description of p -primary strict polynomial functors.

THEOREM 1.2. *Let R be a commutative ring, let p be a prime integer and let $F \in \mathcal{P}_{s,R}$ be a nonzero p -primary strict polynomial functor. Then:*

- (1) *The degree of F is an element of $\mathcal{I}(p, s)$.*
- (2) *If $\deg(F) < s$, then the integral torsion of F is bounded by p^r with $r = \lceil \frac{\deg(F)+1-\Sigma_p(s)}{p-1} \rceil$, where the brackets denote the ceiling function ⁽²⁾.*

In the second step, we show that certain homological invariants are values of strict polynomial functors. Our method applies to those homological invariants which involve strict polynomial functors in their definition. This may seem very restrictive, but one should keep in mind that many classical algebraic constructions yield strict polynomial functors.

- Schur and Weyl functors, as defined in [1], are strict polynomial functors of fundamental importance in representation theory.

⁽²⁾ That is, for all $x \in \mathbb{R}$, $\lceil x \rceil = \min\{n \in \mathbb{Z} \mid x \leq n\}$.

- Free algebras associated to a symmetric operad in differential graded R -modules (free commutative algebras, free Lie algebras, etc.) yield strict polynomial functors (compare Example 2.9(2) and [23, 5.1]).
- Some groups or group rings of fundamental importance are filtered, with strict polynomial functors as associated graded pieces. This is so for free groups by the Magnus–Witt isomorphism [24, 35], or for the group ring of a free abelian group (the graded pieces are symmetric powers).

The homology with coefficients $H_*(G, F(M))$ of a group G , where M is a RG -module and $F \in \mathcal{P}_{s,R}$, provides a simple example of homological invariant depending on a strict polynomial functor. There are also less trivial examples, where strict polynomial functors are hidden, like the stable homology of Eilenberg–Mac Lane spaces.

Given a homological invariant $\Psi(F)$ depending on a strict polynomial functor $F \in \mathcal{P}_{s,R}$, we introduce a parameter $M \in P_R$ by replacing F by the functor $F_M = F \circ (M \otimes_R -)$ in the construction. In this way, we obtain a ‘new’ invariant $\Psi(F_M)$ which is a homogeneous strict polynomial functor of weight s with respect to M . In Section 4.1, we give conditions which allow one to compute the degree of the functor $M \mapsto \Psi(F_M)$. When they apply, Theorem 1.2 gives us information on the integral torsion of $\Psi(F) = \Psi(F_R)$. We illustrate our method by the following examples:

- The Taylor towers of strict polynomial functors.
- Derived functors of strict polynomial functors.
- Functor homology (computations of Ext or Tor groups in various functor categories).
- The cohomology of reductive algebraic groups.

2. Background. This section collects some basic facts about polynomial and strict polynomial functors which will be needed in our proofs. We refer the reader e.g. to [15, 18, 22, 26] for more details.

2.1. Polynomial functors. Let \mathcal{A} be an additive category, let \mathcal{B} be an abelian category and let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a functor. For all sums $X_1 \oplus X_2$ in \mathcal{A} , $F(X_i)$ canonically identifies with a direct summand of $F(X_1 \oplus X_2)$. We start with the definition of the cross effects of F .

DEFINITION 2.1 ([15]). For all positive d , the d th *cross effect* of F is the functor $\text{cr}_d F : \mathcal{A}^{\times d} \rightarrow \mathcal{B}$ defined inductively as follows:

$$\begin{aligned} \text{cr}_1 F(X) \oplus F(0) &= F(X), \\ \text{cr}_2 F(X_1, X_2) \oplus \text{cr}_1 F(X_1) \oplus \text{cr}_1 F(X_2) &= \text{cr}_1 F(X_1 \oplus X_2), \end{aligned}$$

And more generally, $\text{cr}_d F$ measures the nonadditivity of $\text{cr}_{d-1} F$ with respect

to its first variable:

$$\begin{aligned} \text{cr}_d F(X_1, \dots, X_d) \oplus \text{cr}_{d-1} F(X_1, X_3, \dots, X_d) \oplus \text{cr}_{d-1} F(X_2, X_3, \dots, X_d) \\ = \text{cr}_{d-1} F(X_1 \oplus X_2, X_3, \dots, X_d). \end{aligned}$$

One proves by induction the canonical decomposition

$$(2.1) \quad F(X_1 \oplus \dots \oplus X_d) = F(0) \oplus \bigoplus_{k=1}^d \left(\bigoplus_{j_1 < \dots < j_k} \text{cr}_k F(X_{j_1}, \dots, X_{j_k}) \right).$$

By letting the symmetric group \mathfrak{S}_d act on the expression above by permuting the terms of the coproduct, we see that the d th cross effect is symmetric:

$$\text{cr}_d F(X_1, \dots, X_d) \simeq \text{cr}_d F(X_{\sigma(1)}, \dots, X_{\sigma(d)}).$$

It also follows from the definition that $\text{cr}_d F$ is *reduced*, i.e. it is zero as soon as one of its arguments is zero. We denote by $\text{Func}_*(\mathcal{C}^{\times d}, \mathcal{A})$ the category of reduced functors. The d th cross effect yields an exact functor

$$\text{cr}_d : \text{Func}(\mathcal{A}, \mathcal{B}) \rightarrow \text{Func}_*(\mathcal{A}^{\times d}, \mathcal{B}).$$

Precomposition with the diagonal functor $\Delta_d : \mathcal{A} \rightarrow \mathcal{A}^{\times d}, X \mapsto (X, \dots, X)$, yields an exact functor

$$\Delta_d^* : \text{Func}_*(\mathcal{A}^{\times d}, \mathcal{B}) \rightarrow \text{Func}(\mathcal{A}, \mathcal{B}).$$

One easily proves that for all positive d , cr_d and Δ_d^* are (left and right) adjoint to each other.

DEFINITION 2.2 ([15]). A functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is *polynomial of degree less than or equal to d* if $\text{cr}_{d+1} F$ is zero. It is *polynomial of degree d* if $\text{cr}_{d+1} F$ equals zero and $\text{cr}_d F$ is nonzero. By convention the zero functor has degree $-\infty$. We denote by $\text{deg}(F)$ the degree of a functor F .

In particular, a functor of degree zero is constant, and a reduced functor of degree 1 is an additive functor. Moreover, F is polynomial of degree d if and only if $\text{cr}_d F$ is nonzero and additive with respect to each of his arguments. Since the d th cross effect functor is exact, any subfunctor or quotient of a degree d functor has degree $\leq d$.

DEFINITION 2.3. Let P_R be the category of finitely generated projective R -modules. We denote by \mathcal{F}_R the category of functors from P_R to all R -modules.

We have the following classical examples of polynomial functors in \mathcal{F}_R .

EXAMPLE 2.4. Let R be a commutative ring.

(1) If $F, G \in \mathcal{F}_R$ are polynomial functors of degree d , resp. e , their tensor product $F \otimes G : M \mapsto F(M) \otimes_R G(M)$ is polynomial of degree $d + e$.

(2) The d th tensor product functor $\otimes^d : M \mapsto M^{\otimes d}$, the d th symmetric power functor $S^d : M \mapsto S^d(M) = (M^{\otimes n})_{\mathfrak{S}_d}$, the d th exterior power functor

$\Lambda^d : M \mapsto \Lambda^d(M)$ and the d th divided power functor $\Gamma^d : M \mapsto \Gamma^d(M) = (M^{\otimes d})^{\mathfrak{S}_d}$ are examples of polynomial functors of degree d of the category \mathcal{F}_R .

The fact that Δ_d^* is left adjoint to cr_d implies a useful vanishing lemma (see [25], [26] or [3]).

LEMMA 2.5. *Let $G \in \mathcal{F}_R$ be a polynomial functor of degree d . Assume that $(F_i)_{1 \leq i \leq d+1}$ are reduced functors. Then*

$$\text{Hom}_{\mathcal{F}_R}(F_1 \otimes \cdots \otimes F_{d+1}, G) = 0.$$

2.2. Strict polynomial functors. Let R be a commutative ring and let $M \in \text{P}_R$. The symmetric group \mathfrak{S}_d acts on $M^{\otimes d}$ by permuting the factors of the tensor product. The Schur category $\Gamma^s \text{P}_R$ is the R -linear category defined as follows. It has the same objects as P_R , the morphisms from M to N are the \mathfrak{S}_s -equivariant R -linear maps from $M^{\otimes s}$ to $N^{\otimes s}$, and the composition law is composition of equivariant morphisms (if $s = 0$, we let $M^{\otimes 0} = R$ and $\mathfrak{S}_0 = \{1\}$). The notation $\Gamma^s \text{P}_R$ is justified by the following identification of homomorphisms:

$$\begin{aligned} \text{Hom}_{\Gamma^s \text{P}_R}(M, N) &= \text{Hom}_{\mathfrak{S}_s}(M^{\otimes s}, N^{\otimes s}) = \text{Hom}_R(M^{\otimes s}, N^{\otimes s})^{\mathfrak{S}_s} \\ &\simeq (\text{Hom}_R(M, N)^{\otimes s})^{\mathfrak{S}_s} = \Gamma^s(\text{Hom}_R(M, N)). \end{aligned}$$

DEFINITION 2.6. The category of *homogeneous strict polynomial functors of weight s* , denoted by $\mathcal{P}_{s,R}$, is the abelian category of R -linear functors from $\Gamma^s \text{P}_R$ to R -modules.

REMARK 2.7. This presentation of $\mathcal{P}_{s,R}$ follows [4, 26]. The category $\mathcal{P}_{s,R}$ is equivalent to the category of ‘homogeneous strict polynomial functors of degree s ’ introduced by Friedlander and Suslin [18]. We already have a notion of degree recalled in Definition 2.2, so we rather call the integer s the *weight* of strict polynomial functors to avoid confusion.

There is a functor $\gamma_s : \text{P}_R \rightarrow \Gamma^s \text{P}_R$ which is the identity on objects, and whose action on morphisms is given by $\gamma_s(f) = f^{\otimes s}$. Precomposition with γ_s yields a forgetful functor

$$\gamma_s^* : \mathcal{P}_{s,R} \rightarrow \mathcal{F}_R.$$

The following properties of the forgetful functor are an easy check.

LEMMA 2.8. *The forgetful functor γ_s^* is faithful. Let $f : F \rightarrow G$ be a morphism of strict polynomial functors. Then f is a monomorphism (resp. epimorphism, resp. isomorphism) if and only if $\gamma_s^*(f)$ is.*

Thus, the notion of strict polynomial functors can be thought of as an enrichment of the usual notion of functor from P_R to R -modules. If F is a strict polynomial functor, we call $F \circ \gamma_s$ the *underlying ordinary functor*. We often abuse notation and denote by the same letter a strict polynomial functor and the underlying ordinary functor.

EXAMPLE 2.9. We have the following examples of homogeneous strict polynomial functors of weight s .

