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Obstruction sets and extensions of groups

by

Francesca Balestrieri (Oxford)

1. Introduction

1.1. General notation and background. In this paper, k will always be a number field and $\overline{k}$ a fixed algebraic closure of k. Let Ω_k be the set of places of k. If $v \in \Omega_k$, we denote by k_v the completion of k at v. By a *variety X over k*, we will mean a reduced, separated scheme of finite type over k. We say that a variety is *nice* if it is smooth, projective, and geometrically integral. If l/k is any field extension, we write X_l for $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(l)$; when $l = \overline{k}$, we simply write $\overline{X}$ for $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$.

Let X be a variety over k. If l is any field containing k, we denote by

$$X(l) := \mathrm{Hom}_{\mathrm{Spec}(k)}(\mathrm{Spec}(l), X)$$

the *set of l-rational points of X*. There is a diagonal embedding $X(k) \subset X(\mathbb{A}_k)$, where $X(\mathbb{A}_k)$ is the *set of adelic points of X*, which is endowed with the restricted product topology. We remark that if X is proper (e.g. if X is projective), then $X(\mathbb{A}_k) = \prod_{v \in \Omega_k} X(k_v)$.

Given a linear algebraic group G over k, a k-scheme Y with a G-action $\mu : Y \times G \to Y$, and a G-morphism $f : Y \to X$, we say that Y is a (*right*) *G-torsor over X* if f is faithfully flat and if the morphism $Y \times G \to Y \times_X Y$ given by $(y, g) \mapsto (y, \mu(y, g))$ is an isomorphism.

Let $\mathcal{L}_k := \{G : G \text{ is a linear algebraic group over } k\}/\sim$, where $G_1 \sim G_2$ if and only if G_1 is k-isomorphic to G_2 as algebraic k-groups. If $\mathcal{S} \subset \mathcal{L}_k$, we write "$(f, Y, G) \in \mathcal{S}(X)$" to mean "$f : Y \to X$ is a G-torsor over X with $G \in \mathcal{S}$"; when the emphasis on f is not needed, we just write "(Y, G)" instead of "(f, Y, G)". The pointed set $H^1(X, G) := \check{H}^1_{\mathrm{fppf}}(X, G)$, which is a group whenever G is abelian, classifies G-torsors over X up to k-isomorphism

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[Sko01, §2.2]; we usually denote by $[Y]$ the k-isomorphism class in $H^1(X,G)$ of the G-torsor $f : Y \to X$.

Given a G-torsor $f : Y \to X$ over X and a 1-cocycle $\tau \in Z^1(k,G)$, we can *twist* Y *by* τ: we can construct a G^τ-torsor $f^\tau : Y^\tau \to X$ over X, where G^τ and Y^τ are "twisted" versions of G and Y (see [Sko01, Chapter 2] for more details); we remark that, up to k-isomorphism, this construction only depends on the class $[\tau] \in H^1(k,G)$. For more on the theory of torsors, we refer the reader to [Sko01, Chapter 2].

For any $(f,Y,G) \in \mathcal{S}(X)$, we write

$$X(\mathbb{A}_k)^f := \bigcup_{[\tau]\in H^1(k,G)} f^\tau(Y^\tau(\mathbb{A}_k)),$$

and for any $\mathcal{S} \subset \mathcal{L}_k$, we write

$$X(\mathbb{A}_k)^{\mathcal{S}} := \bigcap_{G\in\mathcal{S}} \bigcap_{[Y]\in H^1(X,G)} X(\mathbb{A}_k)^f.$$

The *Brauer group functor* $\mathrm{Br} : \mathbf{Sch}_k^{\mathrm{op}} \to \mathbf{Ab}$ is given by $Y \mapsto \check{H}^2_{\text{ét}}(Y,\mathbb{G}_{\mathrm{m}})$, and the *Brauer group of* X is simply $\mathrm{Br}(X)$. The *algebraic Brauer group of* X is given by $\mathrm{Br}_1(X) := \ker(\mathrm{Br}(X) \to \mathrm{Br}(\bar{X}))$, and one can check that $\mathrm{Br}_1 : \mathbf{Sch}_k^{\mathrm{op}} \to \mathbf{Ab}$ is a subfunctor of Br. Recall that the *Brauer–Manin pairing* $\langle -,- \rangle_{\mathrm{BM}} : X(\mathbb{A}_k) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}$ is defined by $((x_v),\alpha) \mapsto \sum_{v\in\Omega_k} \mathrm{inv}_v\, \alpha(x_v)$, where $\mathrm{inv}_v : \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$, for $v \in \Omega_k$, is the invariant map coming from local class field theory. The *Brauer–Manin set* is then given by

$$X(\mathbb{A}_k)^{\mathrm{Br}} := \bigcap_{\alpha\in\mathrm{Br}(X)} \{(x_v) \in X(\mathbb{A}_k) : \langle (x_v),\alpha \rangle_{\mathrm{BM}} = 0\}.$$

We define the *algebraic Brauer–Manin set* $X(\mathbb{A}_k)^{\mathrm{Br}_1}$ similarly, by taking the intersection over $\alpha \in \mathrm{Br}_1(X)$ instead. We have $\overline{X(k)} \subset X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k)^{\mathrm{Br}_1} \subset X(\mathbb{A}_k)$, where the first inclusion follows from global class field theory and the continuity of $\langle -,\alpha \rangle_{\mathrm{BM}} : X(\mathbb{A}_k) \to \mathbb{Q}/\mathbb{Z}$ for each $\alpha \in \mathrm{Br}(X)$.

More generally, let $\mathcal{O} : \mathbf{Sch}_k^{\mathrm{op}} \to \mathbf{Sets}$ be any functor from the opposite category of schemes over k to the category of sets. Let $W \in \mathbf{Sch}_k$. For each $\alpha \in \mathcal{O}(W)$, there is a commutative diagram

$$\begin{array}{ccc} W(k) & \longrightarrow & W(\mathbb{A}_k) \\ {\scriptstyle \mathrm{ev}_\alpha}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_\alpha} \\ \mathcal{O}(k) & \longrightarrow & \mathcal{O}(\mathbb{A}_k) \end{array}$$

where the horizontal maps are the diagonal maps and the vertical maps are the evaluation maps. We set

$$W(\mathbb{A}_k)^{\mathcal{O}} := \bigcap_{\alpha\in\mathcal{O}(W)} \{(w_v) \in W(\mathbb{A}_k) : \mathrm{ev}_\alpha((w_v)) \in \mathrm{Im}(\mathcal{O}(k) \to \mathcal{O}(\mathbb{A}_k))\}.$$

Hence, for any subset $\mathcal{S} \subset \mathcal{L}_k$, we can define

$$X(\mathbb{A}_k)^{\mathcal{S},\mathcal{O}} := \bigcap_{G\in\mathcal{S}} \bigcap_{[Y]\in H^1(X,G)} \bigcup_{[\tau]\in H^1(k,G)} f^\tau(Y^\tau(\mathbb{A}_k)^{\mathcal{O}}). \tag{1.1}$$

Let $\mathcal{F}_k \subset \mathcal{L}_k$ be the set of finite algebraic groups over k (up to k-isomorphisms). Following the refinements studied in [Sto07], we consider the chain of inclusions $\mathcal{F}_k^{\mathrm{Ab}} \subset \mathcal{F}_k^{\mathrm{Sol}} \subset \mathcal{F}_k$, where $\mathcal{F}_k^{\mathrm{Ab}}$ and $\mathcal{F}_k^{\mathrm{Sol}}$ are, respectively, the sets of (k-isomorphism classes of) commutative and solvable finite linear algebraic groups over k. By taking $\mathcal{S} = \mathcal{F}_k$ and $\mathcal{O} = \mathrm{Br}$ or $\mathcal{O} = \mathrm{Br}_1$ in (1.1), we get, respectively, the *étale-Brauer set*, usually denoted by $X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}}$, and the *algebraic étale-Brauer set*, denoted by $X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}_1}$. Similarly, by taking $\mathcal{S} = \mathcal{F}_k^{\mathrm{Sol}}$ or $\mathcal{S} = \mathcal{F}_k^{\mathrm{Ab}}$ and $\mathcal{O} = \mathrm{Br}$, we obtain sets that we will denote by $X(\mathbb{A}_k)^{\mathrm{Sol},\mathrm{Br}}$ and $X(\mathbb{A}_k)^{\mathrm{Ab},\mathrm{Br}}$, respectively; by taking $\mathcal{S} = \mathcal{F}_k^{\mathrm{Sol}}$ or $\mathcal{S} = \mathcal{F}_k^{\mathrm{Ab}}$ and $\mathcal{O} = \mathrm{Br}_1$, we obtain sets that we will denote by $X(\mathbb{A}_k)^{\mathrm{Sol},\mathrm{Br}_1}$ and $X(\mathbb{A}_k)^{\mathrm{Ab},\mathrm{Br}_1}$, respectively.

1.2. Motivation. Let k be a number field. In general, a family $\{X_\omega\}$ of nice varieties over k can have interesting arithmetic properties depending on the interplay between the set $X(\mathbb{A}_k)$ of adelic points and the set $X(k)$ of rational points, for all $X \in \{X_\omega\}$. For example, we say that $\{X_\omega\}$ *satisfies the Hasse principle* if $X(\mathbb{A}_k) \neq \emptyset$ implies that $X(k) \neq \emptyset$, for all $X \in \{X_\omega\}$; we remark that the opposite implication is clear, as we always have $X(k) \subset X(\mathbb{A}_k)$. Another example, if one is more interested in density properties, is the following: we say that $\{X_\omega\}$ *satisfies weak approximation* if $X(k) \neq \emptyset$ and $\overline{X(k)} = X(\mathbb{A}_k)$, for all $X \in \{X_\omega\}$.

Often, however, some of these arithmetic properties fail to hold, since $X(\mathbb{A}_k)$ is, in some sense, too big to detect whichever feature we are looking for. (To give an idea, it is common to have $X(\mathbb{A}_k) \neq \emptyset$ but $X(k) = \emptyset$—a clear failure of the Hasse principle.)

When this happens, we can try to refine $X(\mathbb{A}_k)$ by cutting out *obstruction sets* $X(\mathbb{A}_k)^{\mathfrak{O}} \subset X(\mathbb{A}_k)$ in a suitable way, where "suitable" depends on the context. For example, if we are considering the Hasse principle, we need the inclusion $X(k) \subset X(\mathbb{A}_k)^{\mathfrak{O}}$ to hold; we then say that there is an $\mathfrak{O}$-*obstruction* to the Hasse principle if $X(\mathbb{A}_k) \neq \emptyset$ and $X(\mathbb{A}_k)^{\mathfrak{O}} = \emptyset$, and that the $\mathfrak{O}$-*principle holds* if $X(\mathbb{A}_k)^{\mathfrak{O}} \neq \emptyset$ implies $X(k) \neq \emptyset$ (i.e. if the Hasse principle holds with "$X(\mathbb{A}_k)$" replaced by "$X(\mathbb{A}_k)^{\mathfrak{O}}$"). Similarly, if we are interested in weak approximation, we want $\overline{X(k)} \subset X(\mathbb{A}_k)^{\mathfrak{O}}$; we then say that there is an $\mathfrak{O}$-*obstruction to weak approximation* if $X(\mathbb{A}_k)^{\mathfrak{O}} \neq X(\mathbb{A}_k)$, and that the $\mathfrak{O}$-*weak approximation holds* if $X(k) \neq \emptyset$ and $\overline{X(k)} = X(\mathbb{A}_k)^{\mathfrak{O}}$ (assuming $X(\mathbb{A}_k)^{\mathfrak{O}} = \overline{X(\mathbb{A}_k)^{\mathfrak{O}}}$).

Obstruction sets are thus useful objects to study, as they help us measure how far varieties are from satisfying interesting arithmetic properties concerning rational points. The general study of these objects took off in the 1970s with the work of Manin [Man71] and is still a very active area of research in arithmetic geometry—see, for example, [Har96], [Sko99], [Poo10], [HS13] for some more recent developments.

There are two different, somewhat competing, approaches in defining obstruction sets: we can either make use of the Brauer group, as is the case for e.g. $X(\mathbb{A}_k)^{\text{ét},\text{Br}}$, or use the more classical "pure" descent on torsors under linear algebraic groups (as is the case for e.g. $X(\mathbb{A}_k)^{\mathcal{L}_k}$), as introduced by Colliot-Thélène and Sansuc [CTS87]. An interesting task, then, is to try to reconcile these different approaches, that is, to provide a "translation" between obstruction sets defined in a "Brauer-type" language and obstruction sets defined in a "descent-type" language. There are some important results in the literature in this direction, whenever X is a nice variety over a number field k: Skorobogatov has shown that $X(\mathbb{A}_k)^{\text{Br}_1} = X(\mathbb{A}_k)^{\mathcal{M}_k}$, where $\mathcal{M}_k \subset \mathcal{L}_k$ is the set of (k-isomorphism classes of) linear algebraic groups of multiplicative type over k (see [Sko99, Theorem 3]; a less general result, requiring $\operatorname{Pic} \overline{X}$ to be torsion-free, had been proven by Colliot-Thélène and Sansuc [CTS87]); a result by Harari [Har02, Théorème 2(ii)], together with a result of Gabber (cf. [dJ]) and [Sko01, Proposition 5.3.4], implies that $X(\mathbb{A}_k)^{\text{Br}} = X(\mathbb{A}_k)^{\mathcal{C}_k}$, where $\mathcal{C}_k \subset \mathcal{L}_k$ is the set of (k-isomorphism classes of) connected linear algebraic groups over k; the articles by Demarche [Dem09] and Skorobogatov [Sko09] show that $X(\mathbb{A}_k)^{\text{ét},\text{Br}} = X(\mathbb{A}_k)^{\mathcal{L}_k}$.

Now, any linear algebraic group can be decomposed into simpler building blocks: finite étale algebraic groups, connected linear algebraic groups, reductive linear algebraic groups, solvable linear algebraic groups, semisimple linear algebraic groups, linear algebraic tori, and unipotent linear algebraic groups. A natural question then is:

Question 1.1. If $\mathcal{S} \subset \mathcal{L}_k$ is the set of any of the above building blocks, what is (if any) the translation of $X(\mathbb{A}_k)^{\mathcal{S}}$ in "Brauer-type" terms?

Symmetrically, one can also ask:

Question 1.2. Given any natural enough Brauer-type obstruction set (such as, for example, $X(\mathbb{A}_k)^{\text{Br}}$, $X(\mathbb{A}_k)^{\text{ét},\text{Br}}$, $X(\mathbb{A}_k)^{\text{Sol},\text{Br}_1}$, and so on), what is (if any) its translation in pure "descent-type" terms?

