ACTA ARITHMETICA
173.2 (2016)

On the subset sums of exponential type sequences

by

YONG-GAO CHEN (Nanjing), JIN-HUI FANG (Nanjing)
and NORBERT HEGYVÁRI (Budapest)

1. Introduction. Let $\mathbb{N}$ denote the set of all positive integers. For a sequence $A \subseteq \mathbb{N}$, let $P(A)$ be the set of all sums of distinct terms taken from A. Here $0 \in P(A)$. The sequence A is said to be *complete* if $P(A)$ contains all sufficiently large integers. The sequence A is said to be *subcomplete* if $P(A)$ contains an infinite arithmetic progression. The simplest example for a complete sequence is the powers of two: $S_2 = \{2^n : n = 0, 1, \dots\}$, where $P(S_2)$ contains all nonnegative integers; furthermore, the sequence $S_p = \{p^n : n = 0, 1, \dots\}$ ($p \in \mathbb{N}$, $p > 1$) is complete if and only if $p = 2$. In 1959 Birch [1] confirmed the following conjecture of Erdős: the sequence $S_{p,q} = \{p^n q^m : n, m = 0, 1, \dots\}$ is complete, where p and q are coprime integers greater than 1.

Cassels [3] proved a theorem more general than Birch's.

THEOREM (Cassels, 1960). *Let $A \subseteq \mathbb{N}$ and assume that*

$$\lim_{n\to\infty} \frac{A(2n) - A(n)}{\log\log n} = \infty,$$

and for every $0<\theta<1$, $\sum_{i=1}^{\infty} \|a_i\theta\|^2 = \infty$. Then the sequence A is complete.

Later H. Davenport remarked that there is a stronger version of Erdős' conjecture: there should be a positive integer $K = K(p,q)$ for which the sequence

$$S_{p,q}(K) = \{p^n q^m : n = 0, 1, \dots;\ 0 \le m \le K\}$$

will be complete.

In [7] it was proved that

$$K(p,q) \le 2p^{2c^{2^{2q^{4p+3}}}},$$

2010 *Mathematics Subject Classification*: Primary 11B13; Secondary 11A07.
Key words and phrases: Birch's theorem, complete sequence, exponential type sequence.
Received 15 April 2015; revised 20 December 2015 and 11 March 2016.
Published online 11 May 2016.

where $c = 1152 \log_2 p \log_2 q$, and this result was improved in [5], [4] to

$$K(p,q) \le c^{2^{q^{2p+3}}},$$

where p and q are coprime integers greater than 1. For related results, see [2], [8] and [9].

The Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$. Let $\mathcal{F} = \{F_0, F_1, \dots\}$ and $\mathcal{F}_k = \{F_k, F_{k+1}, \dots\}$. For any sequence $\{G_i\}$ and two integers $n \ge k$, define $\mathcal{G}_k(n) = \{G_i\}_{i=k}^n$. For (finite or infinite) sequences $A = \{a_1 \le a_2 \le \cdots\}$ and $B = \{b_1 \le b_2 \le \cdots\}$, define the sequence $AB = \{a_i b_j : i = 1, 2, \dots; j = 1, 2, \dots\}$.

The aim of this paper is to investigate the completeness of $S_p A$, where $p > 1$ is an integer.

If $A_{p-1} = \{a_1, \dots, a_{p-1}\}$ with $a_i = 1$ $(1 \le i \le p-1)$, then $P(S_p A_{p-1}) = \mathbb{N} \cup \{0\}$ since every nonnegative integer has a p-ary expansion. It is known that almost all numbers, when expressed in any scale, contain every possible digit (see [6, Theorem 143]). It follows that, if $t \le p-2$ and $A_t = \{a_1, \dots, a_t\}$ with $a_i = 1$ $(1 \le i \le t)$, then $P(S_p A_t)$ has asymptotic density zero. What about $S_p A$ for general A? In this paper, we solve this problem.

It is not too hard to see that the sequence $S_p \mathcal{F}_1(n)$ is complete provided $n \gg \log p$ (more precisely, when $F_n > p$). So it is reasonable to ask the following question: fixing the integer $k > 1$, what is the minimum of $n = n(k)$ for which the sequence $S_p \mathcal{F}_k(n)$ is complete or subcomplete or has positive lower asymptotic density?