(1) We denote by \otimes^s the strict polynomial functor which sends a projective module $M \in \mathcal{P}_R$ to the tensor product $M^{\otimes s}$, and whose effect on morphisms is just given by the canonical inclusion

$$\mathrm{Hom}_{\mathfrak{S}_s}(M^{\otimes s}, N^{\otimes s}) \hookrightarrow \mathrm{Hom}_R(M^{\otimes s}, N^{\otimes s}).$$

(2) The symmetric group \mathfrak{S}_s acts on \otimes^s by permuting the factors of the tensor product. We denote by Γ^s the intersection of the kernels of the maps $\sigma - \mathrm{Id} : \otimes^s \rightarrow \otimes^s$ for $\sigma \in \mathfrak{S}_s$. More generally, a \mathfrak{S}_s -module M yields strict polynomial functors $(M \otimes_R \otimes^s)^{\mathfrak{S}_s}$ and $M \otimes_{R\mathfrak{S}_s} \otimes^s$.

(3) If M is a finitely generated projective R -module, we let $\Gamma^{s,M}$ be the strict polynomial functor defined by $\Gamma^{s,M}(N) = \mathrm{Hom}_{\Gamma^s \mathcal{P}_R}(M, N)$. In particular, $\Gamma^{s,R}$ is isomorphic to Γ^s .

The functors $\Gamma^{s,M}$ generate the category $\mathcal{P}_{s,R}$. To be more specific, the Yoneda lemma yields a natural isomorphism

$$\mathrm{Hom}_{\mathcal{P}_{s,R}}(\Gamma^{s,M}, F) \simeq F(M),$$

and the canonical map

$$\bigoplus_{n \geq 1} \Gamma^{s,R^n} \otimes F(R^n) \rightarrow F$$

is an epimorphism. Since the ordinary functor $N \mapsto \Gamma^s(\mathrm{Hom}_R(R^n, N))$ is polynomial of degree s , and quotients of polynomial functors of degree s are polynomial of degree $\leq s$, we have:

LEMMA 2.10. *If F is a homogeneous strict polynomial functor of weight s , the underlying ordinary functor is polynomial of degree $\leq s$.*

REMARK 2.11. Not all polynomial functors of \mathcal{F}_R lie in the essential image of the forgetful functor. See for example Remark 3.6.

2.3. The divided power algebra. We recall the algebra structure on divided powers, which will be crucial later on. The sum of all the (s, t) -shuffles induces a morphism $\mathrm{sh}_{s,t} : (\otimes^t) \otimes (\otimes^s) \rightarrow \otimes^{s+t}$ of strict polynomial functors. These morphisms endow the tensor powers with the structure of a commutative algebra, called the shuffle algebra [14]. View $\Gamma^s \otimes \Gamma^t$ and Γ^{s+t} as subfunctors of \otimes^{s+t} . Then $\mathrm{sh}_{s,t}$ restricts to a morphism $\mathrm{sh}_{s,t} : \Gamma^s \otimes \Gamma^t \rightarrow \Gamma^{s+t}$. So the divided powers form a subalgebra of the shuffle algebra. One checks the following properties from the definition of the divided power algebra.

LEMMA 2.12.

(1) *Let (s_1, \dots, s_n) be an n -tuple of positive integers with sum s . Then the composite, where the first map is the canonical inclusion and the*

second one is induced by the multiplication

$$\Gamma^s \hookrightarrow \Gamma^{s_1} \otimes \dots \otimes \Gamma^{s_n} \rightarrow \Gamma^s,$$

is equal to multiplication by the multinomial $\binom{s}{s_1, \dots, s_n} = \frac{s!}{s_1! \dots s_n!}$.

- (2) For all $M, N \in \mathcal{P}_R$, multiplication of the divided power algebra induces an isomorphism

$$\bigoplus_{t=0}^s \Gamma^t(M) \otimes \Gamma^{s-t}(N) \simeq \Gamma^s(M \oplus N).$$

3. Degrees, weights and integral torsion

3.1. The classification of additive strict polynomial functors.

Given a positive integer s , we denote by $Q_R^s \in \mathcal{P}_{s,R}$ the cokernel of the morphism induced by multiplication (if $s = 1$, the sum on the left is zero, so $Q_R^1 = \Gamma^1$ is the identity functor):

$$(3.1) \quad \bigoplus_{0 < k < s} \Gamma^k \otimes \Gamma^{s-k} \rightarrow \Gamma^s.$$

LEMMA 3.1. *The functor Q_R^s is additive.*

Proof. As recalled in Lemma 2.12, multiplication induces an isomorphism $\bigoplus_{t=0}^s \Gamma^t(M) \otimes \Gamma^{s-t}(N) \simeq \Gamma^s(M \oplus N)$. Thus, it suffices to prove that $\bigoplus_{t=1}^{s-1} \Gamma^t(M) \otimes \Gamma^{s-t}(N)$ is in the image of the map (3.1), which is obvious. ■

LEMMA 3.2. *Let $F \in \mathcal{P}_{s,R}$ be additive. There is an isomorphism of strict polynomial functors $F \simeq Q_R^s \otimes F(R)$.*

Proof. By the Yoneda lemma there is a morphism $\phi : \Gamma^s \otimes F(R) \rightarrow F$, which induces an isomorphism $\phi_R : \Gamma^s(R) \otimes F(R) \xrightarrow{\simeq} F(R)$. We claim that the following composite is zero (where the morphism on the left is induced by multiplication of the divided power algebra):

$$\bigoplus_{0 < k < s} \Gamma^k \otimes \Gamma^{s-k} \otimes F(R) \rightarrow \Gamma^s \otimes F(R) \xrightarrow{\phi} F.$$

Indeed, the forgetful functor is faithful, so it suffices to prove that the same composite, viewed in \mathcal{F}_R , is zero. Since F is additive, the composite is zero by Lemma 2.5. Thus, ϕ induces a morphism of additive functors $\bar{\phi} : Q_R^s \otimes F(R) \rightarrow F$, and we have a commutative diagram of R -modules

$$\begin{array}{ccc} \Gamma^s(R) \otimes F(R) & & \\ \downarrow & \searrow \phi_R & \\ Q_R^s(R) \otimes F(R) & \xrightarrow{\bar{\phi}_R} & F(R) \end{array}$$

In particular $\bar{\phi}_R$ is an isomorphism. By additivity, this implies that $\bar{\phi}$ is an isomorphism of strict polynomial functors. ■

LEMMA 3.3. *Let $s \geq 2$. If s is not a power of a prime integer, then Q_R^s is zero. If s is a power of a prime integer p , then the R -module $Q_R^s(R)$ is isomorphic to R/pR .*

Proof. For all i , there is a canonical isomorphism $\Gamma^i(R) \simeq R$, and the map

$$R \simeq \Gamma^k(R) \otimes \Gamma^{s-k}(R) \xrightarrow{\text{mult}} \Gamma^s(R) \simeq R$$

is equal to multiplication by $\binom{s}{k}$. Thus $Q_R^s(R)$ is isomorphic to R/nR , where n is the greatest common denominator of the integers $\binom{s}{k}$, $0 < k < s$. By elementary arithmetic (see e.g. [12, Hilfsatz 10.10] or our Lemma 3.14), n equals p if s is a power of a prime p , and 1 otherwise. In the latter case, since $Q_R^s(R) = 0$, we have $Q_R^s = 0$ by additivity. ■

When $s = p^r$ is a power of a prime p (with $r > 0$), one would like to have a more concrete description of Q_R^s . For this, we use the Frobenius twist functors defined in [29]. Given an \mathbb{F}_p -algebra A , the r th Frobenius twist functor $I_A^{(r)} \in \mathcal{P}_{p^r, A}$ is an additive functor, whose values can be concretely described as follows. The A -module $I_A^{(r)}(M)$ is the quotient of the free A -module on M modulo the relations (for all $m, n \in M$ and $a \in A$)

$$(m + n)^{(r)} = m^{(r)} + n^{(r)}, \quad (am)^{(r)} = a^{p^r} m^{(r)},$$

where $m^{(r)}$ denotes the basis element of the free A -module on M indexed by $m \in M$. In particular, $I_A^{(r)}(A) \simeq A$.

We extend the definition of Frobenius twists functors to arbitrary commutative ground rings R in the following way. Given a prime integer p , we define the r th Frobenius twist functor $I_{p,R}^{(r)} \in \mathcal{P}_{p^r, R}$ as the composite

$$\Gamma^{p^r} P_R \xrightarrow{\otimes \mathbb{F}_p} \Gamma^{p^r} P_{R/pR} \xrightarrow{I_{R/pR}^{(r)}} R\text{-Mod.}$$

In particular $I_{p,R}^{(r)}(M) = I_{R/pR}^{(r)}(M/pM)$, so $I_{p,R}^{(r)}$ is additive and $I_{p,R}^{(r)}(R)$ is isomorphic to R/pR .

LEMMA 3.4. *Let p be a prime and let r be a positive integer. Then $Q_R^{p^r}$ is isomorphic to the r th Frobenius twist functor $I_{p,R}^{(r)}$.*

Proof. Since $Q_R^{p^r}(R) = R/pR$, we have an isomorphism $Q_R^{p^r} \simeq Q_R^{p^r} \otimes R/pR$. The result now follows from Lemma 3.2, applied to $F = I_{p,R}^{(r)}$. ■

Combining Lemmas 3.2–3.4, we obtain the following classification of additive strict polynomial functors over an arbitrary ring R .

PROPOSITION 3.5. *Let R be a commutative ring, let s be a positive integer and let $F \in \mathcal{P}_{s,R}$ be additive.*

- (1) *If $s = 1$, then there is an isomorphism $F \simeq \text{Id} \otimes F(R)$.*
- (2) *If $s \geq 2$, then there exists a prime p and a positive integer r such that $s = p^r$, and an isomorphism $F \simeq I_{p,R}^{(r)} \otimes F(R)$. In particular the values of F only consist of p -torsion elements.*

REMARK 3.6. Additive functors from \mathbf{P}_R to R -modules are well-known: they are all of the form $M \mapsto N \otimes_R M$ for some R - R -bimodule N which characterizes the functor [13, 33]. If $\phi : R \rightarrow R/pR$ is a ring morphism, we can consider R/pR as an R - R -bimodule by letting $r \cdot x \cdot s = rx\phi(s)$. In this special case, tensoring yields an additive functor which we denote by I^ϕ . Proposition 3.5 implies that I^ϕ is the underlying ordinary functor of a strict polynomial functor if and only if $\phi(x) = x^{p^r}$ for some prime p and some nonnegative integer r . In particular, some functors of the form I^ϕ may not be in the image of any of the forgetful functors $\mathcal{P}_{s,R} \rightarrow \mathcal{F}_R$ (for example if $R = R/pR$ is the algebraic closure of \mathbb{F}_p and ϕ is the inverse of the Frobenius homomorphism).

3.2. Weight vs degree for p -primary functors. In this section, we prove Theorem 1.2(1). The case of additive (or degree one) functors is already handled by Proposition 3.5. To study the case of higher degree functors, we need to lift the cross effect functors cr_n to the level of the categories of strict polynomial (multi)functors. This is already used in [4, 16, 29, 30]. We briefly recall how it works.

