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On the Dirichlet problem associated with the Dunkl Laplacian

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Abstract. This paper deals with the questions of the existence and uniqueness of a solution to the Dirichlet problem associated with the Dunkl Laplacian Δ_k as well as the hypoellipticity of Δ_k on noninvariant open sets.

1. Introduction. Let R be a root system in $\mathbb{R}^d$, $d \geq 1$. We fix a positive subsystem R_+ of R and a nonnegative multiplicity function $k : R \to \mathbb{R}_+$. For every $\alpha \in R$, let H_α be the hyperplane orthogonal to α, and σ_α be the reflection with respect to H_α, that is, for every $x \in \mathbb{R}^d$,

$$\sigma_\alpha x = x - 2\frac{\langle x, \alpha\rangle}{|\alpha|^2}\alpha$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product of $\mathbb{R}^d$. The set of hyperplanes H_α, $\alpha \in R_+$, divides $\mathbb{R}^d$ into connected open components, called *Weyl chambers* generated by R. We consider the differential-difference operators T_i, $1 \leq i \leq d$, defined in [6], for $f \in C^1(\mathbb{R}^d)$, by

$$T_i f(x) = \frac{\partial f}{\partial x_i}(x) + \sum_{\alpha \in R_+} k(\alpha)\alpha_i \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle},$$

and called *Dunkl operators* in the literature. These operators are used in the study of certain exactly solvable models of quantum mechanics, namely the Calogero–Moser–Sutherland type (see [10, 12]). Dunkl operators bear a rich analytic structure, not only because of their commutativity [6], but also thanks to the existence of an intertwining operator between Dunkl operators and the usual partial derivatives [8]. In particular, a counterpart of Fourier analysis, called Dunkl analysis, was developed (see [7, 11]). It was shown in

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[9, 13], using tools from Dunkl analysis, that the Dunkl Laplacian defined by

$$\Delta_k = \sum_{i=1}^{d} T_i^2$$

is hypoelliptic on every *invariant* open subset D of $\mathbb{R}^d$, i.e., $\sigma_\alpha(D) = D$ for every $\alpha \in R_+$. However, the question of whether Δ_k is hypoelliptic on noninvariant open sets remains open.

The Dunkl Laplacian is a prototype of differential-difference (nonlocal) operators. It generates a Hunt process [15], and therefore it generates a balayage space in the sense of [3] where a general framework of balayage spaces has been developed. Some specific elements of potential theory related to the Dunkl Laplacian, namely the Green function, harmonic measures, regular sets and the Dirichlet problem, have been studied in [2]. In particular, the authors consider, for an invariant bounded open set D, the following Dirichlet problem:

$$\begin{cases} \Delta_k h = 0 & \text{on } D, \\ h = f & \text{on } \mathbb{R}^d \setminus D, \end{cases} \tag{1}$$

where the function f is assumed to be continuous. To establish the existence of a twice continuously differentiable function h on D such that both equations of (1) are pointwise fulfilled, they use the hypoellipticity of Δ_k on D, and therefore they need to assume that D is invariant. So, we are led to the following questions: If we consider a noninvariant open set D, what can be said about the hypoellipticity of Δ_k on D as well as on the existence of a solution to problem (1), bearing in mind that when Δ_k is not hypoelliptic on D, the technical approach used in [2] becomes invalid? The present paper deals with these two questions. Let us give a short description of our results.

By a solution of problem (1), we mean every function $h : \mathbb{R}^d \to \mathbb{R}$ which is continuous on $\mathbb{R}^d$ such that $h = f$ on $\mathbb{R}^d \setminus D$ and

$$\int_{\mathbb{R}^d} h(x)\Delta_k\varphi(x)w_k(x)\,dx = 0 \quad \text{for every } \varphi \in C_c^\infty(D),$$

where $C_c^\infty(D)$ denotes the space of infinitely differentiable functions on D with compact support and w_k is the invariant weight function defined on $\mathbb{R}^d$ by

$$w_k(x) = \prod_{\alpha\in R_+} |\langle x, \alpha\rangle|^{2k(\alpha)}.$$

Note that $\Delta_k = \Delta$, the classical Laplacian, when $k \equiv 0$. The set D is called Δ_k-*regular* if, for every continuous function f on $\mathbb{R}^d \setminus D$, problem (1) admits a unique solution; this solution will be denoted by $H_D^{\Delta_k} f$. Given a noninvariant bounded open set D, assume that $D \subset \mathbb{R}^d \setminus \bigcup_{\alpha\in R_+} H_\alpha$.