The above mentioned results answer these questions for all linear algebraic groups of multiplicative type (symmetrically, for the algebraic Brauer–Manin set), for all connected linear algebraic groups (symmetrically, for the Brauer–Manin set), and for all linear algebraic groups (symmetrically, for the étale-Brauer set). The aim of this article is to answer Questions 1.1 and 1.2 in some other cases: for unipotent, reductive, and solvable linear algebraic groups, and for the Brauer-type obstruction sets $X(\mathbb{A}_k)^{\mathrm{Sol,Br}}$, $X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}_1}$, and $X(\mathbb{A}_k)^{\mathrm{Sol,Br}_1}$.

1.3. Main results and structure of the paper. In §2, we give some properties of linear algebraic groups. In §3 and §4, we follow closely techniques and ideas from [Dem09], [Sko09], and [Sto07] to prove the following comparison theorem for obstruction sets.

Theorem (Theorem 3.1). *Let X be a nice variety over k. Let $\mathcal{O}$: $\mathbf{Sch}_k^{\mathrm{op}} \to \mathbf{Sets}$ be any functor such that $X(\mathbb{A}_k)^{\mathcal{O}} \subset X(\mathbb{A}_k)^{\mathrm{Br}_1}$, and let $\mathcal{S}_k \subset \mathcal{L}_k$ be subject to Conditions* 1, 2 *and* 3 *in* §3. *Then*

$$X(\mathbb{A}_k)^{\mathcal{F}_k,\mathcal{O}} = X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k,\mathcal{S}_k)},$$

where $\mathrm{Ext}(\mathcal{F}_k,\mathcal{S}_k) \subset \mathcal{L}_k$ *is the subset of linear algebraic groups over k that can be written as extensions of elements in $\mathcal{F}_k$ by elements in $\mathcal{S}_k$ (up to k-isomorphisms). The same result holds if we replace "$\mathcal{F}_k$" by "$\mathcal{F}_k^{\mathrm{Sol}}$".*

In §5, by applying Theorem 3.1, we are able to translate the natural "Brauer-type" obstruction sets $X(\mathbb{A}_k)^{\mathrm{Sol,Br}}$, $X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}_1}$, and $X(\mathbb{A}_k)^{\mathrm{Sol,Br}_1}$ into a pure "descent-type" language—see, respectively, Theorem 5.1, Corollary 5.10, and Corollary 5.10.

Let $\mathcal{U}_k \subset \mathcal{L}_k$ be the set of (k-isomorphism classes of) unipotent linear algebraic groups over k. In §6, we prove the following.

Theorem (Theorem 6.1). *Let X be a smooth, proper, geometrically integral variety over k. Let $\mathcal{A} \subset \mathcal{B} \subset \mathcal{L}_k$ be such that $\mathcal{B} \subset \mathrm{Ext}(\mathcal{A},\mathcal{U}_k)$. (Here, $\mathrm{Ext}(\mathcal{A},\mathcal{U}_k) \subset \mathcal{L}_k$ is the set of (k-isomorphism classes of) linear algebraic groups over k that can be written as extensions of elements in $\mathcal{A}$ by elements in $\mathcal{U}_k$.) Then $X(\mathbb{A}_k)^{\mathcal{A}} = X(\mathbb{A}_k)^{\mathcal{B}}$.*

From Theorem 6.1, we then get a series of corollaries (cf. Corollaries 6.8, 6.10, 6.12, 6.14, 6.15, 6.17). In particular:

(1) $X(\mathbb{A}_k)^{\mathcal{U}_k} = X(\mathbb{A}_k)$ (Corollary 6.8).
(2) $X(\mathbb{A}_k)^{\mathcal{L}_k} = X(\mathbb{A}_k)^{\mathcal{R}_k}$ (Corollary 6.10). Here, $\mathcal{R}_k \subset \mathcal{L}_k$ is the set of (k-isomorphism classes of) reductive linear algebraic groups over k.
(3) $X(\mathbb{A}_k)^{\mathrm{Sol,Br}_1} = X(\mathbb{A}_k)^{\mathrm{Sol}_k}$ (Corollary 6.14). Here, $\mathrm{Sol}_k \subset \mathcal{L}_k$ is the set of (k-isomorphism classes of) solvable linear algebraic groups over k.

Finally, in §7, we summarise the relations between the different obstruction sets considered in this paper.

2. Extensions of linear algebraic groups. Linear algebraic groups play an essential role in this paper, so we start by recalling some of their basic properties. Although some of these properties hold for more general fields, throughout this section we will just consider linear algebraic groups over fields of characteristic 0; this will spare us many technical complications (not needed in this paper anyway).

Let K be a field of characteristic 0.

DEFINITION 2.1. An (*algebraic*) *K-group* is a group object in the category $\mathbf{Var}_K$ of varieties over K. A *linear algebraic K-group* is a group object in the category $\mathbf{AffVar}_K$ of affine varieties over K. If G_1, G_2 are algebraic K-groups, a map $\phi : G_1 \to G_2$ is a *morphism of algebraic K-groups* if it is a K-morphism of K-varieties which is also a homomorphism of groups.

REMARK 2.2. In characteristic 0, any algebraic group is always smooth (see [Car62]). Smooth affine algebraic groups are the same as closed algebraic subgroups of GL_n for some n—hence the name "linear algebraic groups".

EXAMPLE 2.3. The *multiplicative group* $\mathbb{G}_{\mathrm{m},K} = (\mathrm{Spec}(K[x,x^{-1}]),\cdot)$ and the *additive group* $\mathbb{G}_{\mathrm{a},K} = (\mathrm{Spec}(K[x]),+)$ are linear algebraic K-groups that can be represented, respectively, as

$$\left\{\begin{pmatrix} x & 0 \\ 0 & x^{-1}\end{pmatrix} : x \in K^\times\right\} \subset \mathrm{GL}_2(K) \quad \text{and} \quad \left\{\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} : x \in K\right\} \subset \mathrm{GL}_2(K).$$

2.1. Closure properties of linear algebraic groups. For convenience, we recall some definitions and, without proofs, some "closure" properties of algebraic groups over K.

- (C1) Let G be a K-group. By a (*closed*) *K-subgroup H* of G, written $H \leq_K G$, we mean a K-group H (with multiplication and identity induced from those of G) that is also a K-closed subvariety of G (with respect to the Zariski topology).
- (C2) Let $1 \to A \to B \to C \to 1$ be a short exact sequence of algebraic K-groups. Then B is affine if and only if A and C are [DGA, VI.B, 9.2(viii)]. Hence, since we are in characteristic 0, B is linear if and only if A and C are linear.
- (C3) If G is an algebraic K-group and H is a normal (closed) K-subgroup of G, then G/H exists and is an algebraic K-group [DGA, VI.A, Théorèmes 3.2 and 5.2], which is linear if G is linear.

(C4) Let G be a K-group. We denote by G^0 the *connected component of the identity of* G. This is a (closed) normal K-subgroup of G of finite index, whose cosets are the connected (equivalently, irreducible) components of G (see [Bor91, (1.2)]). Note that if G is a linear algebraic K-group, then so is G^0.

(C5) Let G be a K-group. Then the *radical* $R(G)$ of G is the largest connected solvable normal (closed) K-subgroup of G; the set $R_u(G)$ of all unipotent elements of $R(G)$ is the *unipotent radical* of G, which is the largest connected unipotent normal closed K-subgroup of G. Note that $R(G) = R(G^0)$ and $R_u(G) = R_u(G^0)$.

(C6) Let G, G' be K-groups, and let $\phi : G \to G'$ be a morphism of K-groups. Then $\phi(G)$ is a (closed) normal K-subgroup of G' [Spr98, Proposition 2.2.5], and similarly $\ker(\phi)$ is a (closed) K-subgroup of G.

(C7) If G is a connected linear algebraic K-group, then its commutator $\mathcal{D}G$ is a connected linear K-subgroup of G [Spr98, Corollary 2.2.8].

Fix an algebraic closure $\overline{K}$ of K.

Definition 2.4. A linear algebraic group G over K is

(D1) *finite* if $G(\overline{K})$ is finite (this definition uses the fact that $\operatorname{char} K = 0$);

(D2) *connected* if its underlying topological space is connected in the Zariski topology;

(D3) *solvable* if its derived series $G \supset \mathcal{D}G \supset \mathcal{D}^2 G \supset \cdots$ has $\mathcal{D}^i G$ trivial for some $i \in \mathbb{Z}_{\geq 0}$;

(D4) *reductive* if its geometric unipotent radical is trivial, i.e. $R_u(G^0_{\overline{K}}) = R_u(G^0)_{\overline{K}}$ is trivial; note that we do not require G to be connected;

(D5) *semisimple* if its geometric radical is trivial, i.e. $R(G^0_{\overline{K}}) = R(G^0)_{\overline{K}}$ is trivial; again we do not require G to be connected;

(D6) *of multiplicative type* if $G_{\overline{K}}$ is $\overline{K}$-isomorphic to a closed subgroup of $\mathbb{G}^N_{\mathrm{m},\overline{K}}$ for some $N \in \mathbb{Z}_{\geq 0}$;

(D7) a *torus* if $G_{\overline{K}}$ is $\overline{K}$-isomorphic to $\mathbb{G}^N_{\mathrm{m},\overline{K}}$ for some $N \in \mathbb{Z}_{\geq 0}$;

(D8) *unipotent* if $G_{\overline{K}}$ admits a (finite) composition series over $\overline{K}$ such that each successive quotient is isomorphic (over $\overline{K}$) to a closed subgroup of $\mathbb{G}_{\mathrm{a},\overline{K}}$; equivalently (cf. [DGA, XVII, Théorème 3.5]), if G admits a (finite) central series over K such that each successive quotient is isomorphic (over K) to $\mathbb{G}_{\mathrm{a}}$ (here we use the fact that $\operatorname{char} K = 0$); equivalently, if $G = R_u(G)$. In particular, if G is unipotent, then G is connected. Note that "being unipotent" is a geometric property, i.e. is stable under base-extending K (cf. [DGA, XVII, Proposition 2.2(i)]).

We define the following sets:

$$\begin{aligned}
\mathcal{L}_K &= \{G : G \text{ is a linear algebraic } K\text{-group}\}/{\sim},\\
\mathcal{F}_K &= \{G \in \mathcal{L}_K : G \text{ is finite}\},\\
\mathcal{C}_K &= \{G \in \mathcal{L}_K : G \text{ is connected}\},\\
\mathcal{A}b_K &= \{G \in \mathcal{L}_K : G \text{ is commutative}\},\\
\mathrm{Sol}_K &= \{G \in \mathcal{L}_K : G \text{ is solvable}\},\\
\mathcal{M}_K &= \{G \in \mathcal{L}_K : G \text{ is of multiplicative type}\},\\
\mathcal{T}_K &= \{G \in \mathcal{L}_K : G \text{ is a torus}\},\\
\mathcal{R}_K &= \{G \in \mathcal{L}_K : G \text{ is reductive}\},\\
\mathcal{RR}_K &= \{G \in \mathcal{R}_K : H \leq_K G \Rightarrow H \in \mathcal{R}_K\},\\
\mathcal{U}_K &= \{G \in \mathcal{L}_K : G \text{ is unipotent}\},
\end{aligned}$$

where $G_1 \sim G_2$ if and only if G_1 is K-isomorphic to G_2 as a K-group.

NOTATION. For any $\mathcal{S} \subset \mathcal{L}_K$, we write $\mathcal{S}^{\mathrm{Sol}}$ for $\mathcal{S} \cap \mathrm{Sol}_K$, $\mathcal{S}^{\mathrm{Ab}}$ for $\mathcal{S} \cap \mathcal{A}b_K$, and so on. Also, by abuse of notation we write "$G \in \mathcal{S}$" to mean "$[G] \in \mathcal{S}$".

Recall that, if A, B, and G are linear algebraic groups over K, we say that G is *an extension of A by B* if G fits into a short exact sequence $1 \to B \to G \to A \to 1$ of linear algebraic K-groups.

DEFINITION 2.5. Let $\mathcal{A}, \mathcal{B} \subset \mathcal{L}_K$. We define

$$\mathrm{Ext}(\mathcal{A}, \mathcal{B}) = \{G \in \mathcal{L}_K : G \text{ is an extension of } A \text{ by } B \text{ for some } A \in \mathcal{A}, B \in \mathcal{B}\}.$$

REMARK 2.6. If $G \in \mathrm{Ext}(\mathcal{A}, \mathcal{B})$, then, by definition, G fits into a short exact sequence $1 \to B \to G \to A \to 1$ for some $B \in \mathcal{B}$ and $A \in \mathcal{A}$. If $(f, Y, G) \in \mathrm{Ext}(\mathcal{A}, \mathcal{B})(X)$, then f naturally decomposes into $(Y, B) \in \mathcal{B}(Z)$ and $(Z, A) \in \mathcal{A}(X)$, where $Z := Y/B$ is the push-forward of $f : Y \to X$ along the morphism $G \to A$ (see [Sko01, §2.2] for more details).

It might be worth mentioning that some of the sets defined above can be recast in terms of extensions of linear algebraic groups. For example, it is easy to check that $\mathcal{L}_K = \mathrm{Ext}(\mathcal{F}_K, \mathcal{C}_K)$; perhaps less obviously, $\mathcal{RR}_K = \mathrm{Ext}(\mathcal{F}_K, \mathcal{T}_K)$ (cf. Lemma 5.9).

2.2. Some properties of extensions. We describe here some properties of $\mathrm{Ext}(\mathcal{A}, \mathcal{B})$ for the special cases when $\mathcal{A} = \mathcal{F}_K$, $\mathcal{F}_K^{\mathrm{Sol}}$, or $\mathcal{F}_K^{\mathrm{Ab}}$.

In general, we say that some $\mathcal{S} \subset \mathcal{L}_K$ is *closed*

- (E1) *under taking direct products* if $S_1, S_2 \in \mathcal{S}$ implies $S_1 \times_K S_2 \in \mathcal{S}$;
- (E2) *under taking* (*closed*) *K-subgroups* if $S \in \mathcal{S}$ and $H \leq_K S$ imply $H \in \mathcal{S}$;

(E3) *under taking K-twists* if whenever $S \in \mathcal{S}$ and $S' \in \mathcal{L}_K$ are such that $S'_{\overline{K}} \cong S_{\overline{K}}$ as $\overline{K}$-groups, then $S' \in \mathcal{S}$;

(E4) *under "base changing/restricting"* if $S \in \mathcal{S}$ and L/K a finite field extension imply $\mathfrak{R}_{L/K}(S_L) \in \mathcal{S}$, where $\mathfrak{R}_{L/K}$ denotes the Weil restriction.