For $n \geq 1$, the Schur category $\Gamma^s(\mathbf{P}_R^{\times n})$ is the R -linear category defined as follows. It has the same objects as $\mathbf{P}_R^{\times n}$. The R -modules of morphisms are defined by

$$\text{Hom}_{\Gamma^s(\mathbf{P}_R^{\times n})}(M, N) = \Gamma^s(\text{Hom}_{\mathbf{P}_R^{\times n}}(M, N)).$$

The composition law \circ is the unique R -linear morphism fitting into the diagram

$$\begin{array}{ccc} \Gamma^s(\text{Hom}_{\mathbf{P}_R^{\times n}}(N, P)) \otimes \Gamma^s(\text{Hom}_{\mathbf{P}_R^{\times n}}(M, N)) & \xrightarrow{\circ} & \Gamma^s(\text{Hom}_{\mathbf{P}_R^{\times n}}(M, P)) \\ \downarrow & & \downarrow \\ (\text{Hom}_{\mathbf{P}_R^{\times n}}(N, P) \otimes \text{Hom}_{\mathbf{P}_R^{\times n}}(M, N))^{\otimes s} & \longrightarrow & \text{Hom}_{\mathbf{P}_R^{\times n}}(M, P)^{\otimes s} \end{array}$$

where the vertical maps are the canonical inclusions and the bottom horizontal map is induced by composition in $\mathbf{P}_R^{\times n}$. If $n = 1$, this is the same as the Schur category defined in Section 2.2. As in the case $n = 1$, there is a functor $\gamma_s : \mathbf{P}_R^{\times n} \rightarrow \Gamma^s(\mathbf{P}_R^{\times n})$, which is the identity on objects, and whose effect on morphisms is given by $\gamma_s(f) = f^{\otimes s}$.

DEFINITION 3.7. The category $\mathcal{P}_{s,R}(n)$ of *strict polynomial n -functors of total weight s* is the category of R -linear functors from $\Gamma^s(\mathbb{P}_R^{\times n})$ to R -modules. The functor γ_s induces a forgetful functor

$$\gamma_s^* : \mathcal{P}_{s,R}(n) \rightarrow \text{Func}_*(\mathbb{P}_R^{\times n}, R\text{-Mod}).$$

To extend cross effects to the category of strict polynomial functors, we use the following two observations. First, there is a functor $\Gamma^s(\mathbb{P}_R^{\times n}) \rightarrow \Gamma^s(\mathbb{P}_R^{\times n-1})$ which sends $(M_i)_{1 \leq i \leq n}$ to $(M_1 \oplus M_2, M_3, \dots, M_n)$ (and which is the canonical inclusion on morphisms). Thus, if $F \in \mathcal{P}_{s,R}(n-1)$ then

$$F(M_1 \oplus M_2, M_3, \dots, M_n)$$

can be seen as an element of $\mathcal{P}_{s,R}(n)$, with variables M_1, \dots, M_n . Moreover, if $m \leq n$, the canonical projections $\mathbb{P}_R^{\times n} \rightarrow \mathbb{P}_R^{\times m}$ and inclusions $\mathbb{P}_R^{\times m} \rightarrow \mathbb{P}_R^{\times n}$ on a chosen set of m factors of $\mathbb{P}_R^{\times n}$ induce functors between the corresponding Schur categories. In particular, for $i = 1, 2$ the morphisms induced by the canonical projection from $M_1 \oplus M_2$ to $M_i \oplus 0$ induce a morphism of strict polynomial n -functors

$$F(M_1 \oplus M_2, M_3, \dots, M_n) \rightarrow F(M_i \oplus 0, M_3, \dots, M_n).$$

Thus, there is a well defined functor $\text{cr}_n : \mathcal{P}_{s,R} \rightarrow \mathcal{P}_{s,R}(n)$ fitting into a commutative diagram

$$\begin{array}{ccc} \mathcal{P}_{s,R} & \xrightarrow{\text{cr}_n} & \mathcal{P}_{s,R}(n) \\ \downarrow \gamma_s^* & & \downarrow \gamma_s^* \\ \mathcal{F}_R & \xrightarrow{\text{cr}_n} & \text{Func}_*(\mathbb{P}_R^{\times n}, R\text{-Mod}) \end{array}$$

To analyze cross effects of strict polynomial functors, we use the subcategories of homogeneous strict polynomial functors with weight an n -tuple (s_1, \dots, s_n) of nonnegative integers with $\sum_{i=1}^n s_i = s$. To be more specific, let $\Gamma^{(s_1, \dots, s_n)}(\mathbb{P}_R^{\times n})$ denote the R -linear category with the same objects as $\mathbb{P}_R^{\times n}$, whose morphisms from $M = (M_i)_{1 \leq i \leq n}$ and $N = (N_i)_{1 \leq i \leq n}$ are $\prod_{i=1}^n \mathfrak{S}_{s_i}$ -equivariant morphisms from $\otimes_{i=1}^n (M_i)^{\otimes s_i}$ to $\otimes_{i=1}^n (N_i)^{\otimes s_i}$, and whose composition law is composition of equivariant morphisms. (By convention, $M^{\otimes 0} = R$ and $\mathfrak{S}_0 = \{1\}$ in this definition).

DEFINITION 3.8. The category $\mathcal{P}_{s_1, \dots, s_n, R}(n)$ of *strict polynomial n -functors of weight (s_1, \dots, s_n)* is the category of R -linear functors from $\Gamma^{(s_1, \dots, s_n)}(\mathbb{P}_R^{\times n})$ to R -modules.

There is a canonical isomorphism, compatible with composition of morphisms, where the sum is taken over all n -tuples (s_1, \dots, s_n) of nonnegative integers such that $\sum s_i = s$:

$$(3.2) \quad \text{Hom}_{\Gamma^s(\mathbb{P}_R^{\times n})}(M, N) \simeq \bigoplus \text{Hom}_{\Gamma^{(s_1, \dots, s_n)}(\mathbb{P}_R^{\times n})}(M, N).$$

Thus we have the following functorial analogue of the usual decomposition of modules over a ring with a finite set of central idempotents.

LEMMA 3.9. *The category $\mathcal{P}_{s_1, \dots, s_n, R}(n)$ identifies with an abelian subcategory of $\mathcal{P}_{s, R}(n)$ and there is a decomposition, where the sum is taken over all n -tuples (s_1, \dots, s_n) of nonnegative integers such that $\sum_{i=1}^n s_i = s$:*

$$\mathcal{P}_{s, R}(n) = \bigoplus \mathcal{P}_{s_1, \dots, s_n, R}(n).$$

If we partially evaluate a strict polynomial n -functor $F \in \mathcal{P}_{s_1, \dots, s_n, R}(n)$ on a family $(M_1, \dots, M_{i-1}, M_{i+1}, \dots, M_n) \in \mathbb{P}_R^{\times n-1}$, we get a homogeneous strict polynomial functor of weight s_i of the remaining variable. In particular, we obtain the following result.

PROPOSITION 3.10. *Let $F \in \mathcal{P}_{s, R}$ be a nonzero p -primary functor of degree d . Then there exist d nonnegative integers r_i such that $\sum_{i=1}^d p^{r_i} = s$.*

Proof. Since F has degree d , its d th cross effect is a nonzero element of $\mathcal{P}_{s, R}(d)$. Let G be a nonzero homogeneous summand of weight (s_1, \dots, s_d) of $\text{cr}_d F$. Then G is additive with respect to each of its variables (indeed $\text{cr}_d F$ is so because $\text{cr}_{d+1} F = 0$). By the classification of additive functors in Proposition 3.5, this implies that each s_i has the form p^{r_i} for some nonnegative r_i , whence the result. ■

The following elementary lemma, together with Proposition 3.10, provides a proof of the first statement of Theorem 1.2.

LEMMA 3.11. *Let d be a positive integer. Then $d \in \mathcal{I}(p, s)$ if and only if there exist d nonnegative integers r_i such that $\sum_{i=1}^d p^{r_i} = s$.*

Proof. If x and y are positive integers, we have $\Sigma_p(x + y) + c(p - 1) = \Sigma_p(x) + \Sigma_p(y)$, where c is the number of carries which appear when we add x and y using p -adic expansions. In particular, if $s = \sum_{i=1}^d p^{r_i}$, there exists a nonnegative integer c' such that $\Sigma_p(s) + c'(p - 1) = d$. Thus $d \in \mathcal{I}(p, s)$. Conversely, we prove by induction on k that for all $\Sigma_p(s) + k(p - 1) \in \mathcal{I}(p, s)$, there exists a decomposition of s as a sum of $\Sigma_p(s) + k(p - 1)$ powers of p . The decomposition for $k = 0$ is given by the p -adic decomposition of s . If $\Sigma_p(s) + k(p - 1) < s$ and $s = \sum p^{r_i}$ for $\Sigma_p(s) + k(p - 1)$ nonnegative integers r_i , then one of the r_i is strictly positive, and by replacing p^{r_i} by p terms p^{r_i-1} , we see that the property is valid for $k + 1$. ■

3.3. The integral torsion of strict polynomial functors. In this section, we prove Theorem 1.2(2) and Proposition 1.1. A *composition* of s into n parts is an n -tuple $\lambda = (\lambda_1, \dots, \lambda_n)$ of positive integers with $\sum \lambda_i = s$. We denote by $\text{Comp}(s, n)$ the finite set of all compositions of s into n parts. The set $\text{Comp}(s, n)$ is empty if and only if $n > s$. If $n \leq s$, we denote by

$\tau(s, n)$ the greatest common divisor of the set of multinomials $\binom{s}{\lambda_1, \dots, \lambda_n}$, for all compositions $(\lambda_1, \dots, \lambda_n)$ of s into n parts:

$$\tau(s, n) = \text{g.c.d.} \left\{ \binom{s}{\lambda_1, \dots, \lambda_n} \mid (\lambda_1, \dots, \lambda_n) \in \text{Comp}(s, n) \right\}.$$

DEFINITION 3.12. Let R be a commutative ring. Let d and s be two positive integers satisfying $d < s$. We denote by $C_{s,d+1} \in \mathcal{P}_{s,R}$ the cokernel of the map induced by multiplication of the divided power algebra

$$\bigoplus_{\lambda \in \text{Comp}(s,d+1)} \Gamma^\lambda \xrightarrow{\sum \text{mult}} \Gamma^s.$$

LEMMA 3.13. *The functor $C_{s,d+1} \in \mathcal{P}_{s,R}$ takes values in torsion abelian groups. Its integral torsion is bounded by $\tau(s, d + 1)$.*

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_{d+1})$ be a composition of s , and let $n_\lambda := \binom{s}{\lambda_1, \dots, \lambda_{d+1}}$. Then the following composite is zero:

$$(*) \quad \Gamma^p \xrightarrow{\times n_\lambda} \Gamma^p \twoheadrightarrow C_{s,d+1}.$$

Indeed, by Lemma 2.12, multiplication by n_λ is equal to the composite $\Gamma^s \hookrightarrow \Gamma^\lambda \xrightarrow{\text{mult}} \Gamma^s$, and by definition of $C_{s,d+1}$, the composite $\Gamma^\lambda \rightarrow \Gamma^s \rightarrow C_{s,d+1}$ equals zero. Since $(*)$ is zero, the torsion of $C_{s,d+1}$ is bounded by n_λ , for all compositions λ . Hence the torsion of $C_{s,d+1}$ is bounded by $\tau(s, d + 1)$. ■

Our next task is to explicitly compute $\tau(s, n)$. Given a nonnegative integer x , we denote by $N_p(x)$ the exponent of the prime p in the prime decomposition of x . In particular we have

$$N_p(\tau(s, n)) = \min \left\{ N_p \left(\binom{s}{\lambda_1, \dots, \lambda_n} \right) \mid \lambda \in \text{Comp}((s), n) \right\}.$$

LEMMA 3.14. *Let s, n be positive integers with $n \leq s$.*

$$N_p(\tau(s, n)) = \begin{cases} 0 & \text{if } n \leq \Sigma_p(s), \\ \left\lceil \frac{n - \Sigma_p(s)}{p - 1} \right\rceil & \text{if } n > \Sigma_p(s), \end{cases}$$

where the brackets denote the ceiling function.