We first prove that u is a solution of problem (1) if and only if u is a solution of the Schrödinger equation $\Delta - q = g$ on D for some functions q and g. It is known (see [4, 5]) that to solve this equation, we need to assume that q is in the Kato class of D, which requires, in our case, that the intersection of $\overline{D}$, the closure of D, with every hyperplane H_α, $\alpha \in R_+$, should be empty.

By means of the above Schrödinger equation, we show that D is Δ_k-regular provided it is Δ-regular, and we give an analytic formula characterizing the solution $H_D^{\Delta_k} f$ (see Theorem 1 below). We deduce from this formula that, for every $x \in D$, $H_D^{\Delta_k} f(x)$ depends only on the values of f on $\bigcup_{\alpha\in R_+} \sigma_\alpha(D)$ and on ∂D, the Euclidean boundary of D. If, in addition, we assume that f is locally Hölder continuous on $\bigcup_{\alpha\in R_+} \sigma_\alpha(D)$ then $H_D^{\Delta_k} f$ is continuously twice differentiable on D and therefore the first equation in (1) is fulfilled by $H_D^{\Delta_k} f$ not only in the distributional sense but also pointwise.

Finally, we show that, for a Δ_k-regular open set D, if D is noninvariant then Δ_k is not hypoelliptic on D. Therefore the condition "D is invariant" is necessary and sufficient for the hypoellipticity of Δ_k on D.

2. Main results. We first present some various facts on the Dirichlet boundary value problem associated with Schrödinger's operator which are needed for our approach. We refer to [4, 5] for details. Let G be the Green function on $\mathbb{R}^d$, but without the constant factors:

$$G(x,y) = \begin{cases} |x-y|^{2-d} & \text{if } d \geq 3, \\ \ln \frac{1}{|x-y|} & \text{if } d = 2, \\ |x-y| & \text{if } d = 1. \end{cases}$$

Let D be a bounded domain of $\mathbb{R}^d$ and let $q \in J(D)$, the Kato class on D, i.e., q is a Borel measurable function on $\mathbb{R}^d$ such that the function $G(1_D|q|)$ defined, for $x \in \mathbb{R}^d$, by

$$G(1_D|q|)(x) := \int_D G(x,y)|q(y)|\,dy$$

is continuous on $\mathbb{R}^d$. Note that the Kato class $J(D)$ contains all bounded Borel measurable functions on D. Assume that D is Δ-regular. Then, for every continuous function f on ∂D, there exists a unique continuous function h on $\overline{D}$ such that $h = f$ on ∂D and

$$\int h(x)(\Delta - q)\varphi(x)\,dx = 0 \quad \text{for every } \varphi \in C_c^\infty(D). \tag{2}$$

Moreover, $h \geq 0$ when $f \geq 0$. We denote by $H_D^{\Delta-q} f$ the unique continuous extension of f to $\overline{D}$ which satisfies the Schrödinger equation (2). Let G_D^{Δ} and $G_D^{\Delta-q}$ denote, respectively, the Green potential operator of $\Delta_{|D}$ and of

$\Delta_{|D} - q$, i.e., for every Borel bounded function g on D and every $x \in D$,

$$G_D^{\Delta} g(x) := \int_0^\infty P_t^{\Delta} g(x) \quad \text{and} \quad G_D^{\Delta - q} g(x) := \int_0^\infty P_t^{\Delta - q} g(x),$$

where P_t^{Δ} (resp. $P_t^{\Delta-q}$) is the semigroup generated by $\Delta_{|D}$ (resp. $\Delta_{|D} - q$) (see [5] for more details). Then $G_D^{\Delta} g \in C_0(D)$, the set of all continuous functions on D vanishing at ∂D. Moreover, $G_D^{\Delta} g \in C^{n+1}(D)$ whenever $g \in C^n(D)$ (see for example [1, Corollary 5.5.4] or [14, Theorem 4.6.6]). Also, it was shown in [5, pp. 86–88] that $G_D^{\Delta-q} g \in C_0(D)$ and that, for every $x \in D$,

$$G_D^{\Delta-q} g(x) = G_D^{\Delta} g(x) - G_D^{\Delta}(q G_D^{\Delta-q} g)(x). \tag{3}$$