REMARK 2.7. The sets $\mathcal{F}_K$, $\mathcal{F}_K^{\mathrm{Sol}}$, and $\mathcal{F}_K^{\mathrm{Ab}}$ have all the properties above. Note, however, that $\mathcal{F}_K^{\mathrm{Ab}}$ is not closed under extensions, while both $\mathcal{F}_K$ and $\mathcal{F}_K^{\mathrm{Sol}}$ are.

LEMMA 2.8. *If $\mathcal{B} \subset \mathcal{L}_K$ is closed under taking closed K-subgroups, then so is* $\mathrm{Ext}(\mathcal{F}_K, \mathcal{B})$. *The same is true with "$\mathcal{F}_K$" replaced by "$\mathcal{F}_K^{\mathrm{Sol}}$" or "$\mathcal{F}_K^{\mathrm{Ab}}$".*

Proof. Let $G \in \mathrm{Ext}(\mathcal{F}_K, \mathcal{B})$, and say it fits into the short exact sequence $1 \to B \xrightarrow{\beta} G \xrightarrow{\alpha} A \to 1$ for some $A \in \mathcal{F}_K$ and $B \in \mathcal{B}$. Let $H \leq_K G$. Consider the short exact sequence

$$1 \to \beta(B) \cap H \xrightarrow{\beta'} H \xrightarrow{\alpha'} \alpha(H) \to 1,$$

where α' and β' are the obvious maps. This is indeed exact at the ends, and to see exactness at the middle, just notice that $\beta(B) \cap H \cong \ker(\alpha) \cap H \cong \ker(\alpha')$. Since α is a K-homomorphism, it follows that $\alpha(H) \leq_K A$, and hence $\alpha(H) \in \mathcal{F}_K$. Moreover, by assumption, $\mathcal{B}$ is closed under taking K-subgroups, meaning that $\beta(B) \cap H \in \mathcal{B}$. Hence, $H \in \mathrm{Ext}(\mathcal{F}_K, \mathcal{B})$. The same proof works also if we replace "$\mathcal{F}_K$" by "$\mathcal{F}_K^{\mathrm{Sol}}$" or "$\mathcal{F}_K^{\mathrm{Ab}}$". ∎

LEMMA 2.9. *If $\mathcal{B} \subset \mathcal{L}_K$ is closed under taking direct products, then so is* $\mathrm{Ext}(\mathcal{F}_K, \mathcal{B})$. *The same is true with "$\mathcal{F}_K$" replaced by "$\mathcal{F}_K^{\mathrm{Sol}}$" or "$\mathcal{F}_K^{\mathrm{Ab}}$".*

Proof. Easy. ∎

LEMMA 2.10. *Suppose* $\mathrm{Ext}(\mathcal{F}_K, \mathcal{B})$ *is closed under taking direct products and closed K-subgroups. Let $G, G', G'' \in$* $\mathrm{Ext}(\mathcal{F}_K, \mathcal{B})$*, and suppose $\phi' : G' \to G$ and $\phi'' : G'' \to G$ are morphisms of algebraic K-groups. Then $G' \times_G G'' \in$* $\mathrm{Ext}(\mathcal{F}_K, \mathcal{B})$*, where the fibred product is with respect to ϕ and ϕ'. The same is true with "$\mathcal{F}_K$" replaced by "$\mathcal{F}_K^{\mathrm{Sol}}$" or "$\mathcal{F}_K^{\mathrm{Ab}}$".*

Proof. This follows from the assumption that $\mathrm{Ext}(\mathcal{F}_K, \mathcal{B})$ is closed under taking (closed) K-subgroups and direct products, since $G' \times_G G'' \cong \ker(\phi : G' \times_K G'' \to G)$ with $\phi((g', g'')) = \phi'(g')\phi''^{-1}(g'')$ by Yoneda's lemma, and so $G' \times_G G''$ is a closed K-subgroup of $G' \times_K G''$. The same proof works also if we replace "$\mathcal{F}_K$" by "$\mathcal{F}_K^{\mathrm{Sol}}$" or "$\mathcal{F}_K^{\mathrm{Ab}}$". ∎

Let L/K be a finite field extension. Recall that, for any quasi-projective L-scheme W, the *Weil restriction* $\mathfrak{R}_{L/K}(W)$ exists and represents the contravariant functor $T \mapsto W(T \times_K L)$ from $\mathbf{Sch}_K$ to $\mathbf{Set}$. We list here some of the properties of $\mathfrak{R}_{L/K}$ (see [Vos11, Chap. 1, §3.12]).

(W1) If X is a K-variety, there is a functorial bijection

$$\mathrm{Hom}_K(X, \mathfrak{R}_{L/K}(X \times_K L)) \cong \mathrm{Hom}_L(X \times_K L, X \times_K L).$$

This gives, by considering $\mathrm{id}_{X \times_K L} : X \times_K L \to X \times_K L$ on the right-hand side, the canonical embedding $\iota : X \to \mathfrak{R}_{L/K}(X \times_K L)$.

(W2) If X is affine (respectively, smooth) variety over L, then $\mathfrak{R}_{L/K}(X)$ is also affine (respectively, smooth). If G is an L-group, then $\mathfrak{R}_{L/K}(G)$ is a K-group. Moreover, $\mathfrak{R}_{L/K}(G)$ is connected if and only if G is connected, and if G is commutative, then so is $\mathfrak{R}_{L/K}(G)$.

(W3) Given a short exact sequence $1 \to G' \to G \to G'' \to 1$ of algebraic L-groups,

$$1 \to \mathfrak{R}_{L/K}(G') \to \mathfrak{R}_{L/K}(G) \to \mathfrak{R}_{L/K}(G'') \to 1$$

is a short exact sequence of algebraic K-groups. In other words, the functor $\mathfrak{R}_{L/K}(-)$ preserves short exactness.

Lemma 2.11. *If $\mathcal{B} \subset \mathcal{L}_K$ is closed under "base changing/restricting" and under taking (closed) K-subgroups, then $\mathrm{Ext}(\mathcal{F}_K, \mathcal{B})$ is closed under K-twists. The same is true with "$\mathcal{F}_K$" replaced by "$\mathcal{F}_K^{\mathrm{Sol}}$" or "$\mathcal{F}_K^{\mathrm{Ab}}$".*

Proof. We need to show that if $G \in \mathrm{Ext}(\mathcal{F}_K, \mathcal{B})$ and $\tilde{G} \in \mathcal{L}_K$ is such that $\tilde{G}_{\overline{K}} \cong G_{\overline{K}}$ (over $\overline{K}$), then $\tilde{G} \in \mathrm{Ext}(\mathcal{F}_K, \mathcal{B})$. Consider the short exact sequence

$$1 \to B \xrightarrow{b} G \xrightarrow{a} A \to 1.$$

Let $\tilde{G}$ be a K-twist of G, so that $\tilde{G}_{\overline{K}} \cong G_{\overline{K}}$ over $\overline{K}$. In particular, there is a finite field extension L/K such that $G_L \cong \tilde{G}_L$. Let $\phi : G_L \xrightarrow{\sim} \tilde{G}_L$ be an L-isomorphism of these L-groups with $\varphi : \tilde{G}_L \xrightarrow{\sim} G_L$ as inverse. Since group schemes represent functors of fppf group sheaves, short exact sequences of group schemes are stable under base change. This gives a short exact sequence $1 \to B_L \xrightarrow{b_L} G_L \xrightarrow{a_L} A_L \to 1$, which, using the L-isomorphisms ϕ and φ above, in turn gives the short exact sequence

$$1 \to B_L \xrightarrow{\phi \circ b_L} \tilde{G}_L \xrightarrow{a_L \circ \varphi} A_L \to 1. \tag{2.1}$$

Applying $\mathfrak{R}_{L/K}(-)$ to (2.1) gives a short exact sequence of algebraic K-groups

$$1 \to \mathfrak{R}_{L/K}(B_L) \xrightarrow{\beta} \mathfrak{R}_{L/K}(\tilde{G}_L) \xrightarrow{\alpha} \mathfrak{R}_{L/K}(A_L) \to 1.$$

Now, by assumption $\mathfrak{R}_{L/K}(B_L) \in \mathcal{B}$, and clearly $\mathfrak{R}_{L/K}(A_L) \in \mathcal{F}_K$. Moreover, $\iota : \tilde{G} \to \mathfrak{R}_{L/K}(\tilde{G}_L)$ is an embedding. Hence, we can consider the short exact sequence

$$1 \to \mathfrak{R}_{L/K}(B_L) \cap \iota(\tilde{G}) \xrightarrow{\beta'} \iota(\tilde{G}) \cong \tilde{G} \xrightarrow{\alpha'} \alpha(\iota(\tilde{G})) \to 1,$$

where α' and β' are the obvious maps. Since $\mathcal{B}$ is closed under taking (closed)

K-subgroups, and since $\alpha(\iota(\tilde{G})) \leq_K \mathfrak{R}_{L/K}(A_L)$, meaning that $\alpha(\iota(\tilde{G})) \in \mathcal{F}_K$, we find that $\tilde{G} \in \operatorname{Ext}(\mathcal{F}_K, \mathcal{B})$, as required. The same proof works also if we replace "$\mathcal{F}_K$" by "$\mathcal{F}_K^{\mathrm{Sol}}$" or "$\mathcal{F}_K^{\mathrm{Ab}}$". ∎

Let G be an abstract group. By the *constant K-group $\tilde{G}$ induced by G* we mean the disjoint union of copies of $\operatorname{Spec}(K)$, indexed by G, together with the obvious group structure induced by G. If W is a K-scheme, by a (*left*) *action of the abstract group G on W* we mean a group homomorphism $\phi : G \to \operatorname{Aut}_K(W)$. One can show that the group homomorphisms $\phi : G \to \operatorname{Aut}_K(W)$ defining an action of G on W correspond precisely to the K-morphisms $\tilde{\phi} : \tilde{G} \times W \to W$ defining an action of $\tilde{G}$ on W; we will use these two notions of "action" interchangeably. For this reason, with abuse of notation, we will denote by "G" both the abstract group G and the constant K-group induced by G.

LEMMA 2.12. *Suppose that $\mathcal{B} \subset \mathcal{L}_K$ is closed under direct products and $1 \in \mathcal{B}$. Let $G \in \operatorname{Ext}(\mathcal{F}_K, \mathcal{B})$. Let F be a finite abstract group and, by abusing notation, let $F \in \mathcal{F}_K$ be the induced constant finite K-group. Let F act on $G^{|F|}$ (the direct product of G with itself $|F|$ times) by permuting the coordinates, and let $\phi : F \to \operatorname{Aut}_K(G^{|F|})$ be the group homomorphism coming from this action. Then $G^{|F|} \rtimes_\phi F \in \operatorname{Ext}(\mathcal{F}_K, \mathcal{B})$. The same is true with "$\mathcal{F}_K$" replaced by "$\mathcal{F}_K^{\mathrm{Sol}}$".*

Proof. Let $\phi : F \to \operatorname{Aut}_K(G^{|F|})$ denote the morphism induced by the action of F (with F seen as a subgroup of the symmetric group $S_{|F|}$, by Cayley's theorem). Since $G \in \operatorname{Ext}(\mathcal{F}_K, \mathcal{B})$, we see that G fits, say, into the short exact sequence

$$1 \to B \xrightarrow{b} G \xrightarrow{a} A \to 1,$$

where $B \in \mathcal{B}$ and $A \in \mathcal{F}_K$. By Lemma 2.9, we know that $G^{|F|}$ is in $\operatorname{Ext}(\mathcal{F}_K, \mathcal{B})$ and fits into the exact sequence

$$1 \to B^{|F|} \xrightarrow{b^{|F|}:=(b,\ldots,b)} G^{|F|} \xrightarrow{a^{|F|}:=(a,\ldots,a)} A^{|F|} \to 1$$

with $A^{|F|} \in \mathcal{F}_K$ and $B^{|F|} \in \mathcal{B}$. Since $a^{|F|} : G^{|F|} \to A^{|F|}$ is surjective and its kernel $\ker(a^{|F|}) = (\ker(a), \ldots, \ker(a)) \cong B^{|F|}$ is ϕ-characteristic (i.e. $\phi(\ker(a^{|F|})) = \ker(a^{|F|})$) as ϕ just acts by permuting the coordinates, there is a $\varphi : F \to \operatorname{Aut}_K(A)$ making the following diagram commute:

$$\begin{array}{ccc} G^{|F|} & \xrightarrow{\phi} & G^{|F|} \\ {\scriptstyle a^{|F|}}\downarrow & & \downarrow{\scriptstyle a^{|F|}} \\ A^{|F|} & \xrightarrow{\varphi} & A^{|F|} \end{array}$$

In particular, for any $f \in F$ and $g \in G^{|F|}$, we have

$$a^{|F|}(\phi_f(g)) = \varphi_f(a^{|F|}(g)), \tag{2.2}$$

where $\phi_f := \phi(f)$ and $\varphi_f := \varphi(f)$.

Since, by assumption, $1 \in \mathcal{B}$, we find that F is in $\mathrm{Ext}(\mathcal{F}_K, \mathcal{B})$, as it fits into the short exact sequence $1 \to 1 \to F \xrightarrow{\mathrm{id}} F \to 1$. Now consider the sequence

$$1 \to \underbrace{B^{|F|} \rtimes_\phi 1}_{\cong B^{|F|}} \xrightarrow{(b^{|F|},1)} G^{|F|} \rtimes_\phi F \xrightarrow{(a^{|F|},\mathrm{id})} A^{|F|} \rtimes_\varphi F \to 1,$$

where $B^{|F|} \rtimes_\phi 1 \cong B^{|F|} \in \mathcal{B}$ and $A^{|F|} \rtimes_\varphi F \in \mathcal{F}_K$, since $\mathcal{F}_K$ is closed under extensions. Clearly, $(b^{|F|}, 1)$ is a K-group homomorphism. One can check that $(a^{|F|}, \mathrm{id})$ is also a K-group homomorphism, given our choice of φ and (2.2). Finally, it is clear that the sequence is exact. The same proof works if we replace "$\mathcal{F}_K$" by "$\mathcal{F}_K^{\mathrm{Sol}}$". ■

3. Statement of the first result. Let k be a number field and let X be a nice variety over k. We define

$$X(\mathbb{A}_k)^{\mathcal{F}_k,\mathcal{O}} := \bigcap_{F\in\mathcal{F}_k} \bigcap_{[Z]\in H^1(X,F)} \bigcup_{[\tau]\in H^1(k,F)} f^\tau(Z^\tau(\mathbb{A}_k)^{\mathcal{O}}), \tag{3.1}$$

where $\mathcal{O} : \mathbf{Sch}_k^{\mathrm{op}} \to \mathbf{Sets}$ is any functor such that $X(\mathbb{A}_k)^{\mathcal{O}} \subset X(\mathbb{A}_k)^{\mathrm{Br}_1}$; the "solvable" and "abelian" versions of $X(\mathbb{A}_k)^{\mathcal{F}_k,\mathcal{O}}$ are defined by replacing "$\mathcal{F}_k$" by "$\mathcal{F}_k^{\mathrm{Sol}}$" and "$\mathcal{F}_k^{\mathrm{Ab}}$", respectively. We suppose further that there exists a set $\mathcal{S}_k \subset \mathcal{L}_k$ such that the following three conditions hold:

Condition 1. $\mathcal{S}_k$ is contained in $\mathcal{C}_k$, closed under k-twists, and such that, for any nice k-variety W, we have $W(\mathbb{A}_k)^{\mathcal{O}} \subset W(\mathbb{A}_k)^{\mathcal{S}_k}$.