The following results are proved.

Theorem 1.1. *Let $p > 1$ and $t \ge 1$ be two integers.*

(i) *If $t \ge p-1$, then, for any sequence $A_t = \{a_1 \le \cdots \le a_t\}$ of positive integers (not necessarily distinct), $P(S_p A_t)$ has positive lower asymptotic density not less than $1/a_{p-1}$. Furthermore, the lower bound $1/a_{p-1}$ is the best possible.*

(ii) *If $2^t < p$, then, for any sequence $A_t = \{a_1 \le \cdots \le a_t\}$ of positive integers (not necessarily distinct), $P(S_p A_t)$ has asymptotic density zero.*

(iii) *If $t < p-1$ and $2^t \ge p$, then there exist two sequences A and B of positive integers with length t such that $P(S_p A)$ has positive lower asymptotic density and $P(S_p B)$ has asymptotic density zero.*

Theorem 1.2. *For any integers $p > 1$ and $k \ge 1$, $S_p \mathcal{F}_k(n)$ is complete, where $n = p^2 F_{k+2p-1}^2$.*

For $p = 3$ we obtain the following result for general sequences $\{G_i\}$.

Theorem 1.3. *Let $\mathcal{G} = \{G_i\}$ be a sequence of integers with $1 \le G_i < \eta^i$ for some $1 < \eta < 2$ and all i. Then, for any given positive integer $k \ge 1$,* *there is an integer* n *with* $k \le n \le ck$ *such that the sequence* $S_3\mathcal{G}_k(n)$ *is subcomplete, where* $c = c(\eta)$ *is a positive constant depending only on* η.

Motivated by Theorem 1.3, we pose the following problem:

PROBLEM 1.4. *Let* $p > 1$ *be an integer and* $\mathcal{G} = \{G_i\}$ *be a sequence of integers, where* $1 \le G_i < \eta^i$ *for some* $1 < \eta < 2$ *and all* i. *Given an integer* k, *is there an integer* $n > k$ *such that* $S_p\mathcal{G}_k(n)$ *is subcomplete?*

2. Proof of Theorem 1.1. We need the following lemma.

LEMMA 2.1. *Let* $B = \{b_1 \le b_2 \le \cdots\}$ *be a sequence of positive integers. Assume that there exists an integer* n_0 *such that for every* $n \ge n_0$, $b_n \le b_1 + \cdots + b_{n-1} + b_{n_0}$. *Then* $P(B)$ *has bounded gaps, i.e. if* $P(B) = \{x_1 < x_2 < \cdots\}$, *then* $x_{k+1} - x_k \le b_{n_0}$ *for every* k.

Proof. We will prove the following stronger proposition: For any positive integer m with $m \le b_1 + \cdots + b_n$, there exists an integer $x \in P(\{b_1, \ldots, b_n\})$ with $0 < m - x \le b_{n_0}$.

For $1 \le m \le b_1$, the proposition is true for $x = 0 \in P(\{b_1\})$. Now we assume that $m > b_1$, $b_1 + \cdots + b_{n-1} < m \le b_1 + \cdots + b_n$ and the proposition is true for all positive integers less than m. If $n \le n_0$ or $m \le b_1 + \cdots + b_{n-1} + b_{n_0}$, then the proposition is true for $x = b_1 + \cdots + b_{n-1} \in P(\{b_1, \ldots, b_n\})$. Now we assume that $n > n_0$ and $b_1 + \cdots + b_{n-1} + b_{n_0} < m \le b_1 + \cdots + b_n$. Then

$$0 \le b_1 + \cdots + b_{n-1} + b_{n_0} - b_n < m - b_n \le b_1 + \cdots + b_{n-1}.$$

By the inductive hypothesis, there exists $x \in P(\{b_1, \ldots, b_{n-1}\})$ such that $0 < (m - b_n) - x \le b_{n_0}$. This implies that $0 < m - (b_n + x) \le b_{n_0}$ and $b_n + x \in P(\{b_1, \ldots, b_n\})$. This completes the proof of the proposition. ■

Now we use Lemma 2.1 to prove Theorem 1.1(i).