Proof. We first observe that if $n \leq m$, then

$$(3.3) \quad N_p(\tau(s, n)) \leq N_p(\tau(s, m)).$$

Indeed, we can assume $m = n + 1$. Let $\lambda = (\lambda_1, \dots, \lambda_{n+1})$ be a composition of s such that $N_p(\tau(s, n + 1)) = N_p(\binom{s}{\lambda})$, and let $\lambda' = (\lambda_1, \dots, \lambda_n + \lambda_{n+1})$. Then $\binom{s}{\lambda} = \binom{\lambda_n + \lambda_{n+1}}{\lambda_n, \lambda_{n+1}} \binom{s}{\lambda'}$. So we have

$$N_p(\tau(s, n)) \leq N_p \left(\binom{s}{\lambda'} \right) \leq N_p \left(\binom{s}{\lambda} \right) = N_p(\tau(s, n + 1)).$$

Let $s = s_k p^k + \dots + s_1 p + s_0$ be the p -adic expansion of s . By Kummer's theorem [28, p. 32], the multinomial associated to the composition $(p^k, \dots, p^k, \dots, p^0, \dots, p^0)$, where each p^i has multiplicity s_i , is prime to p . Thus, $N_p(\tau(s, \Sigma_p(s))) = 0$. Hence (3.3) implies

$$(3.4) \quad N_p(\tau(s, n)) = 0 \quad \text{if } 1 \leq n \leq \Sigma_p(s).$$

There is only one composition of s into s parts, namely $(1, \dots, 1)$. So by Legendre's theorem [28, p. 32], we have

$$(3.5) \quad N_p(\tau(s, s)) = \frac{s - \Sigma_p(s)}{p - 1}.$$

We have determined $N_p(\tau(s, n))$ for extremal values of n . We are now going to determine how $N_p(\tau(s, n))$ grows with n . Let $(\lambda_1, \dots, \lambda_n)$ be a composition of s such that $N_p(\tau(s, n)) = N_p\left(\binom{s}{\lambda}\right)$. We distinguish two cases.

First case: $n \not\equiv \Sigma_p(s) \pmod{p-1}$. Then by Lemma 3.11, one of the λ_i is not a power of p . For such a λ_i , one can find positive integers k, ℓ such that $k + \ell = \lambda_i$ and $N_p\left(\binom{\lambda_i}{k, \ell}\right) = 0$ (use the p -adic expansion of λ_i in a fashion similar to the proof of (3.4)). Thus if $\lambda' = (\lambda_1, \dots, \lambda_{i-1}, k, \ell, \lambda_{i+1}, \dots, \lambda_n)$, we have

$$N_p(\tau(s, n)) = N_p\left(\binom{s}{\lambda}\right) = N_p\left(\binom{s}{\lambda'}\right) \geq N_p(\tau(s, n + 1)).$$

Hence (3.3) implies that

$$(3.6) \quad N_p(\tau(s, n)) = N_p(\tau(s, n + 1)) \quad \text{if } n - \Sigma_p(s) \not\equiv 0 \pmod{p-1}.$$

Second case: $n \equiv \Sigma_p(s) \pmod{p-1}$ with $n < s$. If one of the λ_i is not a power of p , then we may argue as in the first case to obtain

$$N_p(\tau(s, n)) = N_p(\tau(s, n + 1)).$$

Assume now that all the λ_i are powers of p . Since $n < s$, one of the λ_i is not equal to 1, i.e. is of the form $\lambda_i = p^r$ with $r > 0$. By Kummer's theorem, $N_p\left(\binom{p^r}{p^{r-1}, (p-1)p^{r-1}}\right)$ equals 1, so if we denote by λ' the composition $(\lambda_1, \dots, \lambda_{i-1}, p^{r-1}, (p-1)p^{r-1}, \lambda_{i+1}, \dots, \lambda_n)$, we have

$$N_p(\tau(s, n)) + 1 = N_p\left(\binom{s}{\lambda}\right) + 1 = N_p\left(\binom{s}{\lambda'}\right) \geq N_p(\tau(s, n + 1)).$$

Hence

$$(3.7) \quad N_p(\tau(s, n + 1)) \leq N_p(\tau(s, n)) + 1 \quad \text{if } n - \Sigma_p(s) \equiv 0 \pmod{p-1}.$$

By (3.4) and (3.5), the difference $N_p(\tau(s, s)) - N_p(\Sigma_p(s))$ is equal to $(s - \Sigma_p(s))/(p - 1)$. In view of (3.6), this implies that (3.7) must be an equality. Lemma 3.14 follows by induction on n . ■

The following proposition provides a proof of Proposition 1.1 and of Theorem 1.2(2).

PROPOSITION 3.15. *Let $F \in \mathcal{P}_{s,R}$ be a strict polynomial functor of degree $d < s$. Then the values of F , considered as abelian groups, are torsion groups. Moreover the torsion of F is bounded by $\tau(s, d + 1)$. In particular, if F takes values in p -primary abelian groups, then the torsion of F is bounded by p^r with $r = \lceil \frac{d+1-\Sigma_p(s)}{p-1} \rceil$.*

Proof. Given $F \in \mathcal{P}_{s,R}$, the functor F_M defined by $F_M(N) = F(M \otimes N)$ is also an object of $\mathcal{P}_{s,R}$, with the same degree as F . Moreover, $F_M(R) = F(M)$. So it actually suffices to prove that for all $F \in \mathcal{P}_{s,R}$, $F(R)$ is a torsion abelian group, with torsion bounded by $\tau(s, d + 1)$. By the Yoneda lemma, for all $x \in F(R)$ we can find a morphism $f_x : \Gamma^s \rightarrow F$ such that x lies in the image of f_x . Since F has degree d , the composite

$$\bigoplus_{\lambda \in \text{Comp}(s, d+1)} \Gamma^\lambda \xrightarrow{\sum \text{mult}} \Gamma^s \xrightarrow{f_x} F$$

is zero, hence f_x factors into a morphism $\bar{f}_x : C_{s, d+1} \rightarrow F$ which contains x in its image. Since the torsion of $C_{s, d+1}$ is bounded by $\tau(s, d + 1)$, we obtain $\tau(s, d + 1)x = 0$. This holds for all $x \in F(R)$, so the conclusion follows. ■

4. Applications to integral torsion in homology. In Section 4.1, we describe conditions which imply polynomiality of an abstract ‘homological invariant’ Ψ . These conditions are then applied together with Theorem 1.2 to study integral torsion in concrete examples in Sections 4.2–4.5.

4.1. Functor categories and parameterization. In this subsection, we fix a commutative ring R and we denote by \mathcal{F} one of the functor categories \mathcal{F}_R or $\mathcal{P}_{s,R}$. Given $F \in \mathcal{F}$ and $M \in \text{P}_R$, we define the parameterized functors F_M and F^M by

$$F_M : N \mapsto F(M \otimes_R N), \quad F^M : N \mapsto F(\text{Hom}_R(M, N)).$$

Parameterization yields exact functors $-^M, -_M : \mathcal{F} \rightarrow \mathcal{F}$, which are left and right adjoint to each other. By taking degreewise parameterization of chain complexes in \mathcal{F} , we get endofunctors of the (unbounded) derived category, which are left and right adjoint to each other:

$$-^M, -_M : \mathbf{D}(\mathcal{F}) \rightarrow \mathbf{D}(\mathcal{F}).$$

We will work in the following framework.

FRAMEWORK 4.1. We fix a category \mathcal{C} which is a full additive subcategory of $\mathbf{D}(\mathcal{F})$, idempotent complete (i.e. stable under direct summands) and stable under parameterization (i.e. for all $M \in \text{P}_R$, the parameterization functor $-_M$ restricts to $-_M : \mathcal{C} \rightarrow \mathcal{C}$). We consider an additive functor

$$\Psi : \mathcal{C} \rightarrow R\text{-Mod.}$$

Our purpose is to obtain some information on the integral torsion of the values of Ψ . To this end, we study the functors $\Psi_C \in \mathcal{F}$, defined for all $C \in \mathcal{C}$ by

$$\Psi_C(M) := \Psi(C_M).$$

In the remainder of Section 4.1, we study the degree of the functors Ψ_C .

The definition of degree extends to complexes of functors. To be more specific, the n th cross effect of a complex of functors is defined by taking n th cross effects degreewise. This yields an exact functor (where $\mathcal{F}(n)$ is the category of functors with n variables)

$$\text{cr}_n : \mathbf{Ch}(\mathcal{F}) \rightarrow \mathbf{Ch}(\mathcal{F}(n))$$

and also a functor between the corresponding derived categories. A complex C has degree less than d if $\text{cr}_d C \simeq 0$ in $\mathbf{D}(\mathcal{F}(d))$, or equivalently if all its homology groups $H_i(C)$ are functors of degree less than d in the sense of Definition 2.2.

Given a complex of functors C and $M_1, \dots, M_n \in \text{P}_R$, we introduce a complex of functors $C[M_1, \dots, M_n]$:

$$C[M_1, \dots, M_n] : N \mapsto \text{cr}_n C(M_1 \otimes N, \dots, M_n \otimes N).$$

By definition, $C[M_1, \dots, M_n]$ is a direct summand of the parameterized complex $C_{M_1 \oplus \dots \oplus M_n}$. In particular, if C is an object of the category \mathcal{C} of Framework 4.1, then so is $C[M_1, \dots, M_n]$. The following result is a straightforward consequence of the additivity of Ψ .

LEMMA 4.2. *Let $C \in \mathcal{C}$. Then for all $M_1, \dots, M_n \in \text{P}_R$,*

$$\text{cr}_n(\Psi_C)(M_1, \dots, M_n) = \Psi(C[M_1, \dots, M_n]).$$

In particular, if C is polynomial of degree less than d , then so is Ψ_C .

Lemma 4.2 gives a condition which ensures that Ψ_C , for a given $C \in \mathcal{C}$, is polynomial. In concrete cases, we often encounter functors Ψ such that Ψ_C is polynomial of degree less than d for all $C \in \mathcal{C}$. Such global polynomiality is linked with the vanishing of Ψ on tensor products.