Condition 2. For any nice variety W' over k, we have $W'(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k,\mathcal{S}_k)} \subset W'(\mathbb{A}_k)^{\mathcal{O}}$. The "solvable" and "abelian" versions of this condition are obtained by replacing "$\mathcal{F}_k$" by "$\mathcal{F}_k^{\mathrm{Sol}}$" and "$\mathcal{F}_k^{\mathrm{Ab}}$", respectively.

Condition 3. $\mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)$ is closed under taking closed k-subgroups, under taking k-twists, under taking direct products, and such that the conclusion of Lemma 2.12 (with $\mathcal{B} = \mathcal{S}_k$) holds. The "solvable" and "abelian" versions of this condition are obtained by replacing "$\mathcal{F}_k$" by "$\mathcal{F}_k^{\mathrm{Sol}}$" and "$\mathcal{F}_k^{\mathrm{Ab}}$", respectively.

The following theorem is then a generalisation of the results in [Dem09] and [Sko09].

THEOREM 3.1. *Let X be a nice variety over k. Let $X(\mathbb{A}_k)^{\mathcal{F}_k,\mathcal{O}}$ be defined as above, and assume Conditions* 1–3 *hold. Then*

$$X(\mathbb{A}_k)^{\mathcal{F}_k,\mathcal{O}} = X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k,\,\mathcal{S}_k)}.$$

If we consider the "solvable version" of $X(\mathbb{A}_k)^{\mathcal{F}_k,\mathcal{O}}$, and the "solvable versions" of Conditions 1–3 *hold, then the result is also true if we replace "$\mathcal{F}_k$" by "$\mathcal{F}_k^{\mathrm{Sol}}$".*

4. Proof of Theorem 3.1. We will prove the "standard version" of the theorem; for the "solvable version", the proof is identical, modulo replacing "$\mathcal{F}_k$" by "$\mathcal{F}_k^{\mathrm{Sol}}$" when necessary. Most of the results in this section are just restatements of results in [Dem09] and [Sko09]; we often sketch the proofs of these results, for the reader's convenience. We remark, however, that we use a *finesse*—Proposition 4.14—not present (and not needed) in [Dem09] and [Sko09], which gives us flexibility in the applications of Theorem 3.1 (see §5).

4.1. Proof that $X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k,\,\mathcal{S}_k)} \supset X(\mathbb{A}_k)^{\mathcal{F}_k,\mathcal{O}}$. In this subsection, it suffices to assume that $\mathcal{S}_k$ satisfies Conditions 1 and 3 (without Lemma 2.12).

LEMMA 4.1. *Let X be a smooth, projective variety over a number field k. Let $(f,Z,F) \in \mathcal{F}_k(X)$. Then Z is smooth and projective.*

Proof. This follows as f is finite and étale. ■

DEFINITION 4.2. Let $(Y,G),(Z,H) \in \mathcal{L}_k(X)$. An *$X$-torsor morphism* $f = (\phi,\psi) : (Y,G) \to (Z,H)$ is a pair (ϕ,ψ) where $\phi : Y \to Z$ is an X-morphism and $\psi : G \to H$ is a homomorphism, compatible in the sense that the diagram

$$\begin{array}{ccc} Y \times_X G_X & \xrightarrow{\phi\times\psi} & Z \times_X H_X \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\phi} & Z \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

commutes. We say that the X-torsor morphism $f = (\phi,\psi)$ is *surjective* if ϕ (equivalently, ψ) is surjective.

REMARK 4.3. If $f = (\phi,\psi) : (Y,G) \to (Z,H)$ is surjective, then $Y \to Z$ is a $(\ker\psi)$-torsor over Z.

PROPOSITION 4.4 ([Dem09, Lemme 3]). *Let X be a nice variety over a number field k. Let $(P_v) \in X(\mathbb{A}_k)^{\mathcal{F}_k,\mathcal{O}}$ and $(f,Z,F) \in \mathcal{F}_k(X)$. Then there exist:*

(i) *an $F' \in \mathcal{F}_k$,*
(ii) *an $(X',F') \in \mathcal{F}_k(X)$ with X' geometrically integral,*

(iii) *a* 1*-cocycle* $\sigma \in Z^1(k, F)$,

(iv) *a morphism of* X*-torsors* $(\phi, \psi) : (X', F') \to (Z^\sigma, F^\sigma)$,

such that (P_v) *lifts to some* $(Q_v) \in X'(\mathbb{A}_k)^{\mathcal{O}}$.

Proof. The proof is similar to the one in [Dem09] with "Br" and "ét, Br" replaced by "$\mathcal{O}$" and "$\mathcal{F}_k, \mathcal{O}$", respectively. ∎

The next proposition, which allows us to lift cocycles, is essentially due to Demarche [Dem09, Proposition 4], with the caveat that, while he requires $(P_v) \in X(\mathbb{A}_k)^{\text{ét,Br}}$, we take $(P_v) \in X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}}$.

Proposition 4.5 (Demarche, [Dem09, Proposition 4]). *Let* X *be a nice variety over a number field* k. *Let* $(P_v) \in X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}}$ *and* $(f, Y, G) \in \mathcal{L}_k(X)$. *Let*

$$1 \to H \to G \to F \to 1$$

be a short exact sequence in $\mathcal{L}_k$ *with* $H \in \mathcal{C}_k$ *and* $F \in \mathcal{F}_k$. *Denote the pushforward of* $f : Y \to X$ *under the morphism* $G \to F$ *by* $(Z, F) \in \mathcal{F}_k(X)$. *Let* $\sigma \in Z^1(k, F)$ *be the cocycle coming from Proposition* 4.4 *applied to the torsor* $Z \to X$ *and the point* (P_v). *Then the class* $[\sigma] \in H^1(k, F)$ *lifts to a class* $[\tau] \in H^1(k, G)$.

Remark 4.6. Let $H \in \mathcal{C}_k$. Since H contains a k-point (the identity), it is also geometrically connected (see [Gro65, 4.5.14]).

Proof. roof of Proposition 4]demarche2009, the only place where Demarche uses the fact that $(P_v) \in X(\mathbb{A}_k)^{\text{ét,Br}}$ is in a passage of the proof of "Le lemme 7 implique la proposition 4". In this passage, Demarche wants to prove the existence of some torsor of type λ' (cf. Remark 5.3 for more on the type of a torsor). To do so, he appeals to [Sko01, Corollary 6.1.3(1)]. Indeed, he knows that (P_v) lifts to some $(Q_v) \in X'(\mathbb{A}_k)^{\text{Br}}$ for some nice X'; this means in particular that $X'(\mathbb{A}_k)^{\text{Br}_1} \neq \emptyset$. But X' is proper and geometrically integral (and, in particular, $\overline{k}[X']^* = \overline{k}^*$), and $\emptyset \neq X'(\mathbb{A}_k)^{\text{Br}_1} = \bigcap_\lambda X'(\mathbb{A}_k)^{\text{Br}_\lambda}$ (see [Sko01] for the definitions). Hence, since $X'(\mathbb{A}_k)^{\text{Br}_{\lambda'}} \neq \emptyset$, the required torsor of type λ' exists by [Sko01, Corollary 6.1.3(1)].

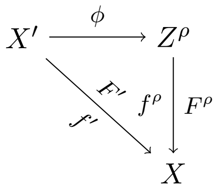
In our case, $(P_v) \in X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}}$. By Proposition 4.4, (P_v) lifts to some $(Q_v) \in X'(\mathbb{A}_k)^{\mathcal{O}}$ for some nice X'. Hence, $\emptyset \neq X'(\mathbb{A}_k)^{\mathcal{O}} \subset X'(\mathbb{A}_k)^{\text{Br}_1}$ (where the inclusion holds by assumption on $\mathcal{O}$), and since X' is proper and geometrically integral, we can appeal to [Sko01, Corollary 6.1.3(1)] as well to conclude that the relevant torsor of type λ' exists. All the rest of the proof of [Dem09, Proposition 4] remains unchanged. ∎

We have an analogue of [Dem09, Théorème 1].

Theorem 4.7. *Let* X *be a nice variety over* k. *Then*

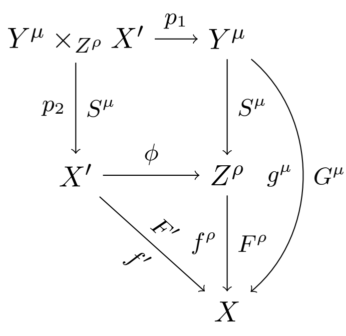
$$X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}} \subset X(\mathbb{A}_k)^{\text{Ext}(\mathcal{F}_k, \mathcal{S}_k)}.$$

Proof. Let $(P_v) \in X(\mathbb{A}_k)^{\mathcal{F}_k,\mathcal{O}}$. We fix $G \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)$ and $(g, Y, G) \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)(X)$. Then G fits, say, into the short exact sequence $1 \to S \to G \to F \to 1$ for some $F \in \mathcal{F}_k$ and $S \in \mathcal{S}_k$. By pushing forward g along $G \to F$, we can decompose $Y \to X$ into $(Y, S) \in \mathcal{S}_k(Z)$ and $(Z, F) \in \mathcal{F}_k(X)$. By applying Proposition 4.4 to (Z, F) and (P_v), we get an $F' \in \mathcal{F}_k$, an $(X', F') \in \mathcal{F}_k(X)$ with X' geometrically integral, a 1-cocycle $\rho \in Z^1(k, F)$, and an X-torsor morphism $\phi : X' \to Z'$ such that (P_v) lifts to a point $(Q_v) \in X'(\mathbb{A}_k)^{\mathcal{O}}$. Hence, we have the commutative triangle



We now apply Proposition 4.5 to the X-torsor (g, Y, G), the short exact sequence $1 \to S \to G \to F \to 1$ (note here that $S \in \mathcal{C}_k$, as $\mathcal{S}_k \subset \mathcal{C}_k$ by Condition 1), and $(P_v) \in X(\mathbb{A}_k)^{\mathcal{F}_k,\mathcal{O}}$ to conclude that $[\rho] \in H^1(k, F)$ lifts to some $[\mu] \in H^1(k, G)$. It follows that the G^μ-torsor $g^\mu : Y^\mu \to X$ can be decomposed naturally into the S^μ-torsor $Y^\mu \to Z^\rho$ and the F^ρ-torsor $Z^\rho \to X$.

Now consider the fibred product $Y^\mu \times_{Z^\rho} X'$. This is naturally an S^μ-torsor over X', and we remark that $(p_2, Y^\mu \times_{Z^\rho} X', S^\mu) \in \mathcal{S}_k(X')$ as $\mathcal{S}_k$ is closed under k-twists. The following diagram summarises the constructions so far:



Note that, since X is smooth and projective and $X' \to X$ is finite and étale, X' is also smooth and projective; moreover, since X' is geometrically connected, it follows that X' is nice. Hence, we can use Condition 1 to deduce that

$$X'(\mathbb{A}_k)^{\mathcal{O}} \subset \bigcup_{[\nu] \in H^1(k, S^\mu)} p_2^\nu\big((Y^\mu \times_{Z^\rho} X')^\nu(\mathbb{A}_k)\big).$$

In particular, there is some $\nu \in Z^1(k, S^\mu)$ such that (Q_v) lifts to $(R_v) \in$

$(Y^\mu \times_{Z^\rho} X')^\nu(\mathbb{A}_k)$. Then, arguing as in [Dem09, Théorème 1], we conclude that

$$(P_v) \in \bigcup_{[\tau]\in H^1(k,G)} g^\tau(Y^\tau(\mathbb{A}_k)).$$

Since $(P_v) \in X(\mathbb{A}_k)^{\mathcal{F}_k,\mathcal{O}}$, $G \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)$, and (g, Y, G) were arbitrary, we get

$$X(\mathbb{A}_k)^{\mathcal{F}_k,\mathcal{O}} \subset X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k,\mathcal{S}_k)},$$

as required. ∎

REMARK 4.8. In this section, we only use Conditions 1 and 3 without Lemma 2.12, and these hold if we replace "$\mathcal{F}_k$" with "$\mathcal{F}_k^{\mathrm{Ab}}$". Hence, we can also conclude that $X(\mathbb{A}_k)^{\mathcal{F}_k^{\mathrm{Ab}},\mathcal{O}} \subset X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Ab}},\mathcal{S}_k)}$.

4.2. Proof that $X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k,\mathcal{S}_k)} \subset X(\mathbb{A}_k)^{\mathcal{F}_k,\mathcal{O}}$. In this subsection, we assume that $\mathcal{S}_k$ satisfies Conditions 1–3.

PROPOSITION 4.9 (based on [Sto07, Proposition 5.17]). *Let X be a smooth, proper variety over a number field k. Let $(Y, F) \in \mathcal{F}_k(X)$. For any $(P_v) \in X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k,\mathcal{S}_k)}$, there exists a twist $(Y', F') \in \mathcal{F}_k(X)$ of (Y, F) with the following property. For any surjective X-torsor morphism (ϕ, ψ) : $(Z, G) \to (Y', F')$, where $(Z, G) \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)(X)$, there exists a twist $Z' \to Y'$ of $(Z, \ker\psi) \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)(Y')$ such that (P_v) lifts to a point in $Z'(\mathbb{A}_k)$.*

Proof. We follow closely the first part of the proof of [Sto07, Proposition 5.17], modifying it when needed.

Let $P \in X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k,\mathcal{S}_k)}$. Let $(Y, F) \in \mathcal{F}_k(X)$. Since X is proper, by [Sko01, Proposition 5.3.2] there are only finitely many twists (Y^σ, F^σ) of (Y, F) such that $Y^\sigma(\mathbb{A}_k) \neq \emptyset$. Moreover, since $P \in X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k,\mathcal{S}_k)}$ and $\mathcal{F}_k \subset \mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)$, we know that P lifts to some point in some $Y^\sigma(\mathbb{A})$. Let $Y_1, \ldots, Y_s$ be the finitely many twists of (Y, F) such that P lifts to some adelic point in them.