Moreover, if in addition we assume that $q \in C^\infty(D)$ then, proceeding by induction, it follows from (3) that, for every $n \in \mathbb{N}$,

$$G_D^{\Delta-q} g \in C^n(D) \quad \text{if and only if} \quad G_D^{\Delta} g \in C^n(D). \tag{4}$$

Now, seeing from (3) that

$$\int G_D^{\Delta-q} g(x)(\Delta - q)\varphi(x)\, dx = -\int g(x)\varphi(x)\, dx \quad \text{for every } \varphi \in C_c^\infty(D),$$

we immediately conclude that the function h defined for $x \in D$ by

$$h(x) = H_D^{\Delta-q} f(x) + G_D^{\Delta-q} g(x) \tag{5}$$

is the unique continuous extension of f to $\overline{D}$ such that

$$\int h(x)(\Delta - q)\varphi(x)\, dx = -\int g(x)\varphi(x)\, dx \quad \text{for every } \varphi \in C_c^\infty(D). \tag{6}$$

Now we are ready to establish our first main result.

Theorem 1. *Let D be a bounded open set such that $\overline{D}$ is contained in one of the Weyl chambers generated by R. If D is Δ-regular then D is Δ_k-regular. Moreover, for every continuous function f on $\mathbb{R}^d \setminus D$ and every $x \in D$,*

$$H_D^{\Delta_k} f(x) = \frac{1}{\sqrt{w_k(x)}} \big(H_D^{\Delta-q}(f\sqrt{w_k})(x) + G_D^{\Delta-q}(\sqrt{w_k}\, Nf)(x) \big), \tag{7}$$

where q and Nf are the functions defined, for $x \in D$, by

$$q(x) := \sum_{\alpha \in R_+} \left(\frac{|\alpha| k(\alpha)}{\langle x, \alpha \rangle} \right)^2,$$

$$Nf(x) := \sum_{\alpha \in R_+} \frac{|\alpha|^2 k(\alpha)}{\langle x, \alpha \rangle^2} f(\sigma_\alpha x).$$

Proof. Let f be a continuous function on $\mathbb{R}^d \setminus D$. We intend to prove the existence and uniqueness of a continuous function h on D such that $h = f$

on $\mathbb{R}^d \setminus D$ and

$$\int h(x)\Delta_k\varphi(x)w_k(x)\,dx = 0 \quad \text{for every } \varphi \in C_c^\infty(D). \tag{8}$$

It follows from [6] that

$$\Delta_k\varphi(x) = \Delta\varphi(x) + 2\sum_{\alpha\in R_+} k(\alpha)\left(\frac{\langle\nabla\varphi(x),\alpha\rangle}{\langle\alpha,x\rangle} - \frac{|\alpha|^2}{2}\,\frac{\varphi(x)-\varphi(\sigma_\alpha x)}{\langle\alpha,x\rangle^2}\right),$$

where ∇ denotes the gradient on $\mathbb{R}^d$. It is clear that for every $x \in D$,

$$\nabla(\sqrt{w_k})(x) = \sqrt{w_k(x)}\sum_{\alpha\in R_+}\frac{k(\alpha)}{\langle x,\alpha\rangle}\alpha$$

and

$$\Delta(\sqrt{w_k})(x) = \sqrt{w_k(x)}\left(\sum_{\alpha,\beta\in R_+} k(\alpha)k(\beta)\frac{\langle\alpha,\beta\rangle}{\langle x,\alpha\rangle\langle x,\beta\rangle} - \sum_{\alpha\in R_+}|\alpha|^2\frac{k(\alpha)}{\langle x,\alpha\rangle^2}\right).$$