Assume that $t \ge p - 1$ and

$$S_pA_t = \{0 = b_1 \le b_2 \le \cdots\}.$$

Our task is to find an integer n_0 such that $b_n \le b_1 + \cdots + b_{n-1} + b_{n_0}$ for every $n \ge n_0$.

Since $1 \in S_p$, it follows that $a_{p-1} \in S_pA_t$. Let n_0 be such that $b_{n_0} = a_{p-1}$. We now show that the n_0 is as required.

Let $n \ge n_0$. For any integer $1 \le l \le p-1$, let T_l be the integer such that

$$p^{T_l} < b_n/a_l \le p^{T_l+1}.$$

Then, for each $1 \le l \le p - 1$, the sum of the integers $p^i a_l$ which are less than b_n is

$$a_l(1 + p + \cdots + p^{T_l}) = \frac{a_l p^{T_l+1} - a_l}{p - 1} \ge \frac{b_n - a_l}{p - 1}.$$

Hence

$$b_1 + \cdots + b_{n-1} \geq \sum_{l=1}^{p-1} a_l(1 + p + \cdots + p^{T_l}) \geq b_n - \frac{1}{p-1}\sum_{l=1}^{p-1} a_l \geq b_n - a_{p-1}.$$

So $b_n \leq b_1 + \cdots + b_{n-1} + b_{n_0}$.

Let $P(S_pA_t) = \{x_1 < x_2 < \cdots\}$. By Lemma 2.1 we have $x_{k+1} - x_k \leq b_{n_0} = a_{p-1}$. Hence the lower asymptotic density of $P(S_pA_t)$ is not less than $1/a_{p-1}$.

If $a_1 = \cdots = a_{p-1} = p$, then S_pA_{p-1} consists of all p^i ($i \geq 1$), where every p^i repeats exactly $p-1$ times. Thus $P(S_pA_{p-1})$ consists of all nonnegative integers which are divisible by p. Hence the asymptotic density of $P(S_pA_{p-1})$ is $1/p = 1/a_{p-1}$. So the bound in Theorem 1.1(i) is the best possible.

Now we prove Theorem 1.1(ii). Assume that $2^t < p$. Let X be a sufficiently large integer.

Suppose that $x \in P(S_pA_t)$. Then

$$x = \sum_{i=0}^{T}\sum_{j=1}^{t} \varepsilon_{i,j} a_j p^i, \quad \varepsilon_{i,j} \in \{0,1\}.$$

Hence the number of integers in $P(S_pA_t)$ which are less than X is at most

$$(2^t)^{\log_p X+1} \leq (p-1)^{\log_p X+1} = (p-1)X^{\log_p(p-1)} = o(X).$$

Therefore, $P(S_pA_t)$ has asymptotic density zero.

Finally we prove Theorem 1.1(iii). Assume that $t < p-1$ and $2^t \geq p$. Let

$$A = \{1, 2, 2^2, \ldots, 2^{t-1}\},$$

and

$$B = \{a_1, \ldots, a_t\}, \quad a_i = 1 \ (1 \leq i \leq t).$$

Since $P(A) = \{0, 1, \ldots, 2^t - 1\}$ and $p - 1 \leq 2^t - 1$, it follows that

$$\{0, 1, \ldots, p-1\} \subseteq P(A).$$

Thus $P(S_pA) = \mathbb{N} \cup \{0\}$ and then $P(S_pA)$ has asymptotic density one. It is mentioned in the introduction that $P(S_pB)$ has asymptotic density zero. This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

3.1. On $P(\mathcal{F}_k)$. In this subsection we establish an interesting fact on the subset sums of $\mathcal{F}_k$, unnoticed in the literature.

PROPOSITION 3.1. *Let $k \geq 2$ be an integer. Then the sequence*

$$P(\mathcal{F}_k) = \{b_1 < b_2 < \cdots\}$$

has bounded gaps. In fact, for every $n \in \mathbb{N}$, we have

$$b_{n+1} - b_n \in \{F_{k-1}, F_k\}.$$

First, we recall a lemma.

LEMMA 3.2. *If*

$$n = F_{i_1} + \cdots + F_{i_t}, \qquad i_1 < \cdots < i_t,$$

then there are integers $i_1 \le j_1 < \cdots < j_s$ *with* $j_{u+1} - j_u \ge 2$ *for all* $1 \le u \le s-1$ *such that*

$$n = F_{j_1} + \cdots + F_{j_s}.$$

This lemma is also known as Zeckendorf's theorem (see [10]). For the sake of completeness we include a short proof.