Let $\tau(j) \subset \{1, \ldots, s\}$ be the set of indices i such that, for every $(Z, G) \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)(X)$ mapping to $Y_j \to X$, there is a twist Z^ξ that lifts P and induces a twist of Y_j that is isomorphic to Y_i. Following Stoll, one can show that

- (i) $\tau(j) \neq \emptyset$ (using Condition 3);
- (ii) if $i \in \tau(j)$, then $\tau(i) \subset \tau(j)$;
- (iii) for some j, we have $j \in \tau(j)$.

We can take Y' to be Y_j, where $j \in \tau(j)$. ∎

REMARK 4.10. For the above proof to make sense, we need $\mathcal{F}_k \subset \mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)$.

Lemma 4.11 (Skorobogatov, [Sko09, Corollary 2.7]). *Let Y be a proper variety over k, and let $(g, Z, G) \in \mathcal{L}_k(Y)$. Then the set $Y(\mathbb{A}_k)^g$ is closed in $Y(\mathbb{A}_k)$.* ∎

The following proposition is essentially due to Skorobogatov.

Proposition 4.12 (Skorobogatov, [Sko09, Proposition 2.3]). *Let X be a variety over k. Let $(Y, F) \in \mathcal{F}_k(X)$ and $(Z, G) \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)(Y)$. Then there exist a $(V, G') \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)(X)$ and a surjective X-torsor morphism $(\theta, \psi) : (V, G') \to (Y, F)$ such that there is a surjective Y-torsor morphism $(\Theta, \Psi) : (V, \ker\psi) \to (Z, G)$ with $\ker\Psi \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)$.*

Proof. As in the proof of [Sko09, Proposition 2.3], let

$$V := \mathfrak{R}_{Y/X}(Z) \times_X Y,$$

where $\mathfrak{R}_{Y/X}(Z)$ is the Weil restriction (which exists since $Y \to X$ is finite étale, cf. [BLR90, §7.6]), and let $\theta : V \to Y$ be the second projection. Consider the group $G^m \rtimes_\rho F(\overline{k})$, where $m = |F(\overline{k})|$ and ρ is the action of the constant group $F(\overline{k})$ on G^m by permutation of coordinates. Since $G \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)$, by Condition 3 we also have $G^m \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)$. Moreover, since $F(\overline{k}) \in \mathcal{F}_k$ is a constant group acting on G^m by permuting the coordinates, and since we are assuming Condition 3, by Lemma 2.12 we know that $G^m \rtimes_\rho F(\overline{k}) \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)$.

Now consider the k-twist of $G^m \rtimes_\rho F(\overline{k})$ given by

$$G' := \mathfrak{R}_{F/\mathrm{Spec}\, k}(G_F) \rtimes_\phi F,$$

where ϕ is the action induced by ρ (and where $\mathfrak{R}_{F/\mathrm{Spec}\, k}(G_F)$ exists: again cf. [BLR90, §7.6]). Since, by Condition 3, $\mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)$ is closed under taking k-twists, it follows that $G' \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)$. Following the proof of [Sko09, Proposition 2.3], one can show that $V \to X$ is a G'-torsor, that θ and the natural projection $\psi : G' \to F$ together give a surjective X-torsor morphism $(\theta, \psi) : (V, G') \to (Y, F)$, and that there exists a surjective Y-torsor morphism $(\Theta, \Psi) : (V, \ker\psi) \to (Z, G)$. Moreover, since $\mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)$ is closed under taking (closed) k-subgroups (by Condition 3), we also deduce that $\ker\psi \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)$. ∎

We can now prove that $X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)}$ is "well-behaved" with respect to torsors under groups in $\mathcal{F}_k$. Our proof is almost verbatim that of [Sko09, Theorem 1.1].

Theorem 4.13. *For any $F \in \mathcal{F}_k$ and any $[Z] \in H^1(X, F)$, we have*

$$X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)} = \bigcup_{[\tau] \in H^1(k, F)} f^\tau(Z^\tau(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)}).$$

Proof. The inclusion "$\supset$" follows by pulling back torsors. Indeed, suppose that

$$(P_v) \in \bigcup_{[\tau]\in H^1(k,F)} f^\tau(Z^\tau(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k,\mathcal{S}_k)}),$$

say $(P_v) = f^\tau((Q_v))$ for some $(Q_v) \in Z^\tau(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k,\mathcal{S}_k)}$ and $[\tau] \in H^1(k,F)$. If $(g,Y,G) \in \mathrm{Ext}(\mathcal{F}_k,\mathcal{S}_k)(X)$, by pulling back $(g,Y,G) \in \mathrm{Ext}(\mathcal{F}_k,\mathcal{S}_k)(X)$ along $f^\tau : Z^\tau \to X$ we obtain the torsor $(p_1, Z^\tau \times_X Y, G) \in \mathrm{Ext}(\mathcal{F}_k,\mathcal{S}_k)(Z^\tau)$. Since $(Q_v) \in Z^\tau(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k,\mathcal{S}_k)}$ and $G \in \mathrm{Ext}(\mathcal{F}_k,\mathcal{S}_k)$, there is a $[\mu] \in H^1(k,G)$ such that (Q_v) lifts to some $(R_v) \in (Z^\tau \times_X Y)^\mu(\mathbb{A}_k) = (Z^\tau \times_X Y^\mu)(\mathbb{A}_k)$. By the commutativity of the obvious pullback diagram, we conclude that (P_v) is in $g^\mu(Y^\mu(\mathbb{A}_k))$. Since $(g,Y,G) \in \mathrm{Ext}(\mathcal{F}_k,\mathcal{S}_k)(X)$ was arbitrary, it follows that $(P_v) \in X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k,\mathcal{S}_k)}$.

For the other inclusion, let $(P_v) \in X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k,\mathcal{S}_k)}$. Let $(\tilde{f},\tilde{Z},\tilde{F}) \in \mathcal{F}_k(X)$ be the twist of $(f,Z,F) \in \mathcal{F}_k(X)$ coming from Proposition 4.9. It is clear that we only need to show that (P_v) lifts to a point in $\tilde{Z}(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k,\mathcal{S}_k)}$. Suppose not. Then

$$\tilde{f}^{-1}((P_v)) \subset \bigcup_{(g',W',G')\in\mathrm{Ext}(\mathcal{F}_k,\mathcal{S}_k)(\tilde{Z})} \tilde{Z}(\mathbb{A}_k) \setminus \tilde{Z}(\mathbb{A}_k)^{g'},$$

where the cover on the right-hand side is an open cover since each $\tilde{Z}(\mathbb{A}_k)^{g'}$ is closed in $\tilde{Z}(\mathbb{A}_k)$, by Lemma 4.11. Using compactness, we have

$$\tilde{f}^{-1}((P_v)) \subset \bigcup_{i=1}^{n} \tilde{Z}(\mathbb{A}_k) \setminus \tilde{Z}(\mathbb{A}_k)^{g_i}$$

for some $(g_i,W_i,G_i) \in \mathrm{Ext}(\mathcal{F}_k,\mathcal{S}_k)(\tilde{Z})$. Let $(g,W,G) \in \mathrm{Ext}(\mathcal{F}_k,\mathcal{S}_k)(\tilde{Z})$ denote the fibred product of the g_i's over $\tilde{Z}$; here $G \in \mathrm{Ext}(\mathcal{F}_k,\mathcal{S}_k)$, since $\mathrm{Ext}(\mathcal{F}_k,\mathcal{S}_k)$ is closed under finite fibred products. It is clear from the construction that $\tilde{f}^{-1}((P_v)) \cap \tilde{Z}(\mathbb{A}_k)^g = \emptyset$. By Proposition 4.12, there is a torsor $(V,L) \in \mathrm{Ext}(\mathcal{F}_k,\mathcal{S}_k)(X)$ and a surjective X-torsor morphism $(\theta,\psi) : (V,L) \to (\tilde{Z},\tilde{F})$ such that there exists a surjective $\tilde{Z}$-torsor morphism $(V,\ker\psi) \to (W,G)$. To get the required contradiction, one can then argue just as in [Sko09, proof of Theorem 1.1]. ∎

PROPOSITION 4.14. *Let X be a nice variety over k. If $X(\mathbb{A}_k)^{\mathcal{F}_k} \neq \emptyset$, then*

$$X(\mathbb{A}_k)^{\mathcal{F}_k,\mathcal{O}} = \bigcap_{F\in\mathcal{F}_k} \bigcap_{\substack{[Z]\in H^1(X,F)\\ Z\ \text{nice}}} \bigcup_{[\tau]\in H^1(k,F)} f^\tau(Z^\tau(\mathbb{A}_k)^{\mathcal{O}}),$$

that is, we can restrict our attention to nice *X-torsors $[Z] \in H^1(X,F)$ only.*

Proof. The inclusion "$\subset$" is clear, so we just prove the opposite inclusion.

Let $(P_v) \in \bigcap_F \bigcap_{[Z]\text{nice}} \bigcup_{[\tau]} f^\tau(Z^\tau(\mathbb{A}_k)^{\mathcal{O}})$. Fix $F \in \mathcal{F}_k$ and $[f : Z \to X] \in H^1(X,F)$. We need to show that $(P_v) \in \bigcup_{[\tau]\in H^1(k,F)} f^\tau(Z^\tau(\mathbb{A}_k)^{\mathcal{O}})$. Consider

$(\bar{Z}, \overline{F}) \in \mathcal{F}_{\overline{k}}(\overline{X})$. Let $\bar{Z}_0$ be a connected component of $\bar{Z}$, and let $\overline{F}_0 \leq_{\overline{k}} \overline{F}$ be its stabiliser. Then we have an $\overline{X}$-torsor morphism $(\bar{Z}_0, \overline{F}_0) \to (\bar{Z}, \overline{F})$, given by inclusion.

By the notion of *cofinal coverings* in [Sto07] and by Lemma 5.7 in [Sto07], using our assumption that $X(\mathbb{A}_k)^{\mathcal{F}_k} \neq \emptyset$, we know there exist an $F' \in \mathcal{F}_k$ and an $(f', W, F') \in \mathcal{F}_k(X)$ with W geometrically connected such that there is an $\overline{X}$-torsor morphism $(\overline{W}, \overline{F'}) \to (\bar{Z}_0, \overline{F}_0)$. Hence, by composition, we get an $\overline{X}$-torsor morphism

$$(\overline{W}, \overline{F'}) \to (\bar{Z}_0, \overline{F}_0) \to (\bar{Z}, \overline{F})$$

with W geometrically connected. We can then apply [Sto07, Lemma 5.6] to conclude that there exists a twist $(f^\tau, Z^\tau, F^\tau) \in \mathcal{F}_k(X)$ of (f, Z, F) such that there is an X-torsor morphism

$$(W, F') \to (Z^\tau, F^\tau)$$

with W geometrically connected. Since W is also smooth and projective (cf. Lemma 4.1), W is nice, and the same is true for all its twists.

Without loss of generality (twisting if necessary), we can assume that (P_v) lifts to some point in $W(\mathbb{A}_k)^{\mathcal{O}}$. We can then use [Sto07, Lemma 5.6] and the functoriality of $\mathcal{O}$ to deduce that $(P_v) \in \bigcup_{\tau \in H^1(k,F)} f^\tau(Z^\tau(\mathbb{A}_k)^{\mathcal{O}})$. Since F and f were arbitrary, we conclude that

$$\bigcap_F \bigcap_{[Z]\,\text{nice}} \bigcup_{[\tau]} f^\tau(Z^\tau(\mathbb{A}_k)^{\mathcal{O}}) \subset X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}}.$$

Hence, $\bigcap_F \bigcap_{[Z]\,\text{nice}} \bigcup_{[\tau]} f^\tau(Z^\tau(\mathbb{A}_k)^{\mathcal{O}}) = X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}}$, meaning that we can restrict our attention to nice torsors only. ∎

Proof of Theorem 3.1. By Theorem 4.7, $X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}} \subset X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)}$.

If $X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)} = \emptyset$, then trivially $X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)} \subset X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}}$, and we are done. So suppose $X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)} \neq \emptyset$. In particular, this implies that $X(\mathbb{A}_k)^{\mathcal{F}_k} \neq \emptyset$. By Theorem 4.13, for any $[Z] \in H^1(X, F)$ and $F \in \mathcal{F}_k$,

$$X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)} = \bigcup_{[\tau] \in H^1(k,F)} f^\tau(Z^\tau(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)}).$$

Hence,

$$\begin{aligned} X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)} &= \bigcap_{F \in \mathcal{F}_k} \bigcap_{[Z]\,\text{nice}} \bigcup_{[\tau]} f^\tau(Z^\tau(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k, \mathcal{S}_k)}) \\ &\subset \bigcap_{F \in \mathcal{F}_k} \bigcap_{[Z]\,\text{nice}} \bigcup_{[\tau]} f^\tau(Z^\tau(\mathbb{A}_k)^{\mathcal{O}}) = X(\mathbb{A}_k)^{\mathcal{F}_k, \mathcal{O}}, \end{aligned}$$

where the inclusion in the second line comes from Condition 2, and the final equality comes from Proposition 4.14 (and the fact that $X(\mathbb{A}_k)^{\mathcal{F}_k} \neq \emptyset$).

A similar proof holds if we replace "$\mathcal{F}_k$" by "$\mathcal{F}_k^{\mathrm{Sol}}$". ∎

Remark 4.15. Proposition 4.14 makes Theorem 3.1 robust enough for applications. Indeed, we usually only have Condition 2 as we have stated it, and not a more general version as, for example, is the case of the proof of $X(\mathbb{A}_k)^{\mathrm{Desc}} \subset X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}}$ in [Sko09]: there, as a corollary to a result by Gabber, one has $W(\mathbb{A}_k)^{\mathcal{L}_k} \subset W(\mathbb{A}_k)^{\text{ét},\mathrm{Br}}$ for *all* smooth, projective varieties over k (cf. [Sko09, Lemma 2.8]). Proposition 4.14 tells us that, in fact, our version of Condition 2 is sufficient.