On the other hand, by formula (1) in [6, Proposition 1.7], we have

$$\sum_{\alpha,\beta\in R_+,\,\sigma_\alpha\sigma_\beta\neq \mathrm{id}_{\mathbb{R}^d}} k(\alpha)k(\beta)\frac{\langle\alpha,\beta\rangle}{\langle x,\alpha\rangle\langle x,\beta\rangle} = 0.$$

Since $\sigma_\alpha\sigma_\beta \neq \mathrm{id}_{\mathbb{R}^d}$ for every $\alpha,\beta \in R_+$ such that $\alpha \neq \beta$, we immediately deduce that

$$\sum_{\alpha,\beta\in R_+,\,\alpha\neq\beta} k(\alpha)k(\beta)\frac{\langle\alpha,\beta\rangle}{\langle x,\alpha\rangle\langle x,\beta\rangle} = 0.$$

Consequently,

$$\Delta(\sqrt{w_k})(x) = \sqrt{w_k(x)}\,q(x) - \sqrt{w_k(x)}\sum_{\alpha\in R_+}|\alpha|^2\frac{k(\alpha)}{\langle x,\alpha\rangle^2}.$$

Thus, for every $\varphi \in C_c^\infty(D)$,

$$\begin{aligned}\Delta(\varphi\sqrt{w_k})(x) &= q(x)\varphi(x)\sqrt{w_k(x)} + \sqrt{w_k(x)}\,\Delta\varphi(x)\\ &\quad + 2\sqrt{w_k(x)}\sum_{\alpha\in R_+}k(\alpha)\left(\frac{\langle\nabla\varphi(x),\alpha\rangle}{\langle\alpha,x\rangle} - \frac{|\alpha|^2}{2}\,\frac{\varphi(x)}{\langle\alpha,x\rangle^2}\right).\end{aligned}$$

Hence, for every $x \in D$,

$$\begin{aligned}&\sqrt{w_k(x)}\,\Delta_k\varphi(x)\\ &\quad = \big(\Delta(\varphi\sqrt{w_k})(x) - q(x)\varphi(x)\sqrt{w_k(x)}\big) + \sqrt{w_k(x)}\,N\varphi(x).\end{aligned}\tag{9}$$

Since the map $\varphi \mapsto \varphi\sqrt{w_k}$ is invertible on the space $C_c^\infty(D)$ and the function $x \mapsto w_k(x)/\langle x,\alpha\rangle^2$ is invariant under the reflection σ_α, equation (8) is equivalent to the following Schrödinger equation: For every $\psi \in C_c^\infty(D)$,

$$\int h(x)\sqrt{w_k(x)}\,(\Delta - q)\psi(x)\,dx = -\int \sqrt{w_k(x)}\,Nf(x)\psi(x)\,dx.$$

Finally, since q is bounded on D and therefore is in $J(D)$, the statements follow from (5) and (6). ■

It is worth noting that a bounded open set D is Δ-regular provided it satisfies the following geometric assumption known as *exterior cone condition*: for every $z \in \partial D$ there exists a cone C of vertex z such that $C \cap B(z,r) \subset \mathbb{R}^d \setminus D$ for some $r > 0$, where $B(z,r)$ is the ball of center z and radius r (see, for example, [5]).

Remark. Note that, in order to obtain $q \in J(D)$, the hypothesis of the above theorem, "$\overline{D} \subset \mathbb{R}^d \setminus \bigcup_{\alpha\in R_+} H_\alpha$", is nearly optimal. Indeed, assume that there exists a cone C_z of vertex $z \in \overline{D} \cap H_\alpha$ for some $\alpha \in R_+$ with $k(\alpha) \neq 0$ such that $C_z^r := C_z \cap B(z,r) \subset D$ for some $r > 0$. Then

$$\begin{aligned} G(1_D q)(z) &\geq |\alpha|^2 k^2(\alpha) \int_{C_z^r} G(z,y)\frac{1}{\langle y,\alpha\rangle^2}\,dy \\ &= |\alpha|^2 k^2(\alpha) \int_{C_z^r} G(z,y)\frac{1}{\langle z-y,\alpha\rangle^2}\,dy \\ &\geq k^2(\alpha) \int_{C_z^r - z} G(0,y)\frac{1}{|y|^2}\,dy = \infty. \end{aligned}$$