Proof of Lemma 3.2. If $i_{u+1} - i_u \ge 2$ for all $1 \le u \le t-1$, then we are done. Now we assume that there is an integer v such that $i_{v+1} - i_v = 1$. Let v_0 be the largest integer with $i_{v_0+1} - i_{v_0} = 1$. Then $i_{v_0+2} - i_{v_0+1} \ge 2$ (if i_{v_0+2} exists). That is, $i_{v_0+2} \ge i_{v_0} + 3$. Thus

$$n = F_{i_1} + \cdots + F_{i_{v_0-1}} + F_{i_{v_0}+2} + F_{i_{v_0+2}} + \cdots + F_{i_t}.$$

Now, by induction on t, we obtain the desired conclusion. ∎

LEMMA 3.3. *Suppose that*

$$m = F_{i_1} + \cdots + F_{i_s}, \qquad i_1 > \cdots > i_s \ge 1$$

and

$$n = F_{j_1} + \cdots + F_{j_t}, \qquad j_1 > \cdots > j_t \ge 1.$$

If $m > n$, *then* $m - n \ge F_{\min\{i_s, j_t - 1\}}$.

Proof. By Lemma 3.2, we may assume that $i_{u-1} - i_u \ge 2$ $(2 \le u \le s)$ and $j_{v-1} - j_v \ge 2$ $(2 \le v \le t)$.

If $s \le t$ and $i_u = j_u$ $(u = 1, \ldots, s)$, then $m \le n$, a contradiction.

If $s > t$ and $i_u = j_u$ $(u = 1, \ldots, t)$, then $m - n \ge F_{i_s} \ge F_{\min\{i_s, j_t-1\}}$.

Now we assume that there exists $1 \le u \le \min\{s, t\}$ with $i_u \ne j_u$. Let w be the least integer with $i_w \ne j_w$. Then $1 \le w \le \min\{s, t\}$ and $i_u = j_u$ $(1 \le u < w)$.

If $i_w < j_w$, then

$$\begin{aligned} n - m &\ge F_{j_w} - (F_{i_w} + \cdots + F_{i_s}) \\ &\ge F_{i_w+1} - (F_{i_w} + \cdots + F_{i_s}) \\ &= F_{i_w-1} - (F_{i_{w+1}} + \cdots + F_{i_s}) \\ &\ge F_{i_{w+1}+1} - (F_{i_{w+1}} + \cdots + F_{i_s}) \\ &\ge \cdots \\ &\ge F_{i_s+1} - F_{i_s} = F_{i_s-1} \ge 0, \end{aligned}$$

contrary to $m > n$.

Similarly, if $i_w > j_w$, then $m - n \ge F_{j_t-1} \ge F_{\min\{i_s, j_t-1\}}$. ∎

Proof of Proposition 3.1. Let $n \in \mathbb{N}$. Since $b_n, b_{n+1} \in P(\mathcal{F}_k)$, it follows from Lemma 3.2 that

$$b_{n+1} = F_{i_1} + \cdots + F_{i_s}, \quad i_1 > \cdots > i_s \geq k,$$

and

$$b_n = F_{j_1} + \cdots + F_{j_t}, \quad j_1 > \cdots > j_t \geq k,$$

where $i_{u-1} - i_u \geq 2$ $(2 \leq u \leq s)$ and $j_{v-1} - j_v \geq 2$ $(2 \leq v \leq t)$.

If $j_t > k$, then $b_n + F_k \in P(\mathcal{F}_k)$. So $b_{n+1} \leq b_n + F_k$. By Lemma 3.3, we have $b_{n+1} - b_n \geq F_{\min\{i_s, j_t - 1\}} \geq F_k$. Hence $b_{n+1} - b_n = F_k$.