5. Some applications of Theorem 3.1

5.1. The étale-Brauer set and its variations. Let X be a nice variety over k. Consider the "solvable" étale-Brauer set

$$X(\mathbb{A}_k)^{\mathrm{Sol},\mathrm{Br}} := \bigcap_{F\in\mathcal{F}_k^{\mathrm{Sol}}} \bigcap_{[Z]\in H^1(X,F)} \bigcup_{[\tau]\in H^1(k,F)} f^\tau(Z^\tau(\mathbb{A}_k)^{\mathrm{Br}}).$$

In our terminology, $\mathcal{O} = \mathrm{Br}$. As we have seen in the introduction, for any nice variety W over k we have $W(\mathbb{A}_k)^{\mathrm{Br}} = W(\mathbb{A}_k)^{\mathcal{C}_k}$. Hence, in our setting, we can take $\mathcal{S}_k = \mathcal{C}_k$. As Condition 1 is clear, we only need to check that the "solvable versions" of Conditions 2 and 3 hold. The latter condition is easy to see by the closure properties of $\mathcal{F}_k^{\mathrm{Sol}}$, and the former holds since $\mathcal{C}_k \subset \mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{C}_k)$. Hence, by applying the "solvable version" of Theorem 3.1, we can deduce the following.

Theorem 5.1. *Let X be a nice variety over k. Then*

$$X(\mathbb{A}_k)^{\mathrm{Sol},\mathrm{Br}} = X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}},\mathcal{C}_k)}. \blacksquare$$

5.2. The algebraic étale-Brauer set and its variations. Let X be a nice variety over k. Consider the algebraic étale-Brauer set

$$X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}_1} := \bigcap_{F\in\mathcal{F}_k} \bigcap_{[Z]\in H^1(X,F)} \bigcup_{[\tau]\in H^1(k,F)} f^\tau(Z^\tau(\mathbb{A}_k)^{\mathrm{Br}_1}).$$

In our terminology, $\mathcal{O} = \mathrm{Br}_1$.

Theorem 5.2 ([Sko01, Theorem 6.1.1]). *Let Z be a variety over a number field k such that $\overline{k}[Z]^* = \overline{k}^*$. Then*

$$Z(\mathbb{A}_k)^{\mathrm{Br}_1} = \bigcap_{\lambda:\hat{S}\hookrightarrow \mathrm{Pic}(\bar{Z})} \bigcup_{\mathrm{type}(Y,f)=\lambda} f(Y(\mathbb{A}_k)), \tag{5.1}$$

where $\hat{S}$ ranges over the finitely generated $\mathrm{Gal}(\overline{k}/k)$*-submodules of* $\mathrm{Pic}(\overline{X})$.

Remark 5.3. We recall that the *type* λ of a torsor $(f, Y, S) \in \mathcal{M}_k(Z)$ is defined by $\chi([Y]) =: \lambda$, where χ is the map in the fundamental exact sequence due to Colliot-Thélène and Sansuc (see [Sko01])

$$0 \to H^1(k,S) \to H^1(Z,S) \xrightarrow{\chi} \mathrm{Hom}_{\mathrm{Gal}_k}(\hat{S}, \mathrm{Pic}\,\bar{Z}) \xrightarrow{\partial} H^2(k,S) \to H^2(Z,S),$$

where $\hat{S} := \mathrm{Hom}_{\overline{k}\text{-groups}}(\overline{S}, \mathbb{G}_{\mathrm{m},\overline{k}})$ is the *module of characters* of S. We remark that there is an anti-equivalence of categories

$$\left\{\begin{array}{c}\text{algebraic linear } k\text{-groups}\\ \text{of multiplicative type}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{finitely generated } \mathbb{Z}\text{-modules with}\\ \text{a continuous action of } \mathrm{Gal}(\overline{k}/k)\end{array}\right\}$$

given by the functor $S \mapsto \hat{S}$, with inverse $M \mapsto \mathrm{Spec}(\overline{k}[M]^{\mathrm{Gal}(\overline{k}/k)})$.

In particular, Theorem 5.2 tells us that, for any nice variety W over k, we have $W(\mathbb{A}_k)^{\mathcal{M}_k} = W(\mathbb{A}_k)^{\mathrm{Br}_1}$. A sensible idea would be to take $\mathcal{S}_k = \mathcal{M}_k$, but unfortunately $\mathcal{M}_k \not\subset \mathcal{C}_k$, so Condition 1 fails. We take instead $\mathcal{S}_k = \mathcal{T}_k \subset \mathcal{C}_k$, and consider $\mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k)$. Since our aim is to apply Theorem 3.1, we check that Conditions 1–3 hold for our choice of $\mathcal{S}_k$.

Condition 1. This holds since $\mathcal{T}_k \subset \mathcal{C}_k$, the k-twist of an algebraic torus is again an algebraic torus ("being an algebraic torus" is a geometric condition), and since Theorem 5.2 (implying that $W(\mathbb{A}_k)^{\mathrm{Br}_1} \subset W(\mathbb{A}_k)^{\mathcal{T}_k}$) holds.

Condition 2. As we have seen, $W(\mathbb{A}_k)^{\mathcal{M}_k} \subset W(\mathbb{A}_k)^{\mathrm{Br}_1}$. But, by definition of groups of multiplicative type,

$$\mathcal{M}_k = \mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k) \cap \mathcal{A}b_k \subset \mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k) \subset \mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k),$$

where, in the first inclusion, we have used the fact that $\mathcal{T}_k \subset \mathcal{A}b_k \subset \mathrm{Sol}_k$ and that Sol_k is closed under taking quotients. Hence, Condition 2 and its "solvable version" also hold.

Condition 3. Notice that $\mathcal{T}_k$ is clearly closed under taking finite direct products, and it contains the trivial group. By Lemma 2.9, $\mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k)$ is closed under taking direct products; moreover, the hypotheses of Lemma 2.12 hold (with $\mathcal{B} = \mathcal{T}_k$), so its conclusion also holds. The same is true for $\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k)$. Hence, it remains to check that $\mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k)$ and $\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k)$ are closed under taking closed k-subgroups and k-twists.

LEMMA 5.4. *We have*

$$\mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k) = \mathrm{Ext}(\mathcal{F}_k, \mathcal{M}_k) \quad \textit{and} \quad \mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k) = \mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{M}_k).$$

Proof. For the first equality, the inclusion "$\subset$" is clear. For the other inclusion, let $G \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{M}_k)$ fit into the short exact sequence $1 \to M \to G \to F \to 1$ with $F \in \mathcal{F}_k$ and $M \in \mathcal{M}_k$. But M contains a maximal subtorus T (cf. [DGA, SGA3.IV, §1.3]), which is a normal subgroup of G (since T is characteristic in M) and with $G/T \in \mathcal{F}_k$. Hence, G fits into the short exact sequence

$$1 \to T \to G \to G/T \to 1$$

with $T \in \mathcal{T}_k$ and $G/T \in \mathcal{F}_k$.

We now prove the second equality. Using the closure properties of solvable groups, we have $\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k) = \mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k) \cap \mathrm{Sol}_k$ and $\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{M}_k) =$

$\mathrm{Ext}(\mathcal{F}_k, \mathcal{M}_k) \cap \mathrm{Sol}_k$. But we have just seen that $\mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k) = \mathrm{Ext}(\mathcal{F}_k, \mathcal{M}_k)$; by intersecting both sides with Sol_k, the result follows. ∎

Lemma 5.5. *$\mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k)$ is closed under taking k-subgroups. The same holds for $\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k)$.*

Proof. Let $G \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k)$ and let $H \leq_k G$ be a k-subgroup of G. Suppose that G fits into a short exact sequence $1 \to T \to G \to F \to 1$, where $T \in \mathcal{T}_k$ and $F \in \mathcal{F}_k$. Then $H \cap T \in \mathcal{M}_k$ and $H/(T \cap H) \in \mathcal{F}_k$, so H fits into the short exact sequence

$$1 \to H \cap T \to H \to H/(H \cap T) \to 1,$$

that is, $H \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{M}_k)$. By Lemma 5.4, we deduce that $H \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k)$. An analogous proof gives the result for $\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k)$, once we notice that $G \in \mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k)$ implies that G is solvable, since solvable groups are closed under extensions; hence, H is also solvable and thus $H/(T \cap H) \in \mathcal{F}_k^{\mathrm{Sol}}$. ∎

Lemma 5.6. *$\mathcal{M}_k$ is closed under taking k-subgroups and under "base changing/restricting".*

Proof. The first statement is clear, by definition of groups of multiplicative type. For the second statement, let $M \in \mathcal{M}_k$ and let l/k be a finite extension. Since "being a group of multiplicative type" is a geometric condition, $M_l \in \mathcal{M}_l$. Moreover, since $\mathcal{M}_l = \mathrm{Ext}(\mathcal{F}_l, \mathcal{T}_l) \cap \mathcal{A}b_l$ and the Weil restriction preserves tori, commutative groups, finite groups, and short exact sequences, it follows that $\mathfrak{R}_{l/k}(M_l) \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k) \cap \mathcal{A}b_k = \mathcal{M}_k$. ∎

Lemma 5.7. *$\mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k)$ is closed under k-twists. The same is true for $\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k)$.*

Proof. By Lemma 5.6, we can apply Lemma 2.11 with $\mathcal{B} = \mathcal{M}_k$ to conclude that $\mathrm{Ext}(\mathcal{F}_k, \mathcal{M}_k)$ is closed under taking k-twists. By Lemma 5.4, we have $\mathrm{Ext}(\mathcal{F}_k, \mathcal{M}_k) = \mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k)$, hence the result. The same proof works for $\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k)$ as well. ∎

Hence, Condition 3 and its "solvable version" hold.

Since all the hypotheses in the statement of Theorem 3.1 are satisfied, we have just proved the following.

Theorem 5.8. *Let X be a nice variety over a number field k. Then*

$$X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}_1} = X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k)}.$$

Similarly,

$$X(\mathbb{A}_k)^{\mathrm{Sol},\mathrm{Br}_1} = X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k)}. \blacksquare$$

LEMMA 5.9. $\mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k) = \mathcal{RR}_k$. *In particular,* $\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k) = \mathcal{RR}_k^{\mathrm{Sol}}$.

Proof. We follow [Mey10, proof of Lemma 2.2]. We first prove that if $G \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k)$, then $G \in \mathcal{R}_k$; since $\mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k)$ is closed under taking k-subgroups, it then follows that $G \in \mathcal{RR}_k$. So let $G \in \mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k)$ with short exact sequence $1 \to T \to G \to F \to 1$. Then G^0/T is trivial, being finite and connected, meaning that $T = G^0$. It follows that $R_u(G) = R_u(G^0) = R_u(T)$ is trivial. Hence, $G \in \mathcal{R}_k$, as required.

For the converse implication, suppose that $G \in \mathcal{L}_k \setminus \mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k)$. Then clearly $G^0 \notin \mathcal{T}_k$. If $G^0 \notin \mathcal{R}_k$, then we are done. So suppose that $G^0 \in \mathcal{R}_k$. Its commutator $\mathcal{D}G^0$ is a connected linear k-subgroup, which is reductive and semisimple (since G^0 is reductive), and non-trivial, since otherwise $G^0 \in \mathcal{T}_k$, a contradiction. Since $\mathcal{D}G^0$ is connected and semisimple, it contains a k-subgroup of type A_1 [Spr98, Theorem 7.2.4]. But groups of type A_1 all have non-reductive k-subgroups, so we are done.

The last claim is clear, since $\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k) = \mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k) \cap \mathrm{Sol}_k$. ■

COROLLARY 5.10. *Let X be a nice variety over a number field k. Then*

$$X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}_1} = X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k, \mathcal{T}_k)} = X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k, \mathcal{M}_k)} = X(\mathbb{A}_k)^{\mathcal{RR}_k}.$$

Similarly,

$$X(\mathbb{A}_k)^{\mathrm{Sol},\mathrm{Br}_1} = X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k)} = X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{M}_k)} = X(\mathbb{A}_k)^{\mathcal{RR}_k^{\mathrm{Sol}}}.$$

Proof. Clear from Theorem 5.8, Lemma 5.4, and Lemma 5.9. ■

6. Interlude

THEOREM 6.1. *Let X be a nice variety over k. Let $\mathcal{A} \subset \mathcal{B} \subset \mathcal{L}_k$ be such that $\mathcal{B} \subset \mathrm{Ext}(\mathcal{A}, \mathcal{U}_k)$. Then $X(\mathbb{A}_k)^{\mathcal{A}} = X(\mathbb{A}_k)^{\mathcal{B}}$.*

Before we can prove the above theorem, we need to recall some preliminary results.

Let $\mathcal{A} \subset \mathcal{B}$ be as in the statement of Theorem 6.1 and let $G \in \mathcal{B} \subset \mathrm{Ext}(\mathcal{A}, \mathcal{U}_k)$, say fitting into the short exact sequence $1 \to U_G \to G \to A_G \to 1$ with $U_G \in \mathcal{U}_k$ and $A_G \in \mathcal{A}$. We want to show that $H^1(K, G) \cong H^1(K, A_G)$ as pointed sets for $K = k$ and $K = k_v$, for all $v \in \Omega_k$.

Since we are dealing with pointed sets, showing that $\ker(H^1(K, G) \to H^1(K, A_G))$ is trivial is not enough to conclude injectivity. Fortunately, U_G is a normal K-subgroup of G, meaning that we can use the following result.

PROPOSITION 6.2 ([Ser01, §I.5.5, Corollary 2]). *Let $[\tau] \in H^1(K, G)$. Then the elements of $H^1(K, G)$ with the same image in $H^1(K, A_G)$ as $[\tau]$ are in bijection with the elements of the quotient of $H^1(K, U_G^{\tau})$ by the action of the group $H^0(K, A_G^{\tau})$.* ■

Corollary 6.3. *The map of pointed sets $H^1(K,G) \to H^1(K,A_G)$ is injective.*

Proof. Since $\mathcal{U}_K$ is closed under K-twists (as "being unipotent" is a geometric property), $U_G^\tau \in \mathcal{U}_K$. But $H^1(K,U^\tau)$ is trivial (cf. [Ser01, §III.2.1, Proposition 6]), meaning that the quotient of $H^1(K,U_G^\tau)$ by the action of the group $H^0(K,A_G^\tau)$ is also trivial. The result then follows from Proposition 6.2. ∎

The surjectivity of $H^1(K,G) \to H^1(K,A_G)$ follows from a "vanishing" theorem for unipotent K-liens by Douai (see [Bor93, Corollary 4.2]), which requires some knowledge of non-abelian second Galois cohomology. We refer the reader to [FSS98, (1.2) and (5.1)], and to [Bor93] for more details on the construction of K-liens and on the theory of non-abelian cohomology.

Finally, since $H^1(K,G) \to H^1(K,A_G)$ preserves base points, we have the following.