It follows from (7) that, for every $x \in D$, the map $f \mapsto H_D^{\Delta_k} f(x)$ defines a positive Radon measure on $\mathbb{R}^d \setminus D$ since both $f \mapsto H_D^{\Delta - q}(f\sqrt{w_k})$ and $f \mapsto G_D^{\Delta-q}(\sqrt{w_k}\,Nf)$ define such measures. We denote this measure by $H_D^{\Delta_k}(x,dy)$.

Corollary 1. *For every $x \in D$, $H_D^{\Delta_k}(x,dy)$ is a probability measure supported by*

$$\partial D \cup \bigcup_{\alpha\in R_+} \sigma_\alpha(D)$$

and satisfies

$$\frac{\sqrt{w_k(x)}}{\sqrt{w_k(y)}} H_D^{\Delta_k}(x,dy) = H_D^{\Delta-q}(x,dy) + \sum_{\alpha\in R_+} \frac{|\alpha|^2 k(\alpha)}{\langle y,\alpha\rangle^2} G_D^{\Delta-q}(x,\sigma_\alpha y)dy.$$

Proof. Since $\Delta_k 1(x) = 0$ for every $x \in D$, the uniqueness of solution of problem (1) implies that $H_D^{\Delta_k} 1(x) = 1$, which means that $H_D^{\Delta_k}(x,dy)$ is a probability measure on $\mathbb{R}^d \setminus D$. Let f be a nonnegative bounded Borel function on $\mathbb{R}^d \setminus D$ and let $x \in D$. By (7), we have

$$\sqrt{w_k(x)} \int_{\mathbb{R}^d\setminus D} f(y)\, H_D^{\Delta_k}(x,dy) = \int_{\partial D} \sqrt{w_k(y)}\, f(y)\, H_D^{\Delta-q}(x,dy)$$
$$+ \sum_{\alpha\in R_+} \int_{\sigma_\alpha(D)} \frac{|\alpha|^2 k(\alpha)}{\langle y,\alpha\rangle^2} f(y) \sqrt{w_k(y)}\, G_D^{\Delta-q}(x,\sigma_\alpha y)\, dy.$$

Hence, $H_D^{\Delta_k}(x,dy)$ is supported by $\partial D \cup \bigcup_{\alpha\in R_+} \sigma_\alpha(D)$ and

$$\frac{\sqrt{w_k(x)}}{\sqrt{w_k(y)}} H_D^{\Delta_k}(x,dy) = H_D^{\Delta-q}(x,dy) + \sum_{\alpha\in R_+} \frac{|\alpha|^2 k(\alpha)}{\langle y,\alpha\rangle^2} G_D^{\Delta-q}(x,\sigma_\alpha y) dy. \ \blacksquare$$

Corollary 2. *Let D be a Δ-regular bounded open set such that $\overline{D}$ is contained in one of the Weyl chambers generated by R. Let f be a continuous function on $\partial D \cup \bigcup_{\alpha\in R_+} \sigma_\alpha(D)$. If f is Hölder continuous on $\bigcup_{\alpha\in R_+} \sigma(D)$ then $H_D^{\Delta_k} f \in C^2(D)$ and, for every $x \in D$,*

$$\Delta_k(H_D^{\Delta_k} f)(x) = 0.$$

Proof. Since $H_D^{\Delta-q}(f\sqrt{w_k})$ is a solution of equation (2), the hypoellipticity of the operator $\Delta - q$ on D implies that $H_D^{\Delta-q}(f\sqrt{w_k}) \in C^\infty(D)$. Moreover, since $\sqrt{w_k}Nf$ is Hölder continuous on D, it follows from [14, Theorem 4.6.6] that $G_D^{\Delta}(\sqrt{w_k}\,Nf) \in C^2(D)$, and consequently, by (4), $G_D^{\Delta-q}(\sqrt{w_k}\,Nf) \in C^2(D)$. Then it follows from (7) that $H_D^{\Delta_k} f \in C^2(D)$. For every $\varphi \in C_c^\infty(D)$, a direct computation using (9) yields

$$\int \Delta_k(H_D^{\Delta_k} f)(x)\varphi(x) w_k(x)\, dx = \int H_D^{\Delta_k} f(x) \Delta_k \varphi(x) w_k(x)\, dx.$$