If $j_t = k$, then $j_{t-1} \geq k+2$. Thus $b_n + F_{k-1} = b_n - F_k + F_{k+1} \in P(\mathcal{F}_k)$. So $b_{n+1} \leq b_n + F_{k-1}$. By Lemma 3.3, we have $b_{n+1} - b_n \geq F_{\min\{i_s, j_t - 1\}} = F_{k-1}$. Hence $b_{n+1} - b_n = F_{k-1}$. ■

3.2. On the completeness of $S_p\mathcal{F}_k(n)$—proof of Theorem 1.2. In this section we prove Theorem 1.2: for any integers $p > 1$ and $k \geq 1$, there exists an integer $n \leq p^2 F_{k+2p-1}^2$ such that the sequence $S_p\mathcal{F}_k(n)$ is complete.

First, we recall a lemma.

LEMMA 3.4. *We have*

$$P(\{F_0, F_1, \dots\}) = \mathbb{N} \cup \{0\}.$$

This lemma is well-known. It is also a special case of Proposition 3.1.

LEMMA 3.5. *For any positive integers m, r with $r > 2m$, there are integers*

$$r - 2m + 2 \leq i_1 < \cdots < i_t$$

such that

$$mF_r = F_{i_1} + \cdots + F_{i_t}.$$

Proof. We use induction on m. It is clear that the assertion is true for $m = 1$. Suppose that it is true for m. Now we consider $(m+1)F_r$ with $r > 2m+2$. Since $r > 2m+2 > 2m$, it follows from the inductive hypothesis that there are integers $r - 2m + 2 \leq i_1 < \cdots < i_t$ such that

$$mF_r = F_{i_1} + \cdots + F_{i_t}.$$

By Lemma 3.2, we may assume that $i_{u+1} - i_u \geq 2$ for all $1 \leq u \leq t-1$.

If $r \notin \{i_1, \dots, i_t\}$, then by rearranging $r, i_1, \dots, i_t$ as $j_1 < \cdots < j_{t+1}$, we have

$$j_1 \geq r - 2m + 2 > r - 2(m+1) + 2$$

and

$$(m+1)F_r = F_{j_1} + \cdots + F_{j_{t+1}}.$$

Now we assume that $r \in \{i_1, \dots, i_t\}$ and say $i_v = r$.

If $i_{u+1} - i_u = 2$ for all $u \le v-1$, then

$$\begin{aligned} F_{i_1} + \cdots + F_{i_v} + F_r &= F_{i_1} + \cdots + F_{i_{v-1}} + F_{i_v - 2} + F_{i_v+1} \\ &= F_{i_1} + \cdots + F_{i_{v-2}} + F_{i_{v-1}-2} + F_{i_{v-1}+1} + F_{i_v+1} \\ &= \cdots \\ &= F_{i_1-2} + F_{i_1+1} + \cdots + F_{i_v+1}. \end{aligned}$$

If $i_{u+1} - i_u > 2$ for some $u \le v-1$, we assume that u is the largest such integer. Similarly, we have

$$F_{i_1} + \cdots + F_{i_v} + F_r = F_{i_1} + \cdots + F_{i_u} + F_{i_{u+1}-2} + F_{i_{u+1}+1} + \cdots + F_{i_v+1}.$$

Hence

$$(m+1)F_r = F_{j_1} + \cdots + F_{j_{t+1}}$$

and $j_{t+1} > \cdots > j_1 \ge i_1 - 2 \ge r - 2(m+1) + 2$. ∎

The next lemma is also known; we include a proof for completeness.

LEMMA 3.6. *For any positive integer d, the sequence $\{F_r\}$ is purely periodic modulo d with period length at most d^2.*

Proof. We consider $(F_r, F_{r+1}) \pmod d$ $(r = 1, 2, \ldots)$. There are positive integers m, T with $m + T \le d^2 + 1$ such that

$$(F_m, F_{m+1}) \equiv (F_{m+T}, F_{m+T+1}) \pmod d.$$

Since

$$F_{r-1} = F_{r+1} - F_r, \qquad F_{r+2} = F_{r+1} + F_r,$$

it follows that

$$(F_r, F_{r+1}) \equiv (F_{r+T}, F_{r+T+1}) \pmod d, \qquad r = 0, 1, \ldots.$$

So $1 \le T \le d^2$ and $F_r \equiv F_{r+T} \pmod d$ for all $r \ge 0$. ∎

Proof of Theorem 1.2. Let $r = k + 2p - 2$. Let u be the integer with $F_u \le pF_r < F_{u+1}$. By Lemma 3.5, for any $1 \le m \le p-1 < r/2$, there are integers $k + 2 \le r - 2m + 2 \le i_1 < \cdots < i_t$ such that