Corollary 6.4. *For $K = k$ or $K = k_v$, for any place v, we have $H^1(K,G) \cong H^1(K,A_G)$ as pointed sets.* ∎

Proof of Theorem 6.1. Let $G \in \mathcal{B}$. Since $\mathcal{B} \subset \operatorname{Ext}(\mathcal{A},\mathcal{U}_k)$, it follows that G fits into a short exact sequence

$$1 \to U_G \to G \to A_G \to 1,$$

where $U_G \in \mathcal{U}_k$ and $A_G \in \mathcal{A}$. Fix $[g : Y \to X] \in H^1(X,G)$. We can push forward Y along $p : G \to A_G$ to obtain $[a_g : Z \to X] \in H^1(X,A_G)$.

The inclusion $X(\mathbb{A}_k)^g \subset X(\mathbb{A}_k)^{a_g}$ is easy. For the opposite inclusion, let $(x_v) \in X(\mathbb{A}_k)^{a_g}$, so that $[Z]((x_v)) \in \operatorname{Im}(H^1(k,A_G) \to \prod_{v\in\Omega_k} H^1(k_v,A_G))$. Our aim is to show that $[Y]((x_v)) \in \operatorname{Im}(H^1(k,G) \to \prod_{v\in\Omega_k} H^1(k_v,G))$, as, using the functorial description of obstructions, this would indeed imply that $(x_v) \in X(\mathbb{A}_k)^g$.

By Corollary 6.4, for any field K containing k,

$$p_* : H^1(K,G) \xrightarrow{\sim} H^1(K,A_G) \tag{6.1}$$

is an isomorphism of pointed sets, where p_* is induced by the projection $p : G \to A_G$. In particular, (6.1) holds for $K = k$ or $K = k_v$, for any place $v \in \Omega_k$.

Now, for each place $v \in \Omega_k$, we have the commutative diagram

$$\begin{array}{ccccc}
H^1(X,G) & \xrightarrow{x_v^*} & H^1(k_v,G) & \xleftarrow{\operatorname{res}_v} & H^1(k,G) \\
\downarrow{\scriptstyle p_*^X} & & {\scriptstyle p_*^{-1}}\uparrow\downarrow{\scriptstyle p_*} & & {\scriptstyle p_*^{-1}}\uparrow\downarrow{\scriptstyle p_*} \\
H^1(X,A_G) & \xrightarrow{x_v^*} & H^1(k_v,A_G) & \xleftarrow{\operatorname{res}_v} & H^1(k,A_G)
\end{array}$$

From the left square, we deduce that

$$p_*([Y](x_v)) = p_*^X[Y](x_v) = [Z](x_v) \in \operatorname{Im}(H^1(k, A_G) \to H^1(k_v, A_G)).$$

But $H^1(k, A_G) = p_*(H^1(k, G))$ and $H^1(k_v, A_G) = p_*(H^1(k_v, G))$, by (6.1). This fact and the commutativity of the right square in the above diagram together imply that

$$p_*([Y](x_v)) \in \operatorname{Im}\big(p_*(H^1(k, G)) \to p_*(H^1(k_v, G))\big) = p_*\big(\operatorname{Im}(H^1(k, G) \to H^1(k_v, G))\big).$$

By applying p_*^{-1}, we get $[Y](x_v) \in \operatorname{Im}(H^1(k, G) \to H^1(k_v, G))$; taking the product over all places $v \in \Omega_k$ then gives the required result.

Hence, $X(\mathbb{A}_k)^{a_g} \subset X(\mathbb{A}_k)^g$, and so

$$X(\mathbb{A}_k)^{a_g} = X(\mathbb{A}_k)^g. \tag{6.2}$$

Intersecting over all torsors in $H^1(X, G)$ and all $G \in \mathcal{B}$ gives

$$\bigcap_{G\in\mathcal{B}} \bigcap_{[Y]\in H^1(X,G)} X(\mathbb{A}_k)^{a_g} = \bigcap_{G\in\mathcal{B}} \bigcap_{[Y]\in H^1(X,G)} X(\mathbb{A}_k)^g. \tag{6.3}$$

For any $G \in \mathcal{A} \subset \mathcal{B}$, we have $G = A_G$ and $g = a_g$. Hence, the left-hand side of (6.3) can be rewritten as

$$\bigcap_{G\in\mathcal{B}} \bigcap_{\substack{[g:Y\to X]\\ \in H^1(X,G)}} X(\mathbb{A}_k)^{a_g} = X(\mathbb{A}_k)^{\mathcal{A}} \cap \underbrace{\Bigg(\bigcap_{G\in\mathcal{B}\setminus\mathcal{A}} \bigcap_{\substack{[a_g:Z\to X]\\ \in H^1(X,A_G)}} X(\mathbb{A}_k)^{a_g}\Bigg)}_{\supset X(\mathbb{A}_k)^{\mathcal{A}}} = X(\mathbb{A}_k)^{\mathcal{A}}.$$

Since the right-hand side of (6.3) is, by definition, $X(\mathbb{A}_k)^{\mathcal{B}}$, we conclude that $X(\mathbb{A}_k)^{\mathcal{A}} = X(\mathbb{A}_k)^{\mathcal{B}}$, as required. ■

Remark 6.5. In Theorem 6.1, we use $\mathcal{U}_k$ just to guarantee that $H^1(K, G) \cong H^1(K, A_G)$ as pointed sets for $K = k$ and $K = k_v$, for all $v \in \Omega_k$ (cf. (6.1)). In theory, we can get a result similar to Theorem 6.1 if we replace $\mathcal{U}_k$ with any other $\mathcal{S} \subset \mathcal{L}_k$ such that (6.1) holds for $K = k$ and $K = k_v$, for all $v \in \Omega_k$: for example, when k is a totally imaginary number field, a good candidate for such an $\mathcal{S}$ is the set of semisimple simply connected linear algebraic groups over k—see [Ser01, §3.1] and [Bor93, Corollary 5.1].

Example 6.6. Any $G \in \mathcal{A}b_k$ can be written as $G = U \times M$, where $U \in \mathcal{U}_k$ and $M \in \mathcal{M}_k$ (cf. [Mil12, 1.21]). Hence, $\mathcal{M}_k \subset \mathcal{A}b_k \subset \operatorname{Ext}(\mathcal{M}_k, \mathcal{U}_k)$, implying that $X(\mathbb{A}_k)^{\mathcal{M}_k} = X(\mathbb{A}_k)^{\mathcal{A}b_k}$.

Corollary 6.7. *Let $G \in \mathcal{L}_k$. Then $X(\mathbb{A}_k)^{H^1(X,G)} = X(\mathbb{A}_k)^{H^1(X,R_G)}$, where $R_G := G/R_u(G)$.*

Proof. Let $\mathcal{A} = \{R_G\}$ and $\mathcal{B} = \{G, R_G\}$ in Theorem 6.1. ■

Corollary 6.8. *Let X be a nice variety over k. Then $X(\mathbb{A}_k)^{\mathcal{U}_k} = X(\mathbb{A}_k)$.*

Proof. Let $\mathcal{A} = \{1\}$ and $\mathcal{B} = \mathcal{U}_k$. Clearly $\mathcal{U}_k \subset \mathrm{Ext}(\{1\},\mathcal{U}_k) = \mathcal{U}_k$, and so the hypotheses of Theorem 6.1 are satisfied. Therefore, $X(\mathbb{A}_k)^{\mathcal{U}_k} = X(\mathbb{A}_k)^{\{1\}} = X(\mathbb{A}_k)$, as required. ∎

Remark 6.9. Since $H^1(K,U)$ is trivial for all $U \in \mathcal{U}_k$, for $K = k$ and $K = k_v$ for all $v \in \Omega_k$ (cf. [Ser01, §III.2.1, Proposition 6]), we deduce that (6.1) in the proof of Corollary 6.8 holds without having to resort to non-abelian second Galois cohomology.

Since linear algebraic unipotent groups over k give no obstruction, and since reductive groups are, in a sense, the "opposite" of unipotent groups, the following should not be too surprising.

Corollary 6.10. *Let X be a nice variety over k. Then $X(\mathbb{A}_k)^{\mathcal{L}_k} = X(\mathbb{A}_k)^{\mathcal{R}_k}$.*

Proof. Let $\mathcal{A} = \mathcal{R}_k$ and $\mathcal{B} = \mathcal{L}_k$. It is clear that $\mathcal{L}_k = \mathrm{Ext}(\mathcal{R}_k,\mathcal{U}_k)$. This means that the hypotheses of Theorem 6.1 are satisfied. Therefore, $X(\mathbb{A}_k)^{\mathcal{L}_k} = X(\mathbb{A}_k)^{\mathcal{R}_k}$, as required. ∎

6.1. Some corollaries. The above also suggests the following proposition.

Proposition 6.11. *Let $\mathcal{B} \subset \mathcal{L}_k$ be such that $B/R_u(B) \in \mathcal{B}$, for any $B \in \mathcal{B}$. Let X be a nice variety over k. Then $X(\mathbb{A}_k)^{\mathcal{B}} = X(\mathbb{A}_k)^{\mathcal{B}\cap\mathcal{R}_k}$.*

Proof. Let $\mathcal{A} = \mathcal{B} \cap \mathcal{R}_k \subset \mathcal{B}$. By assumption $B/R_u(B) \in \mathcal{B} \cap \mathcal{R}_k$ for any $B \in \mathcal{B}$, so we can apply Theorem 6.1 to conclude the proof. ∎

Note that, for example, $\mathcal{B} = \mathcal{R}_k, \mathcal{C}_k, \mathcal{A}b_k, \mathrm{Sol}_k$ all satisfy the hypotheses of Proposition 6.11, as they are all closed under taking quotients. Hence, we can immediately deduce the following.

Corollary 6.12. $X(\mathbb{A}_k)^{\mathcal{C}_k} = X(\mathbb{A}_k)^{\mathcal{C}_k\cap\mathcal{R}_k}$. ∎

Remark 6.13. The above could have also been deduced in the following way. Let $\mathrm{PGL} := \bigcup_{n\geq 1} \mathrm{PGL}_n$. For X nice over k, we know that $X(\mathbb{A}_k)^{\mathcal{C}_k} = X(\mathbb{A}_k)^{\mathrm{Br}} = X(\mathbb{A}_k)^{\mathrm{PGL}}$: the latter equality is [Sko01, Proposition 5.3.4], while the former follows from [Har02]. Since $\mathcal{R}_k \cap \mathcal{C}_k \subset \mathcal{C}_k$, it follows immediately that $X(\mathbb{A}_k)^{\mathcal{C}_k} \subset X(\mathbb{A}_k)^{\mathcal{C}_k\cap\mathcal{R}_k}$. Now, for any $n \geq 1$, PGL_n is both connected and reductive. It follows that $\mathrm{PGL} \subset \mathcal{R}_k \cap \mathcal{C}_k$, meaning that $X(\mathbb{A}_k)^{\mathcal{C}_k\cap\mathcal{R}_k} \subset X(\mathbb{A}_k)^{\mathrm{PGL}} = X(\mathbb{A}_k)^{\mathcal{C}_k}$. Hence, $X(\mathbb{A}_k)^{\mathcal{C}_k} = X(\mathbb{A}_k)^{\mathcal{C}_k\cap\mathcal{R}_k}$.

Corollary 6.14. $X(\mathbb{A}_k)^{\mathrm{Sol}_k} = X(\mathbb{A}_k)^{\mathrm{Sol}_k\cap\mathcal{R}_k}$. *In particular,* $X(\mathbb{A}_k)^{\mathrm{Sol},\mathrm{Br}_1} = X(\mathbb{A}_k)^{\mathrm{Sol}_k}$.

Proof. By Proposition 6.11, the former statement is obvious, so we just need to prove the latter. We show that $\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k) = \mathrm{Sol}_k \cap \mathcal{R}_k$. Let $G \in \mathrm{Sol}_k \cap \mathcal{R}_k$. Then G^0 is reductive, connected, and solvable. Hence, $G^0 \in \mathcal{T}_k$. Moreover, since $G \in \mathrm{Sol}_k$, we have $G/G^0 \in \mathcal{F}_k^{\mathrm{Sol}}$. Conversely, let $G \in \mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k)$. Then $G \in \mathrm{Sol}_k$, as the solvable groups are closed under extensions, and $G \in \mathcal{R}_k$ by Lemma 5.9. Hence, $\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{T}_k) = \mathrm{Sol}_k \cap \mathcal{R}_k$. From Corollary 5.10 and the fact that $X(\mathbb{A}_k)^{\mathrm{Sol}_k} = X(\mathbb{A}_k)^{\mathrm{Sol}_k \cap \mathcal{R}_k}$, we then deduce the required result. ∎

Corollary 6.15. $X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{C}_k)} = X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{C}_k) \cap \mathcal{R}_k}$.

Proof. Let $G \in \mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{C}_k)$. We need to check that $G/R_u(G)$ is in $\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{C}_k)$. We know that G fits into a short exact sequence

$$1 \to C \to G \to F \to 1,$$

where $C \in \mathcal{C}_k$ and $F \in \mathcal{F}_k^{\mathrm{Sol}}$. Notice that $C = G^0$. Indeed, by definition of G^0, we have $C \subset G^0$, and since C is normal in G, it follows that C is also normal in G^0. Consider the short exact sequence $1 \to G^0 \to G \to G/G^0 \to 1$, where $G/G^0 \in \mathcal{F}_k$. This induces the short exact sequence

$$1 \to G^0/C \to G/C \cong F \to G/G^0 \to 1;$$

since G^0/C is connected and injects into the finite group F, it must be trivial, i.e. $C = G^0$. Hence, G fits into the short exact sequence

$$1 \to G^0 \to G \to F \to 1.$$

Now, since $R_u(G)$ is, by definition, a connected normal subgroup of G, it follows that $R_u(G)$ is normal in G^0. Hence, we get the short exact sequence

$$1 \to G^0/R_u(G) \to G/R_u(G) \to F \to 1,$$

where $G^0/R_u(G) \in \mathcal{C}_k$ and $F \in \mathcal{F}_k^{\mathrm{Sol}}$, meaning that $G/R_u(G) \in \mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}}, \mathcal{C}_k)$. By Proposition 6.11, the result follows. ∎

Remark 6.16. Analogously, $X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Ab}}, \mathcal{C}_k)} = X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Ab}}, \mathcal{C}_k) \cap \mathcal{R}_k}$, and similarly for any subset of $\mathcal{F}_k$ extended by $\mathcal{C}_k$.

Corollary 6.17. $X(\mathbb{A}_k)^{\mathcal{T}_k} = X(\mathbb{A}_k)^{\mathcal{A}b_k \cap \mathcal{C}_k} = X(\mathbb{A}_k)^{\mathrm{Sol}_k \cap \mathcal{C}_k}$.