This completes the proof. ■

Let D be an open subset of $\mathbb{R}^d$. The operator Δ_k is said to be *hypoelliptic* on D if, for every $f \in C^\infty(D)$, every continuous function h on $\mathbb{R}^d$ which satisfies

$$\int_{\mathbb{R}^d} h(x)\Delta_k\varphi(x) w_k(x)\, dx = \int f(x)\varphi(x) w_k(x)\, dx \quad \text{for every } \varphi \in C_c^\infty(D)$$

is infinitely differentiable on D. The hypoellipticity of Δ_k on invariant open sets was proved in [9, 13]. However, the question whether Δ_k is hypoelliptic on noninvariant open sets remains open.

Theorem 2. *Let D be a Δ_k-regular open set. Then Δ_k is hypoelliptic on D if and only if D is invariant.*

Proof. Obviously, it suffices to show that if D is noninvariant then Δ_k is not hypoelliptic on D. Assume that D is noninvariant. Since the open set $D \setminus \bigcup_{\alpha\in R_+} H_\alpha$ is also noninvariant, there exists a nonempty open ball B

such that

$$\overline{B} \subset D \setminus \bigcup_{\alpha \in R_+} H_\alpha \quad \text{and} \quad \sigma_\alpha(B) \subset \mathbb{R}^d \setminus D \quad \text{for some } \alpha \in R_+.$$

We choose B small enough that, for every $\alpha \in R_+$,

$$\sigma_\alpha(B) \subset D \quad \text{or} \quad \sigma_\alpha(B) \subset \mathbb{R}^d \setminus D.$$

Let $I := \{\alpha \in R_+ : \sigma_\alpha(B) \subset \mathbb{R}^d \setminus D\}$ and $J := R_+ \setminus I$. Let f be a positive continuous function on $\mathbb{R}^d \setminus D$, and denote $H_D^{\Delta_k} f$ by h. Since B is Δ-regular and h satisfies

$$\int h(x) \Delta_k \varphi(x) w_k(x)\, dx = 0 \qquad \text{for every } \varphi \in C_c^\infty(B),$$

it follows from Theorem 1 that B is Δ_k-regular and, for every $x \in B$,

$$h(x) = \frac{1}{\sqrt{w_k(x)}} \big(H_B^{\Delta - q}(h\sqrt{w_k})(x) + G_B^{\Delta - q}(\sqrt{w_k}\, Nh)(x) \big). \tag{10}$$

For every $x \in B$, we define

$$g_1(x) := \sum_{\alpha \in J} \frac{|\alpha|^2 k(\alpha)}{\langle x, \alpha \rangle^2} h(\sigma_\alpha x) \quad \text{and} \quad g_2(x) := \sum_{\alpha \in I} \frac{|\alpha|^2 k(\alpha)}{\langle x, \alpha \rangle^2} f(\sigma_\alpha x).$$

It is clear that g_2 is nontrivial and $Nh = g_1 + g_2$ on B. Now, assume that $h \in C^\infty(D)$. Then $g_1 \in C^\infty(B)$ and therefore $G_B^{\Delta - q}(\sqrt{w_k}\, g_1) \in C^\infty(B)$. Furthermore, as $H_B^{\Delta - q}(h\sqrt{w_k}) \in C^\infty(B)$, it follows from (10) that $G_B^{\Delta - q}(\sqrt{w_k}\, g_2) \in C^\infty(B)$. Thus $-(\Delta - q) G_B^{\Delta - q}(\sqrt{w_k}\, g_2) = \sqrt{w_k}\, g_2 \in C^\infty(B)$ and so $g_2 \in C^\infty(B)$, a contradiction. Hence h is not infinitely differentiable on D and consequently the Dunkl Laplacian Δ_k is not hypoelliptic on D. ∎

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