$$mF_r = F_{i_1} + \cdots + F_{i_t}.$$

Since $F_{i_t} \le mF_r \le pF_r < F_{u+1}$, it follows that $i_t \le u$. Thus

$$\{0, F_r, \ldots, (p-1)F_r\} \subseteq P(\mathcal{F}_k(u)). \tag{3.1}$$

Let $d = pF_r$. Let s be any integer with

$$s > F_k + \cdots + F_{k+d^2+u}.$$

Suppose that $s \equiv s_1 \pmod d$ with $0 \le s_1 \le d-1$. By Lemma 3.4, there exist $0 \le j_1 < \cdots < j_\ell$ such that

$$s_1 = F_{j_1} + \cdots + F_{j_\ell}.$$

Then $F_{j_\ell} \le s_1 < d = pF_r < F_{u+1}$. So $j_\ell \le u$. By Lemma 3.6, the sequence $\{F_r\}$ is purely periodic modulo d with period length $T \le d^2$. Suppose that v is the integer with $vT \le d^2 < (v+1)T$. Then

$$vT = \frac{v}{v+1}(v+1)T \ge \frac{1}{2}(v+1)T > \frac{1}{2}d^2 = \frac{1}{2}p^2F_r^2 > F_r \ge F_{k+2} > k,$$
$$s \equiv s_1 = F_{j_1} + \cdots + F_{j_\ell} \equiv F_{vT+j_1} + \cdots + F_{vT+j_\ell} \pmod d$$

and

$$vT + j_\ell \le d^2 + u.$$

Let $n = d^2 + u$. Hence there is $s_2 \in P(\mathcal{F}_k(n))$ such that $s \equiv s_2 \pmod d$. Let $s = s_2 + dL = s_2 + pF_rL$. Since

$$s_2 \le F_k + \cdots + F_n < F_k + \cdots + F_{k+d^2+u} < s,$$

it follows that L is a positive integer. Let its p-ary expansion be

$$L = \ell_0 + \ell_1 p + \cdots + \ell_w p^w, \qquad 0 \le \ell_i \le p-1 \ (0 \le i \le w).$$

Then

$$s = s_2 + dL = s_2 + pF_rL = s_2 + (\ell_0 F_r)p + \cdots + (\ell_w F_r)p^{w+1}.$$

It follows from (3.1) and $s_2 \in P(\mathcal{F}_k(n))$ that $s \in P(S_p\mathcal{F}_k(n))$, where

$$\begin{aligned} n = d^2 + u &= p^2F_r^2 + u \le p^2F_r^2 + pF_r \\ &= p^2F_{k+2p-2}^2 + pF_{k+2p-2} \\ &\le p^2(F_{k+2p-1} - 1)^2 + pF_{k+2p-2} \le p^2F_{k+2p-1}^2. \end{aligned}$$

Since $S_p\mathcal{F}_k(n)$ is complete for an integer $k \le n \le p^2F_{k+2p-1}^2$, it follows that $S_p\mathcal{F}_k(m)$ is complete for $m = p^2F_{k+2p-1}^2$. This completes the proof of Theorem 1.2. ∎

4. Proof of Theorem 1.3. In the following, $\log_2 x$ is the logarithm of x to base 2.

In the notation of Theorem 1.3, we choose

$$n = \left\lfloor \frac{k - \log_2(\eta - 1)}{1 - \log_2 \eta} \right\rfloor. \tag{4.1}$$

Then from $1 < \eta < 2$ we have

$$k \le n \le \frac{k - \log_2(\eta - 1)}{1 - \log_2 \eta} \le \frac{k - k\log_2(\eta - 1)}{1 - \log_2 \eta} = c(\eta)k,$$

where

$$c(\eta) = \frac{1 - \log_2(\eta - 1)}{1 - \log_2 \eta}.$$

It follows from (4.1) that

$$n > \frac{k - \log_2(\eta - 1)}{1 - \log_2 \eta} - 1.$$

Therefore

$$2^{n-k+1} > \frac{\eta^{n+1}}{\eta - 1} > 1 + \eta + \eta^2 + \cdots + \eta^n.$$