PROPOSITION 4.3 (Global polynomiality, necessary condition). *Assume that for all $C \in \mathcal{C}$, Ψ_C has degree less than d . Then for all families of degreewise reduced complexes C_k , $1 \leq k \leq d$, such that $C_1 \otimes \dots \otimes C_d \in \mathcal{C}$, we have*

$$\Psi(C_1 \otimes \dots \otimes C_d) = 0.$$

Proof. Since the complexes C_i are degreewise reduced, i.e. the complexes of R -modules $C_i(0)$ are equal to zero, the tensor product $C_1(M_1) \otimes \dots \otimes C_d(M_d)$ is a direct summand of $\text{cr}_n(C_1 \otimes \dots \otimes C_n)(M_1, \dots, M_d)$. So $C_1 \otimes \dots \otimes C_d$ is a direct summand of $(C_1 \otimes \dots \otimes C_d)[R, \dots, R]$. By Lemma 4.2, Ψ vanishes on $(C_1 \otimes \dots \otimes C_d)[R, \dots, R]$, hence on $C_1 \otimes \dots \otimes C_d$. ■

A converse of Proposition 4.3 holds when \mathcal{C} equals $\mathbf{D}_{\geq 0}(\mathcal{F})$, the full subcategory of $\mathbf{D}(\mathcal{F})$ of nonnegatively graded chain complexes, and when Ψ is some homology group of a homological functor $\mathbf{D}_{\geq 0}(\mathcal{F}) \rightarrow \mathbf{D}_{\geq 0}(R\text{-Mod})$ (i.e. an additive functor commuting with suspension and preserving exact triangles).

PROPOSITION 4.4 (Global polynomiality, sufficient condition). *Let*

$$\Phi : \mathbf{D}_{\geq 0}(\mathcal{F}) \rightarrow \mathbf{D}_{\geq 0}(R\text{-Mod})$$

be a homological functor, and let $\Psi_i = H_i\Phi$. Assume that for all $i \leq k$, Ψ_i vanishes on direct sums of d -fold tensor products of reduced functors. Then for all $i \leq k$, and all $C \in \mathbf{D}_{\geq 0}(\mathcal{F})$, $(\Psi_i)_C$ is polynomial of degree less than d .

Proof. Let a *special functor* be a tensor product $F_1 \otimes \cdots \otimes F_d \in \mathcal{F}$ where all the F_i are reduced with R -projective values, and let a *special complex* be a complex of direct sums of special functors.

First, for $i \leq k$, Ψ_i vanishes on special complexes. This follows from a dimension shifting argument. If C is a special chain complex, the exact sequence of complexes $C_0 \rightarrow C \rightarrow C' = C/C_0$ yields an exact triangle in $\mathbf{D}_{\geq 0}(\mathcal{F})$, and since $\Psi_i(C_0) = 0$ for $0 \leq i \leq k$, we get $\Psi_0(C) = 0$, and $\Psi_i(C) = \Psi_i(C') = \Psi_{i-1}(C'[-1])$ for $1 \leq i \leq k$. So the result is obtained by induction on i . Thus, by Lemma 4.2, it suffices to prove that for all complexes C and all R -modules M_i , $C[M_1, \dots, M_d]$ is isomorphic in $\mathbf{D}_{\geq 0}(\mathcal{F})$ to a special complex.

To prove this, we use the following key property. Any reduced (i.e. equal to zero as soon as one variable is zero) functor in $\mathcal{F}(d)$ admits a projective resolution by projective functors of the form

$$(*) \quad (N_1, \dots, N_d) \rightarrow P_1(N_1) \otimes \cdots \otimes P_d(N_d),$$

where the one-variable functors P_i are reduced. Indeed, in the case $\mathcal{F} = \mathcal{P}_{d,R}$, the standard projectives are functors of the form $\Gamma^{s_1, X_1} \otimes \cdots \otimes \Gamma^{s_d, X_d}$, and if $\mathcal{F} = \mathcal{F}_R$, the standard projectives are of the form $R\text{Hom}_R(X_1, -) \otimes \cdots \otimes R\text{Hom}_R(X_d, -)$ (here $R\text{Hom}_R(X, N)$ denotes the free R -module with basis $\text{Hom}_R(X, N)$). Moreover, if F is reduced, one can use the canonical maps of R -modules $0 \rightarrow N_i$ and $N_i \rightarrow 0$ to split off the nonreduced part in a projective resolution.

The key property implies that the Cartan–Eilenberg projective resolution of the complex $\text{cr}_d C \in \mathbf{Ch}_{\geq 0}(\mathcal{F}(d))$ is a complex $D \in \mathbf{Ch}_{\geq 0}(\mathcal{F}(d))$ whose objects are of the form $(*)$. By evaluating the quasi-isomorphism $D \rightarrow C$ on the R -modules $M_i \otimes N$ we get a quasi-isomorphism of complexes of functors with a single variable N :

$$(**) \quad D(M_1 \otimes N, \dots, M_d \otimes N) \rightarrow C[M_1, \dots, M_d](N).$$

The complex on the left-hand side is a special complex. This finishes the proof. ■

REMARK 4.5. The results of this section have obvious variants. For example, we can consider a *contravariant* functor Ψ . In this case, we use upper parameterization $\Psi_C(M) := \Psi(C^M)$ to get a functor $\Psi_C \in \mathcal{F}$. We can also consider a functor $\Psi : \mathcal{C} \rightarrow \text{Ab}$ where \mathcal{C} is a subcategory of $\mathbf{D}(\mathcal{F}_R)$ or $\mathbf{D}(\mathcal{P}_{s,R})$. In this case, we use free abelian groups A as parameters: for all $M \in \text{P}_R$, we set $F_A(M) := F(A \otimes_{\mathbb{Z}} M)$ and $F^A(M) := F(\text{Hom}_{\mathbb{Z}}(A, M))$. Therefore the functors Ψ_C are functors from free abelian groups to abelian groups. In both cases, the results of the section remain valid *mutatis mutandis*.

4.2. The Taylor towers of strict polynomial functors. Given a functor $F \in \mathcal{F}_R$ we still denote by F its simplicial extension

$$F : \text{simp}(\text{P}_R) \rightarrow \text{simp}(R\text{-Mod}).$$

The Taylor tower of this simplicial extension, as defined by Johnson and McCarthy [21], is a diagram of functors with source $\text{simp}(\text{P}_R)$ and target $\text{simp}(R\text{-Mod})$, of the following form:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{q_{n+1}} & P_n F & \xleftarrow{q_n} & \cdots & \xrightarrow{q_2} & P_1 F & \xrightarrow{q_1} & P_0 F \\ & & & & & & & \swarrow p_1 & \uparrow p_0 \\ & & & & & & & & F \\ & & & & & & & \nwarrow p_n & \end{array}$$

The term $P_n F$ is the *n*th polynomial approximation of F . Let us recall its definition when F is a reduced functor (i.e. $F(0) = 0$). As recalled in Section 2, the $(n + 1)$ th diagonal and the $(n + 1)$ th cross effect yield a pair of adjoint functors between the categories of reduced functors:

$$(4.1) \quad \text{cr}_{n+1} : \text{Func}_*(\text{P}_R, R\text{-Mod}) \rightleftarrows \text{Func}_*(\text{P}_R^{\times n+1}, R\text{-Mod}) : \Delta_{n+1}^*.$$

This pair of adjoints yields [34, Chap. 8] a cotriple $\perp_{n+1} = \text{cr}_{n+1} \circ \Delta_{n+1}^*$. Hence for all reduced functors $F \in \mathcal{F}_R$, we have an augmented simplicial object $\perp_{n+1}^{*+1} F \xrightarrow{\epsilon} F$ in the category of reduced functors. Hence, given a simplicial object X , we obtain a morphism of bisimplicial R -modules $\perp_{n+1}^{*+1} F(X) \xrightarrow{\epsilon} F(X)$. The simplicial R -module $P_n F(X)$ is obtained by taking first the diagonal of these bisimplicial R -modules and then the homotopy cofiber of the resulting morphism ϵ of simplicial R -modules.

REMARK 4.6. In [21], the Taylor towers are actually defined for functors with values in chain complexes. The Dold–Kan correspondence asserts that $\text{simp}(R\text{-Mod})$ is equivalent to the category of nonnegative chain complexes of R -modules, so the theory of [21] applies to our case. The definition given above is seen to be equivalent to the definition of [21, p. 768] by using the

Eilenberg–Zilber theorem and the fact that the homotopy cofiber of simplicial R -modules corresponds to the cone in the category of chain complexes.

Let us fix $X \in \text{simp}(\mathbb{P}_R)$ and integers i, n , and denote by $\Psi(F)$ the i th homotopy group of $P_n F(X)$ (i.e. the i th homology group of the normalized chains of $P_n F$ [34, Chap. 8]). Then Ψ yields an additive functor

$$\Psi : \mathcal{F}_R \rightarrow R\text{-Mod.}$$

Following the framework of Section 4.1, we introduce the parameterized functor with parameter $M \in \mathbb{P}_R$:

$$\Psi_F : M \mapsto \Psi(F_M) = \pi_i(P_n F(M \otimes X)).$$

LEMMA 4.7. *Let $F \in \mathcal{F}_R$. The parameterized functor Ψ_F is polynomial of degree $\leq n$. If F is the underlying ordinary functor of a homogeneous strict polynomial functor of weight $s > 0$, then so is Ψ_F .*

Proof. The fact that Ψ_F is polynomial of degree $\leq n$ follows from the fact that $P_n F$ is a degree n approximation of F [21, Lemma 2.11(1)]. Now as explained in Section 3.2, the adjunction (4.1) lifts on the level of strict polynomial functors. Hence the definition of the Taylor tower lifts to the level of strict polynomial functors. In particular, if F is the underlying ordinary functor of a strict polynomial functor $\tilde{F} \in \mathcal{P}_{s,R}$, then Ψ_F is the underlying ordinary functor of the strict polynomial functor $\Psi_{\tilde{F}} \in \mathcal{P}_{s,R}$ defined similarly. ■

As a consequence of Lemma 4.7, Proposition 1.1 and Theorem 1.2, we obtain the following estimation of the integral torsion of $\pi_i(P_n F(X))$.

THEOREM 4.8. *Let R be a commutative ring, let s be a positive integer, let $F \in \mathcal{P}_{s,R}$, and let $n < s$. Then for all integers i and all $X \in \text{simp}(\mathbb{P}_R)$, $\pi_i(P_n F(X))$ is a torsion abelian group. Its p -primary part is zero if $n < \Sigma_p(s)$, and it is bounded by p^r with $r = \lceil \frac{n+1-\Sigma_p(s)}{p-1} \rceil$ otherwise.*

The homotopy fiber of the morphism $p_n : P_n F \rightarrow P_{n-1} F$ is called the n th layer of the Taylor tower of F , and is denoted by $D_n F$ [21, p. 772]. As another consequence of Theorem 1.2, we prove the following lacunarity phenomenon for the homotopy groups of $D_n F$.

THEOREM 4.9. *Let R be a commutative ring, let s be a positive integer, let $F \in \mathcal{P}_{s,R}$, and let $n < s$. Then for all integers i and all $X \in \text{simp}(\mathbb{P}_R)$, $\pi_i(D_n F(X))$ is a torsion abelian group. If $n \not\equiv s \pmod{p-1}$, then its p -primary part is zero.*

Proof. By Theorem 4.8, $\pi_i(P_n F(X))$ and $\pi_{i+1}(P_{n-1} F(X))$ are torsion abelian groups. Hence $\pi_i(D_n F(X))$ is also a torsion abelian group by the long exact sequence of homotopy groups

$$\cdots \rightarrow \pi_{i+1}(P_{n-1} F(X)) \xrightarrow{\partial} \pi_i(D_n F(X)) \rightarrow \pi_i(P_n F(X)) \rightarrow \cdots .$$

To prove that the p -primary part of $\pi_i(D_n F(X))$ is zero, it suffices to prove that the localization $\pi_i(D_n F(X)) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is zero. Since localization is exact, this is equivalent to proving that $\pi_i(D_n G(X))$ is zero, where G denotes the functor which sends X to $F(X) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. Let us define $\bar{G} := G \circ \iota_X$, where $\iota_X : \mathbb{P}_R \rightarrow \text{simp}(\mathbb{P}_R)$ is the functor defined by $\iota_X(M) = M \otimes X$. Since ι_X is additive, we have

$$P_n \bar{G} = (P_n G) \circ \iota_X, \quad D_n \bar{G} = (D_n G) \circ \iota_X.$$

Hence, it suffices to prove that for all $M \in \mathbb{P}_R$, $D_n \bar{G}(M)$ has trivial homotopy groups if $n \neq s \pmod{p-1}$.

To prove this, we use the universal property of $P_n \bar{G}$. To be more specific, let $K : \mathbb{P}_R \rightarrow \text{simp}(R\text{-Mod})$ be a functor which is polynomial of degree $\leq n$ (i.e. all its homotopy groups are polynomial functors of degree $\leq n$ in the sense of Definition 2.2), and let $\tau : \bar{G} \rightarrow K$ be a natural transformation. Then for all $M \in \mathbb{P}_R$, we can consider the morphisms $\tau : \bar{G}(M) \rightarrow K(M)$ and $p_n : \bar{G}(M) \rightarrow P_n \bar{G}(M)$ in the homotopy category of simplicial R -modules, and it is shown in [5, Lemma 2.11(3)] that there exists a unique morphism τ' in this homotopy category which fits into a commutative diagram

$$\begin{array}{ccc} \bar{G}(M) & \xrightarrow{\tau} & K(M) \\ p_n \downarrow & \nearrow \exists! \tau' & \\ P_n \bar{G}(M) & & \end{array}$$

By its definition [21, p. 770], the map $q_n : P_n \bar{G} \rightarrow P_{n-1} \bar{G}$ in the Taylor tower of \bar{G} is a representative of the morphism obtained by factorization of $p_{n-1} : \bar{G} \rightarrow P_{n-1} \bar{G}$ through p_n . Given a functor $K : \mathbb{P}_R \rightarrow \text{simp}(R\text{-Mod})$, we let $\text{deg } K$ be the supremum of the degrees (in the sense of Definition 2.2) of the functors $M \mapsto \pi_i(K(M))$. The universal property of $P_n \bar{G}$ and the construction of q_n have the following consequences:

- (i) for all integers n , $\text{deg}(P_{n-1} \bar{G}) \leq \text{deg}(P_n \bar{G})$,
- (ii) $D_n \bar{G}(M)$ has trivial homotopy groups for all $M \in \mathbb{P}_R$ if and only if $\text{deg}(P_{n-1} \bar{G}) = \text{deg}(P_n \bar{G})$.

Statements (i) and (ii) in turn imply

- (iii) if $\text{deg}(P_n \bar{G}) = d$, then the morphism $q_k : P_k \bar{G} \rightarrow P_{k-1} \bar{G}$ induces an isomorphism at the level of homotopy groups for $d < k \leq n$.

Finally, the functors $M \mapsto \pi_i(P_n \bar{G}(M))$ are strict polynomial of weight s , of degree at most n . Moreover, by the construction of \bar{G} , $\pi_i(P_n \bar{G}(M))$ is a p -primary functor. Hence, Theorem 1.2 implies that

- (iv) $\text{deg}(P_n \bar{G}) \in (\mathcal{I}(p, s) \cap [1, n]) \cup \{-\infty\}$.

To sum up, we have proved that the function

$$\mathbb{N} \rightarrow \{-\infty\} \cup \mathbb{N}, \quad n \mapsto \deg(P_n \overline{G}),$$

is nondecreasing by (i), with values in $\mathcal{I}(p, s) \cup \{-\infty\}$ by (iv). Moreover the jumps, i.e. the values for which $\deg P_n \overline{G} > \deg P_{n-1} \overline{G}$, must occur for $n \in \mathcal{I}(p, s)$ by (iii). In particular, the jumps must occur for $n = s \pmod{p-1}$, and by (ii), the homotopy groups of $D_n \overline{G}$ are zero for $n \not\equiv s \pmod{p-1}$. This finishes the proof. ■

4.3. Derived functors. Given a complex $C \in \mathbf{Ch}_{\geq 0}(\mathcal{F}_R)$ and a simplicial object $X \in \mathbf{Simpl}(\mathcal{P}_R)$ we can form a bigraded object $C_i(X_j)$ equipped with a differential relative to the index i and a simplicial structure relative to the index j . By taking normalized chains and the total complex, we obtain a chain complex, which we denote by $c_X(C)$. If X is fixed, this defines a homological functor

$$\Phi := c_X : \mathbf{D}_{\geq 0}(\mathcal{F}_R) \rightarrow \mathbf{D}_{\geq 0}(R\text{-Mod}).$$

Following the framework of Section 4.1, we introduce the parameterized homology groups with parameter $M \in \mathcal{P}_R$, as an object of \mathcal{F}_R :

$$(\Psi_i)_C : M \mapsto H_i(\Phi_C(M)) = H_i(c_{X \otimes M} C).$$

LEMMA 4.10.

- (1) If C is the underlying complex of a complex in $\mathcal{P}_{s,R}$, then the functor $(\Psi_i)_C$ is the underlying ordinary functor of a homogeneous strict polynomial functor of weight s .
- (2) If X is $(n-1)$ -connected, then for all $i < nd$ and all C , the functor $(\Psi_i)_C$ is polynomial of degree less than d .

Proof. (1) The definition of $c_X \tilde{C}$ above makes sense when \tilde{C} is a complex of strict polynomial functors. Hence if C is the underlying complex of ordinary functors a complex $\tilde{C} \in \mathbf{Ch}_{\geq 0}(\mathcal{P}_{s,R})$, then $(\Psi_i)_C$ is the underlying functor of the strict polynomial functor $(\Psi_i)_{\tilde{C}} \in \mathcal{P}_{s,R}$.

(2) The simplicial R -module X is homotopy equivalent to a simplicial R -module Y with $Y_i = 0$ for $i < n$. So for $F \in \mathcal{F}_R$, $c_X F$ is homotopy equivalent to $c_Y F$. Assume that F is of the form $F_1 \otimes \cdots \otimes F_d$. Then the Eilenberg–Zilber theorem yields a homotopy equivalence

$$(*) \quad c_Y(F_1 \otimes \cdots \otimes F_d) \simeq c_Y(F_1) \otimes \cdots \otimes c_Y(F_d).$$

If the functors F_k are reduced, then $(c_Y F_k)_i = 0$ for $i < n$. Thus, the right-hand side of $(*)$ is a complex which is zero in degrees $i < nd$. Hence

$$\Psi_i(F_1 \otimes \cdots \otimes F_d) = H_i(c_X(F_1 \otimes \cdots \otimes F_d)) = 0$$

for $i < nd$, and the result follows from Proposition 4.4. ■

By Lemma 4.10, if C is a chain complex in $\mathcal{P}_{s,R}$, we can apply Theorem 1.2 to the functor $M \mapsto H_i(c_{X \otimes M} C)$ to obtain the following properties of torsion in the homology of the complex of abelian groups $c_X C$.

THEOREM 4.11. *Let R be a commutative ring, let $C \in \text{Ch}_{\geq 0}(\mathcal{P}_{s,R})$, and let X be an $(n - 1)$ -connected simplicial R -module, which is degreewise finitely generated and projective over R .*

- (1) *For $i < ns$, $H_i(c_X C)$ is a torsion abelian group.*
- (2) *For $i < nd$ with $\Sigma_p(s) \leq d < s$, the p -primary torsion of $H_i(c_X C)$ is bounded by p^r with $r = \lceil \frac{d - \Sigma_p(s)}{p - 1} \rceil$.*
- (3) *In particular, for $i < n\Sigma_p(s)$ the p -primary part of $H_i(c_X C)$ is zero, and for $i < n$, $H_i(c_X C)$ is equal to zero.*

The homology groups $H_i(c_X C)$ studied in Theorem 4.11 are known under various names, depending on the complex C and the simplicial object X considered. To make Theorem 4.11 more concrete, we give some of them below. We will denote by $K(M, n)$ a free simplicial R -module whose homotopy groups are all zero but $\pi_n(K(M, n)) \simeq M$.

EXAMPLE 4.12 (Derived functors). For a finitely generated abelian group A and a functor $F \in \mathcal{F}_{\mathbb{Z}}$ (considered as a complex concentrated in degree zero), the homology groups $H_i(c_{K(A,n)} F)$ are denoted by $L_i F(A; n)$ in [11, 12] and called the *derived functors* of F .

Theorem 4.11 gives bounds for the integral torsion of these derived functors when F is actually strict polynomial. Our result does not say anything about $L_i F(A; n)$ for $i \geq ns$, but it is not hard to show that $L_i F(A; n) = 0$ for $i > ns$ and A free. This can be proved directly by using an explicit model of $K(A, n)$ and cross effects of F . Similarly, if A is not free, one can prove (using cross effects and the fact that abelian groups have two-step free resolutions) that $L_i F(A; n) = 0$ for $i > (n + 1)s + 1$. So, almost all degrees are covered by the results above.

EXAMPLE 4.13 (Stable derived functors). The $L_{i+n} F(A; n)$ for $0 \leq i < n$ are called the *stable derived functors* of F (they do not depend on n), and denoted by $L_i^{\text{st}} F(A)$. Assume that $F \in \mathcal{P}_{s,\mathbb{Z}}$ with $s > 1$. As a particular case of Theorem 4.11, we find that $L_i^{\text{st}} F(A)$ is zero if $s \neq p^r$, and is a \mathbb{F}_p -vector space if $s = p^r$.

This result was proved by Dold and Puppe [12] in the special case $F = S^s$ (their proof does not work for an arbitrary strict polynomial functor). Since there is an isomorphism $\bigoplus_{s \geq 0} L_i^{\text{st}} S^s(A) = H\mathbb{Z}_*(HA)$ [12], this result explains why only prime torsion occurs in the stable homology of Eilenberg–Mac Lane spaces, as alluded to in the introduction.