Noting that $\eta^i > G_i$ for all i, we have

$$2^{n-k+1} - 1 > \eta + \eta^2 + \cdots + \eta^n \geq G_k + G_{k+1} + \cdots + G_n.$$

Hence there exists $d' > 0$ that has two different representations,

$$d' = \sum_{k \leq i \leq n} \varepsilon_i G_i = \sum_{k \leq i \leq n} \varepsilon'_i G_i,$$

where $\varepsilon_i, \varepsilon'_i \in \{0, 1\}$ $(k \leq i \leq m)$ and $\varepsilon_i \neq \varepsilon'_i$ for at least one i. Deleting the identical terms in these two sums, we obtain

$$\sum_{k \leq i \leq n} \eta_i G_i = \sum_{k \leq i \leq n} \eta'_i G_i,$$

where $\eta_i, \eta'_i \in \{0, 1\}$ and $\eta_i \eta'_i = 0$ $(k \leq i \leq n)$. We write this number as d. Then

$$d = \sum_{k \leq i \leq n} \eta_i G_i = \sum_{k \leq i \leq n} \eta'_i G_i.$$

Hence

$$d, 2d \in P(\mathcal{G}_k(n)).$$

For any positive integer N, let its 3-ary expansion be

$$N = \sum \alpha_r 3^r, \qquad \alpha_r \in \{0, 1, 2\}.$$

Then

$$dN = \sum \alpha_r d 3^r \in P(S_3 \mathcal{G}_k(n)).$$

Therefore, $d\mathbb{N} \subseteq P(S_3 \mathcal{G}_k(n))$. This completes the proof of Theorem 1.3.

Acknowledgments. The first draft of this work was prepared in the beginning of 2015 when the second author visited Alfred Renyi Institute of Mathematics, Hungarian Academy of Sciences. She is grateful to the Institute for their hospitality and financial support. The first author is supported by the National Natural Science Foundation of China, Grant No. 11371195, and PAPD. The second author is supported by the National Natural Science Foundation of China, Grant No. 11201237. The third author is supported by OTKA grants K-81658 and K-100291.

References

[1] B. J. Birch, *Note on a problem of Erdős*, Proc. Cambridge Philos. Soc. 55 (1959), 370–373.

[2] S. A. Burr, P. Erdős, R. L. Graham and W. W.-C. Li, *Complete sequences of sets of integer powers*, Acta Arith. 77 (1996), 133–138.

[3] J. W. Cassels, *On the representation of integers as the sums of distinct summands taken from a fixed set*, Acta Sci. Math. (Szeged) 21 (1960), 111–124.

[4] Y.-G. Chen and J.-H. Fang, *Remark on the completeness of an exponential type sequence*, Acta Math. Hungar. 136 (2012), 189–195.

[5] J.-H. Fang, *A note on the completeness of an exponential type sequence*, Chinese Ann. Math. Ser. B. 32 (2011), 527–532.

[6] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 5th ed., Oxford Univ. Press, 1979.

[7] N. Hegyvári, *On the completeness of an exponential type sequence*, Acta Math. Hungar. 86 (2000), 127–135.

[8] N. Hegyvári, *Complete sequences in* $\mathbb{N}^2$, Eur. J. Combin. 17 (1996), 741–749.

[9] U. Zannier, *On a theorem of Birch concerning sums of distinct integers taken from certain sequences*, Math. Proc. Cambridge Philos. Soc. 106 (1989), 199–206.

[10] E. Zeckendorf, *Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas*, Bull. Soc. Roy. Sci. Liège 41 (1972), 179–182.

Yong-Gao Chen
School of Mathematical Sciences and Institute of Mathematics
Nanjing Normal University
Nanjing 210023, P.R. China
E-mail: ygchen@njnu.edu.cn

Jin-Hui Fang
Department of Mathematics
Nanjing University of Information Science and Technology
Nanjing 210044, P.R. China
E-mail: fangjinhui1114@163.com

Norbert Hegyvári
ELTE TTK
Eötvös University
Institute of Mathematics
Pázmány St. 1/c
H-1117 Budapest, Hungary
E-mail: hegyvari@elte.hu