EXAMPLE 4.14 (Singular homology of symmetric spaces). Let G be a subgroup of the symmetric group \mathfrak{S}_s . Given a CW-complex Y we can consider its G -symmetric product $SP^G(Y) := Y^{\times s}/G$ where G acts by permuting the factors of the product. We denote by $S^G \in \mathcal{P}_{s,\mathbb{Z}}$ the algebraic analogue of G -symmetric products, namely $S^G(A) = (A^{\otimes s})_G$. Assume that Y is $(n-1)$ -connected with finitely generated singular homology groups $H_i(Y)$ for all i ($n \geq 2$). Then the simplicial abelian group of singular chains of Y is homotopy equivalent to an $(n-1)$ -connected simplicial abelian group $X \in \text{simp}(\mathbb{P}_{\mathbb{Z}})$. Dold proved [10, proof of Thm. (7.2)] that $H_i(c_X S^G)$ is isomorphic to the singular homology with integral coefficients $H_i(SP^G(Y))$.

EXAMPLE 4.15 (Ringel duality). Let R be a commutative ring. The classical Ringel duality functor for representations of Schur algebras can be reformulated in a nice way by using the language of strict polynomial functors. To be more specific, its n -fold iteration can be described [31, 32] as an equivalence of triangulated categories

$$\Theta^n : \mathbf{D}(\mathcal{P}_{s,R}) \rightarrow \mathbf{D}(\mathcal{P}_{s,R})$$

which sends a complex C to the complex $M \mapsto c_{K(M,n)}C[-ns]$, where the brackets denote suspension, that is, $H_i(C[-ns]) = H_{ns+i}(C)$.

4.4. Functor (co)homology. Let R be a commutative ring. By *functor (co)homology*, we mean Tor or Ext-groups in \mathcal{F}_R , or in related categories. These groups have interpretations in terms of Topological Hochschild Homology [20, 27], and stable homology of classical groups [7, 9]. In this section, we apply our methods to obtain information on Ext-groups, leaving to the reader the completely analogous case of Tor-groups.

From now on, we let \mathcal{A} be a full abelian subcategory of \mathcal{F}_R , with enough projectives and injectives, such that for all $M \in \mathbb{P}_R$ the parameterization functors restrict to functors: $-^M, -_M : \mathcal{A} \rightarrow \mathcal{A}$. For example we can take for \mathcal{A} the category \mathcal{F}_R itself, or its full subcategory whose objects are the polynomial functors of degree at most n . Since the parameterization functors are adjoint and exact, we have a graded isomorphism, natural with respect to $F, G \in \mathcal{A}$ and $M \in \mathbb{P}_R$:

$$\text{Ext}_{\mathcal{A}}^*(F^M, G) \simeq \text{Ext}_{\mathcal{A}}^*(F, G_M).$$

So we denote by $\underline{\text{Ext}}^i(F, G) \in \mathcal{F}_R$ any one of the two isomorphic functors:

$$M \mapsto \text{Ext}_{\mathcal{A}}^i(F^M, G), \quad M \mapsto \text{Ext}_{\mathcal{A}}^i(F, G_M).$$

LEMMA 4.16. *If F or G is polynomial of degree $\leq d$, then $\underline{\text{Ext}}^i(F, G)$ is also polynomial of degree $\leq d$. If F or G is in $\mathcal{P}_{s,R}$, then $\underline{\text{Ext}}^i(F, G)$ can be viewed as an object of $\mathcal{P}_{s,R}$.*

Proof. Assume for example that G is polynomial of degree $\leq d$. Then the statement follows from Lemma 4.2 with \mathcal{C} the full subcategory of $\mathbf{D}(\mathcal{F}_R)$

containing the functors G_M , for $M \in \mathcal{P}_R$, and their direct summands as objects, and $\Psi : \mathcal{C} \rightarrow R\text{-Mod}$ given by $\Psi(G) = \text{Ext}_{\mathcal{A}}^i(F, G)$. The proof of the other statements is similar. ■

Lemma 4.16 shows that we can apply Theorem 1.2 to obtain information about the integral torsion of the functor $\underline{\text{Ext}}^i(F, G)$. If we evaluate this functor on $M = R$, we immediately obtain the following statement.

THEOREM 4.17. *Let $F \in \mathcal{A}$ be a polynomial functor and let $G \in \mathcal{P}_{s,R}$ with underlying ordinary functor in \mathcal{A} . Assume that $\deg(F) < s$.*

- (i) *For all i , $\text{Ext}_{\mathcal{A}}^i(F, G)$ is a torsion abelian group.*
- (ii) *For all prime p , the integral torsion of the p -primary part of $\text{Ext}_{\mathcal{A}}^i(F, G)$ is bounded by p^r with $r = \lceil \frac{\deg(F)+1-\Sigma_p(s)}{p-1} \rceil$.*
- (iii) *Assume furthermore that $F \in \mathcal{P}_{t,R}$. If $\mathcal{I}(p, t) \cap \mathcal{I}(p, s) = \emptyset$, then the p -primary part of $\text{Ext}_{\mathcal{A}}^i(F, G)$ is zero.*

EXAMPLE 4.18 (Mac Lane cohomology with coefficients). Let us denote by I the inclusion of \mathcal{P}_R in $R\text{-Mod}$. If $F \in \mathcal{F}_R$, the extension groups

$$\text{Ext}_{\mathcal{F}_R}^*(I, F)$$

are called the *Mac Lane cohomology of R with coefficients in F* . Assume that $F \in \mathcal{P}_{s,R}$ with $s \geq 2$. Theorem 4.17 shows that $\text{Ext}_{\mathcal{F}_R}^*(I, F)$ is zero if s is not a power of a prime, and if $s = p^r$ then $\text{Ext}_{\mathcal{F}_R}^*(I, F)$ is an \mathbb{F}_p -vector space. This result was already obtained in the special cases $F = S^n$ or $F = A^n$ (for $R = \mathbb{Z}$) in [17], where the corresponding extension groups are actually explicitly computed.

In Theorem 4.17, we can take for \mathcal{A} the category \mathcal{F}_R itself, or its full subcategory of polynomial functors of degree $\leq d$ for a given d . Hence Theorem 4.17 shows similarities between Ext-computations in these categories. Comparison results for Ext and Tor between these categories can be found in [8].

REMARK 4.19. As another common feature of these Exts, we mention a vanishing result which is well-known in the case $\mathcal{A} = \mathcal{F}_R$ (see Lemma 2.5 or [25, 26, 3]) but which does not seem so well-known when \mathcal{A} is the full subcategory of polynomial functors of degree $\leq d$. Namely, if $F \in \mathcal{A}$ is a polynomial functor of degree less than d , and if G_1, \dots, G_d are reduced functors such that $G_1 \otimes \dots \otimes G_d \in \mathcal{A}$, then

$$\text{Ext}_{\mathcal{A}}^*(F, G_1 \otimes \dots \otimes G_d) = 0.$$

This follows from Lemma 4.16 and Proposition 4.3, applied to the functor $\Psi : G \mapsto \text{Ext}_{\mathcal{A}}^i(F, G)$.

4.5. Cohomology of algebraic groups. We refer the reader to [19] for further details on algebraic groups and the related cohomology.

THEOREM 4.20. *Let G be a split connected reductive algebraic group over a PID R . Let M be a finitely generated free R -module on which G acts rationally. Assume that M has a good filtration. Then for all $F \in \mathcal{P}_{s,R}$ and all positive integers i , $H^i(G, F(M))$ is a torsion abelian group whose p -primary part is bounded by p^r with $r = \frac{s - \Sigma_p(s)}{p-1}$.*

Proof. Let us fix a positive integer i , and let $\Psi : \mathcal{P}_{s,R} \rightarrow R\text{-Mod}$ be the functor $\Psi(F) = H^i(G, F(M))$. Since F is a functor of weight s , its s th cross effect is an object of $\mathcal{P}_{1,\dots,1,R}(s)$. This category is equivalent to the category of R -modules: the equivalence is given by sending an s -functor G to the R -module $G(R, \dots, R)$, and its inverse sends an R -module K to the functor $(M_1, \dots, M_d) \mapsto M_1 \otimes \dots \otimes M_d \otimes K$. In particular, for all $F \in \mathcal{P}_{s,R}$, Lemma 4.2 yields the following formula for the s th cross effect of Ψ_F :

$$(*) \quad \text{cr}_s(\Psi_F)(M_1, \dots, M_s) \simeq H^i(G, M_1 \otimes \dots \otimes M_s \otimes K \otimes M^{\otimes s}),$$

where $K = \text{cr}_s F(R, \dots, R)$ and the action of G on the coefficient module on the right-hand side is trivial on the factors M_i and K . We claim that the right-hand side of $(*)$ is zero. Hence Ψ_F is a functor of degree less than s and the bound in Theorem 4.20 follows directly from Theorem 1.2.

Thus, to finish the proof of Theorem 4.20, it remains to prove our claim that the right-hand side of $(*)$ is zero. Since M has a good filtration, $H^*(G, M^{\otimes s})$ is zero in positive degrees for all s . This vanishing implies a slightly more general one, namely that for all s and all R -modules N (considered as trivial G -modules), $H^*(G, M^{\otimes s} \otimes N)$ is zero in positive degrees. Indeed, the Hochschild complex [19, I 4.14] $C(G, M^{\otimes s} \otimes N)$ is isomorphic to $C(G, M^{\otimes s}) \otimes N$. Since $C(G, M^{\otimes s})$ is a complex of free R -modules whose homology groups are free R -modules, we have $H^i(C(G, M^{\otimes s}) \otimes N) = H^i(G, M^{\otimes s}) \otimes N$ for all i . This proves our claim. ■

Appendix: Modules over the Schur algebra. The Schur algebra $S_R(n, s)$ is the endomorphism algebra $\text{End}_{\Gamma^s(\mathbb{P}_R)}(R^n)$. We identify it with the full subcategory of the Schur category with R^n as the only object. Then the functor $S_R(n, s) \rightarrow \Gamma^s(\mathbb{P}_R)$ induces an evaluation functor

$$\text{ev}_n : \mathcal{P}_{s,R} \rightarrow S_R(n, s)\text{-Mod}.$$

It was proved by Friedlander and Suslin [18, Cor. 3.13] (see also [29, 22]) that if $n \geq s$, then the evaluation functor is an equivalence of categories. In this section we explain how to convert Theorem 1.2 into an equivalent statement in the realm of modules over Schur algebras.

To this end, we recall the theory of weights for modules over Schur algebras. The ground ring R is an arbitrary commutative ring. We denote by $D_R(n, s)$ the endomorphism algebra (where $\Gamma^s(\mathbb{P}_R^{\times n})$ is the category defined

in Section 3.2)

$$D_R(n, s) = \text{End}_{\Gamma^s(\mathbb{P}_R^{\times n})}(\underbrace{R, \dots, R}_{n \text{ times}}).$$

Thus $D_R(n, s)$ is a free R -module with basis (e_λ) indexed by the set $\Lambda(n, s)$ of n -tuples $\lambda = (\lambda_1, \dots, \lambda_n)$ of nonnegative integers with $\sum \lambda_i = s$. The product satisfies $e_\lambda e_\mu = 0$ if $\lambda \neq \mu$ and $e_\lambda^2 = e_\lambda$. In particular, if M is a $D_R(n, s)$ -module then $M_\lambda = e_\lambda M$ is a $D_R(n, s)$ -submodule and there is a direct sum decomposition

$$M = \bigoplus_{\lambda \in \Lambda(n, s)} M_\lambda.$$

The M_λ are the *weight subspaces* of M , and the *weights* of M are the tuples $\lambda \in \Lambda(n, s)$ such that $M_\lambda \neq 0$. The number of positive coefficients λ_i in a weight λ will be called the *length* of the weight. By definition of $D_R(n, s)$, we get examples of $D_R(n, s)$ -modules by evaluating strict polynomial n -functors on (R, \dots, R) . We record this fact in the following lemma.

LEMMA A.1. *Let $F \in \mathcal{P}_{s,R}(n)$. Then $F(R, \dots, R)$ is a $D_R(n, s)$ -module. Moreover, if F is an object of the subcategory $\mathcal{P}_{(\lambda_1, \dots, \lambda_n), R}(n) \subset \mathcal{P}_{s,R}(n)$ and if $F(R, \dots, R) \neq 0$, then $(\lambda_1, \dots, \lambda_n)$ is the unique weight of $F(R, \dots, R)$.*

The functor $\Gamma^s(\mathbb{P}_R^{\times n}) \rightarrow \Gamma^s(\mathbb{P}_R)$, $(M_i) \mapsto \bigoplus M_i$, induces a morphism of algebras

$$D_R(n, s) \rightarrow S_R(n, s).$$

Hence every $S_R(n, s)$ -module restricts to a $D_R(n, s)$ -module. By definition, the weights of a $S_R(n, s)$ -module M are the weights of the corresponding $D_R(n, s)$ -module.

REMARK A.2. The $S_R(n, s)$ -modules provide representations of the group scheme $GL_{n,R}$ (see e.g. [18, Section 3]). From that point of view, the $D_R(n, s)$ -module structure obtained by restriction corresponds to the action of the n -torus of diagonal matrices of $GL_{n,R}$. This explains the notation $D_R(n, s)$.

We have the following correspondence between the degree of a strict polynomial functor F and the weights of the $S_R(n, s)$ -module $F(R^n)$.

LEMMA A.3. *Let $F \in \mathcal{P}_{s,R}$ with $s > 0$ and let $n \geq s$. The degree of F is equal to the maximum of the lengths of the weights of the $S_R(n, s)$ -module $F(R^n)$.*

Proof. Let $F \in \mathcal{P}_{s,R}$ of degree $\leq d$. There is an isomorphism of strict polynomial functors with n variables (M_1, \dots, M_n) :

$$F(M_1 \oplus \dots \oplus M_n) \simeq \bigoplus_{k=1}^d \left(\bigoplus_{j_1 < \dots < j_k} \text{cr}_k F(M_{j_1}, \dots, M_{j_k}) \right).$$

Furthermore, by Lemma 3.9, we have a direct sum decomposition $\text{cr}_k F \simeq \bigoplus (\text{cr}_k F)_\mu$ with $\mu \in \Lambda(k, s)$ and $(\text{cr}_k F)_\mu \in \mathcal{P}_{\mu, R}(k)$. So the isomorphism above refines to an isomorphism

$$F(M_1 \oplus \cdots \oplus M_n) \simeq \bigoplus_{k=1}^d \left(\bigoplus_{\mu \in \Lambda(k, s)} \left(\bigoplus_{j_1 < \cdots < j_k} (\text{cr}_k F)_\mu(M_{j_1}, \dots, M_{j_k}) \right) \right).$$

By evaluation on (R, \dots, R) , we get an isomorphism of $D_R(n, s)$ -modules between $F(R^n)$ and a direct sum of modules of the form $(\text{cr}_k F)_\mu(R, \dots, R)$. By Lemma A.1, the latter is a $D_R(n, s)$ -module of weight $\tilde{\mu}$, where $\tilde{\mu} = (\tilde{\mu}_1, \dots, \tilde{\mu}_n)$ with $\tilde{\mu}_i = 0$ if $i \notin \{j_1, \dots, j_k\}$, and $\tilde{\mu}_{j_\ell} = \mu_\ell$. Thus,

(1) if $\text{cr}_k F(R, \dots, R)$ is nonzero, then the lengths of the weights of the $D_R(n, s)$ -module $\text{cr}_k F(R, \dots, R)$ are at most k .

Moreover,

(2) if the degree of F is exactly d , then $\text{cr}_d F(R, \dots, R)$ is nonzero, and the lengths of the weights of this $D_R(n, s)$ -module are all equal to d .

Indeed, $\text{cr}_d F$ is nonzero and additive with respect to each variable. Now Lemma A.3 follows from (i) and (ii). ■

Now we can use Lemma A.3 to translate Proposition 1.1 and Theorem 1.2 into statements for modules over the Schur algebras.

THEOREM A.4. *Let R be a commutative ring, let n, s be positive integers with $n \geq s$, and let M be a nonzero $S_R(n, s)$ -module.*

- (i) *If all the weights of M have length less than s , then the underlying abelian group of M is a torsion group.*
- (ii) *Assume that the underlying abelian group of M is a p -primary abelian group. Let d be the largest integer such that there exists a weight of M of length d . Then $d \in \mathcal{I}(p, s)$. Moreover, if $d < s$ then the integral torsion of the underlying abelian group of M is bounded by p^r with $r = \left\lceil \frac{d+1-\Sigma_p(s)}{p-1} \right\rceil$.*

References

- [1] K. Akin, D. Buchsbaum and J. Weyman, *Schur functors and Schur complexes*, Adv. Math. 44 (1982), 207–278.
- [2] S. Betley, *Stable derived functors, the Steenrod algebra and homological algebra in the category of functors*, Fund. Math. 168 (2001), 279–293.
- [3] S. Betley and T. Pirashvili, *Stable K-theory as a derived functor*, J. Pure Appl. Algebra 96 (1994), 245–258.
- [4] A. K. Bousfield, *Homogeneous functors and their derived functors*, preprint, Brandeis Univ., 1967.
- [5] L. Breen and R. Mikhailov, *Derived functors of non-additive functors and homotopy theory*, Algebr. Geom. Topol. 11 (2011), 327–415.

- [6] H. Cartan, *Algèbres d'Eilenberg–MacLane et homotopie*, Séminaire Henri Cartan 7, no. 1, 1954/1955.
- [7] A. Djament, *Sur l'homologie des groupes unitaires à coefficients polynomiaux*, J. K-Theory 10 (2012), 87–139.
- [8] A. Djament, *Groupes d'extensions et foncteurs polynomiaux*, J. London Math. Soc. (2) 92 (2015), 63–88.
- [9] A. Djament and C. Vespa, *Sur l'homologie des groupes orthogonaux et symplectiques coefficients tordus*, Ann. Sci. École Norm. Sup. (4) 43 (2010), 395–459.
- [10] A. Dold, *Homology of symmetric products and other functors of complexes*, Ann. of Math. (2) 68 (1958), 54–80.
- [11] A. Dold and D. Puppe, *Non-additive functors, their derived functors, and the suspension homomorphism*, Proc. Nat. Acad. Sci. U.S.A. 44 (1958), 1065–1068.
- [12] A. Dold and D. Puppe, *Homologie nicht-additiver Funktoren. Anwendungen*, Ann. Inst. Fourier (Grenoble) 11 (1961), 201–312.
- [13] S. Eilenberg, *Abstract description of some basic functors*, J. Indian Math. Soc. (N.S.) 24 (1960), 231–234 (1961).
- [14] S. Eilenberg and S. MacLane, *On the groups $H(\Pi, n)$. I*, Ann. of Math. 58 (1953), 55–106.
- [15] S. Eilenberg and S. MacLane, *On the groups $H(\Pi, n)$. II. Methods of computation*, Ann. of Math. (2) 60 (1954), 49–139.
- [16] V. Franjou, E. Friedlander, A. Scorichenko and A. Suslin, *General linear and functor cohomology over finite fields*, Ann. of Math. (2) 150 (1999), 663–728.
- [17] V. Franjou and T. Pirashvili, *On the Mac Lane cohomology for the ring of integers*, Topology 37 (1998), 109–114.
- [18] E. Friedlander and A. Suslin, *Cohomology of finite group schemes over a field*, Invent. Math. 127 (1997), 209–270.
- [19] J. C. Jantzen, *Representations of Algebraic Groups*, 2nd ed., Math. Surveys Monogr. 107, Amer. Math. Soc., Providence, RI, 2003.
- [20] M. Jibladze and T. Pirashvili, *Cohomology of algebraic theories*, J. Algebra 137 (1991), 253–296.
- [21] B. Johnson and R. McCarthy, *Deriving calculus with cotriple*, Trans. Amer. Math. Soc. 356 (2004), 757–803.
- [22] H. Krause, *Koszul, Ringel and Serre duality for strict polynomial functors*, Compos. Math. 149 (2013), 996–1018.
- [23] J.-L. Loday and B. Valette, *Algebraic Operads*, Grundlehren Math. Wiss. 346, Springer, Heidelberg, 2012.
- [24] W. Magnus, *Über Beziehungen zwischen höheren Kommutatoren*, J. Reine Angew. Math. 177 (1937), 105–115.
- [25] T. Pirashvili, *Higher additivizations*, Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR 91 (1988), 44–54 (in Russian).
- [26] T. Pirashvili, *Introduction to functor homology*, in: Rational Representations, the Steenrod Algebra and Functor Homology, Panor. Synthèses 16, Soc. Math. France, Paris, 2003, 27–53.
- [27] T. Pirashvili and F. Waldhausen, *Mac Lane homology and topological Hochschild homology*, J. Pure Appl. Algebra 82 (1992), 81–98.
- [28] P. Ribenboim, *The New Book of Prime Number Records*, Springer, New York, 1996.
- [29] A. Suslin, E. Friedlander and C. Bendel, *Infinitesimal 1-parameter subgroups and cohomology*, J. Amer. Math. Soc. 10 (1997), 693–728.
- [30] A. Touzé, *Cohomology of classical algebraic groups from the functorial viewpoint*, Adv. Math. 225 (2010), 33–68.

- [31] A. Touzé, *Ringel duality and derived functors of nonadditive functors*, J. Pure Appl. Algebra 217 (2013), 1642–1673.
- [32] A. Touzé, *Applications of functor (co)homology*, in: An Alpine Expedition through Algebraic Topology, Contemp. Math. 617, Amer. Math. Soc., Providence, RI, 2014, 259–277.
- [33] C. Watts, *Intrinsic characterizations of some additive functors*, Proc. Amer. Math. Soc. 11 (1960), 5–8.
- [34] C. Weibel, *An Introduction to Homological Algebra*, Cambridge Stud. Adv. Math. 38, Cambridge Univ. Press, Cambridge, 1994.
- [35] E. Witt, *Treue Darstellung Liescher Ringe*, J. Reine Angew. Math. 177 (1937), 152–160.

Antoine Touzé
Université Lille 1 – Sciences et Technologies
Laboratoire Painlevé
Cité Scientifique – Bâtiment M2
F-59655 Villeneuve d’Ascq Cedex, France
E-mail: antoine.touze@math.univ-lille1.fr