Proof. Since both Sol_k and $\mathcal{C}_k$ are closed under quotients, so is their intersection, meaning that we can apply Proposition 6.11 to deduce that $X(\mathbb{A}_k)^{\mathrm{Sol}_k \cap \mathcal{C}_k} = X(\mathbb{A}_k)^{\mathrm{Sol}_k \cap \mathcal{C}_k \cap \mathcal{R}_k} = X(\mathbb{A}_k)^{\mathcal{T}_k}$. Similarly, by noticing that $\mathcal{T}_k = \mathcal{C}_k \cap \mathcal{R}_k \cap \mathcal{A}b_k$ and that the abelian groups are also closed under quotients, we get the other equality in the statement. ∎

7. A network of obstructions

7.1. Summary. For any nice variety X over k, Figure 1 summarises the relations between some of the obstruction sets mentioned in this paper. The rest of this section is to remind the reader of why all the inclusions in Figure 1 can be strict.

$$
\begin{array}{ccccc}
 & & & & X(\mathbb{A}_k)^{\emptyset/\mathcal{U}_k} \\
 & & & & \cup \\
X(\mathbb{A}_k)^{\mathcal{C}_k/\mathcal{C}_k\cap\mathcal{R}_k/\mathrm{PGL}/\mathrm{Br}} & \hookrightarrow & X(\mathbb{A}_k)^{\mathcal{R}\mathcal{R}^{\mathrm{Ab}}/\mathcal{A}b_k/\mathcal{M}_k/\mathrm{Br}_1} & \hookrightarrow & X(\mathbb{A}_k)^{\mathcal{F}_k^{\mathrm{Ab}}} \\
\cup & & \cup & & \cup \\
X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}},\mathcal{C}_k)/\mathrm{Sol},\mathrm{Br}} & \hookrightarrow & X(\mathbb{A}_k)^{\mathcal{R}\mathcal{R}_k^{\mathrm{Sol}}/\mathcal{R}_k^{\mathrm{Sol}}/\mathrm{Sol}_k/\mathrm{Sol},\mathrm{Br}_1} & \hookrightarrow & X(\mathbb{A}_k)^{\mathcal{F}_k^{\mathrm{Sol}}} \\
\cup & & \cup & & \cup \\
X(\mathbb{A}_k)^{\mathcal{L}_k/\mathcal{R}_k/\text{ét},\mathrm{Br}} & \hookrightarrow & X(\mathbb{A}_k)^{\mathcal{R}\mathcal{R}_k/\text{ét},\mathrm{Br}_1} & \hookrightarrow & X(\mathbb{A}_k)^{\mathcal{F}_k} \\
\cup & & & & \\
X(k) & & & &
\end{array}
$$

Fig. 1

Proposition 7.1. *In general,* $X(k) \subsetneq X(\mathbb{A}_k)^{\mathcal{L}_k/\mathcal{R}_k/\text{ét},\mathrm{Br}}$ *for* X *a nice variety over* k.

Proof. This is the content of Poonen's paper [Poo10]. ∎

Proposition 7.2. *In general,* $X(\mathbb{A}_k)^{\mathcal{R}\mathcal{R}^{\mathrm{Ab}}/\mathcal{A}b_k/\mathcal{M}_k/\mathrm{Br}_1} \subsetneq X(\mathbb{A}_k)^{\emptyset/\mathcal{U}_k}$ *for* X *a nice variety over* k.

Proof. See [BSD75] for an example over $k = \mathbb{Q}$. ∎

Remark 7.3. More strongly, in general, $X(\mathbb{A}_k)^{\mathcal{T}_k/\mathcal{C}_k^{\mathrm{Ab}}/\mathcal{C}_k^{\mathrm{Sol}}} \subsetneq X(\mathbb{A}_k)^{\emptyset/\mathcal{U}_k}$. Indeed, let X be the (nice) degree 4 del Pezzo surface over $k = \mathbb{Q}$ in [BSD75]. We know that $X(\mathbb{A}_k)^{\mathrm{Br}_1} = \emptyset$ but $X(\mathbb{A}_k) \neq \emptyset$. Moreover, since X is geometrically rational, $\pi_1(\overline{X}) = 0$, meaning that there are no non-trivial finite torsors over X. In particular, $X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k,\mathcal{T}_k)} = X(\mathbb{A}_k)^{\mathcal{T}_k}$. Hence, $\emptyset = X(\mathbb{A}_k)^{\mathrm{Br}_1} = X(\mathbb{A}_k)^{\mathcal{M}_k} = X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k,\mathcal{T}_k)\cap\mathcal{A}b_k} = X(\mathbb{A}_k)^{\mathcal{T}_k} \neq X(\mathbb{A}_k)$.

Proposition 7.4. *In general, for* X *a nice variety over* k,

(a) $X(\mathbb{A}_k)^{\mathrm{Ab},\mathrm{Br}_1} \subsetneq X(\mathbb{A}_k)^{\mathcal{R}\mathcal{R}^{\mathrm{Ab}}/\mathcal{A}b_k/\mathcal{M}_k/\mathrm{Br}_1}$;
(b) $X(\mathbb{A}_k)^{\mathrm{Ab},\mathrm{Br}} \subsetneq X(\mathbb{A}_k)^{\mathcal{C}_k/\mathcal{C}_k\cap\mathcal{R}_k/\mathrm{PGL}/\mathrm{Br}}$;
(c) *the column in Figure* 1 *with endpoints given by* $X(\mathbb{A}_k)^{\mathcal{R}\mathcal{R}_k/\text{ét},\mathrm{Br}_1}$ *and* $X(\mathbb{A}_k)^{\mathcal{R}\mathcal{R}^{\mathrm{Ab}}/\mathcal{A}b_k/\mathcal{M}_k/\mathrm{Br}_1}$ *can be strictly contained in the column with endpoints* $X(\mathbb{A}_k)^{\mathcal{F}_k}$ *and* $X(\mathbb{A}_k)^{\mathcal{F}_k^{\mathrm{Ab}}}$;
(d) $X(\mathbb{A}_k)^{\mathcal{C}_k/\mathcal{C}_k\cap\mathcal{R}_k/\mathrm{PGL}/\mathrm{Br}} \not\subset X(\mathbb{A}_k)^{\mathrm{Ab},\mathrm{Br}_1}$;
(e) $X(\mathbb{A}_k)^{\mathcal{F}_k^{\mathrm{Sol}}} \not\subset X(\mathbb{A}_k)^{\mathcal{R}\mathcal{R}^{\mathrm{Ab}}/\mathcal{A}b_k/\mathcal{M}_k/\mathrm{Br}_1}$.

Proof. Let $Y = Y_{a,b,c}$ and $X = Y_{a,b,c}/\langle\iota\rangle$ be, respectively, the (nice) K3 and Enriques surfaces over $k = \mathbb{Q}$ from [VAV11]. Then $(f, Y, F) \in \mathcal{F}_k^{\mathrm{Ab}}(X)$, where $F = \mathbb{Z}/2\mathbb{Z}$.

(a), (b), (d). Using Kummer theory and following [VAV11], it is not difficult to show that $\bigcup_{[\tau]\in H^1(X,F)} f^\tau(Y^\tau(\mathbb{A}_k)^{\mathrm{Br}_1}) = \emptyset$, meaning, in particular, that $X(\mathbb{A}_k)^{\mathrm{Ab},\mathrm{Br}_1} = \emptyset$. Moreover, in [BBM+], the authors have shown that $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$.

(c), (e). Note that $Y_{a,b,c}(\mathbb{A}_k)^{\mathcal{F}_k} = Y_{a,b,c}(\mathbb{A}_k) \neq \emptyset$ and $Y_{a,b,c}(\mathbb{A}_k)^{\text{ét},\mathrm{Br}_1} = Y_{a,b,c}(\mathbb{A}_k)^{\mathrm{Br}_1} = \emptyset$. ∎

PROPOSITION 7.5. *In general, for X a nice variety over k,*

(a) $X(\mathbb{A}_k)^{\mathcal{F}_k^{\mathrm{Sol}}} \subsetneq X(\mathbb{A}_k)^{\mathcal{F}_k^{\mathrm{Ab}}}$;
(b) $X(\mathbb{A}_k)^{\mathcal{R}\mathcal{R}^{\mathrm{Ab}}/\mathcal{A}b_k/\mathcal{M}_k/\mathrm{Br}_1} \not\subset X(\mathbb{A}_k)^{\mathcal{F}_k}$.

Proof. (a), (b). Let X be the (nice) bielliptic surface over $k = \mathbb{Q}$ from Skorobogatov's counterexample (see [Sko01, Chapter 8]). Then $X(\mathbb{A}_k)^{\mathcal{F}_k^{\mathrm{Sol}}} = \emptyset$ (cf. [HS02, §5.1]), but $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \emptyset$. ∎

PROPOSITION 7.6. *In general, for X a nice variety over k,*

(a) *the column in Figure* 1 *starting with* $X(\mathbb{A}_k)^{\mathcal{L}_k/\mathcal{R}_k/\text{ét},\mathrm{Br}}$ *can be strictly contained in the column starting with* $X(\mathbb{A}_k)^{\mathcal{R}\mathcal{R}_k/\text{ét},\mathrm{Br}_1}$;
(b) $X(\mathbb{A}_k)^{\mathcal{R}\mathcal{R}_k/\text{ét},\mathrm{Br}_1} \not\subset X(\mathbb{A}_k)^{\mathcal{C}_k/\mathcal{C}_k\cap\mathcal{R}_k/\mathrm{PGL}/\mathrm{Br}}$.

Proof. (a), (b). In [HVA13], the authors have constructed a (nice) K3 surface X over $k = \mathbb{Q}$ such that $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$ but $X(\mathbb{A}_k)^{\mathrm{Br}_1} \neq \emptyset$. Now, since any K3 surface is simply connected, we have $\pi_1(\overline{X}) = 0$. In particular, $X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} = X(\mathbb{A}_k)^{\mathrm{Br}}$ and $X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}_1} = X(\mathbb{A}_k)^{\mathrm{Br}_1}$. ∎

PROPOSITION 7.7. *The row in Figure* 1 *starting with* $X(\mathbb{A}_k)^{\mathcal{L}_k/\mathcal{R}_k/\text{ét},\mathrm{Br}}$ *can be strictly contained in the row starting with* $X(\mathbb{A}_k)^{\mathrm{Ext}(\mathcal{F}_k^{\mathrm{Sol}},\mathcal{C}_k)/\mathrm{Sol},\mathrm{Br}}$, *for X a nice variety over k.*

Proof. Let G be a finite, perfect (i.e. with $G = \mathcal{D}G$), non-abelian, simple group; in particular, G is not solvable. Let X and k be as in the conclusion of [Har00, Corollary 6.1]. Since $G = \pi(\overline{X})$ and G is simple, it follows that $X(\mathbb{A}_k)^{\mathcal{F}_k^{\mathrm{Sol}}} = X(\mathbb{A}_k)$, as there are no non-trivial finite solvable covers of X. This implies that $X(\mathbb{A}_k)^{\mathrm{Sol},\mathrm{Br}_1} = X(\mathbb{A}_k)^{\mathrm{Br}_1}$ and $X(\mathbb{A}_k)^{\mathrm{Sol},\mathrm{Br}} = X(\mathbb{A}_k)^{\mathrm{Br}}$. Since G is perfect, $\mathrm{Br}(\overline{X}) = 0$ (cf. [Har00, remark after Corollary 6.1]), meaning that $X(\mathbb{A}_k)^{\mathrm{Br}} = X(\mathbb{A}_k)^{\mathrm{Br}_1}$ and $\mathrm{Br}(X)/\mathrm{Br}(k)$ is finite. By taking k large enough, we can assume that $\mathrm{Br}(X)/\mathrm{Br}(k)$ is trivial, and thus $X(\mathbb{A}_k)^{\mathrm{Br}} = X(\mathbb{A}_k)$. Hence,

$$\begin{aligned} X(\mathbb{A}_k)^{\mathrm{Sol},\mathrm{Br}_1} &= X(\mathbb{A}_k)^{\mathrm{Br}_1} = X(\mathbb{A}_k)^{\mathrm{Sol},\mathrm{Br}} \\ &= X(\mathbb{A}_k)^{\mathrm{Br}} = X(\mathbb{A}_k)^{\mathcal{F}_k^{\mathrm{Sol}}} = X(\mathbb{A}_k). \end{aligned}$$

But, by construction of X, we have $X(\mathbb{A}_k)^{\mathcal{F}_k} \subsetneq X(\mathbb{A}_k)$. Since $X(\mathbb{A}_k)^{\text{ét,Br}} \subset X(\mathbb{A}_k)^{\text{ét,Br}_1} \subset X(\mathbb{A}_k)^{\mathcal{F}_k}$, the result follows. ∎

We conclude with an open question.

QUESTION 7.8. What are, if they exist, the translations of $X(\mathbb{A}_k)^{\text{Ab,Br}}$ and $X(\mathbb{A}_k)^{\text{Ab,Br}_1}$ in pure "descent-type" terms?

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References

[BBM+] F. Balestrieri, J. Berg, M. Manes, J. Park and B. Viray, *Insufficiency of the Brauer–Manin obstruction for rational points on Enriques surfaces*, in: Directions in Number Theory: Proceedings of the 2014 WIN3 Workshop, to appear; arXiv:1501.04974 (2015).

[BSD75] B. J. Birch and H. P. F. Swinnerton-Dyer, *The Hasse problem for rational surfaces*, J. Reine Angew. Math. 274/275 (1975), 164–174.

[Bor91] A. Borel, *Linear Algebraic Groups*, 2nd ed., Grad. Texts in Math. 126, Springer, New York, 1991.

[Bor93] M. V. Borovoi, *Abelianization of the second nonabelian Galois cohomology*, Duke Math. J. 72 (1993), 217–239.

[BLR90] S. Bosch, W. Lütkebohmert and M. Raynaud, *Néron Models*, Ergeb. Math. Grenzgeb. 21, Springer, 1990.

[Car62] P. Cartier, *Groupes algébriques et groupes formels*, in: Colloq. Théorie des Groupes Algébriques, 1962, 87–111.

[CTS87] J.-L. Colliot-Thélène et J.-J. Sansuc, *La descente sur les variétés rationnelles, II*, Duke Math. J. 54 (1987), 375–492.

[Dem09] C. Demarche, *Obstruction de descente et obstruction de Brauer–Manin étale*, Algebra Number Theory 3 (2009), 237–254.

[DGA] M. Demazure et A. Grothendieck, *Schémas en groupes (SGA 3): Structure des schémas en groupes réductifs*, Doc. Math. (Paris) 8, Soc. Math. France, 2011.

[dJ] A. J. de Jong, *A result of Gabber*, http://www.math.columbia.edu/~dejong/papers/2-gabber.pdf.

[FSS98] Y. Z. Flicker, C. Scheiderer and R. Sujatha, *Grothendieck's theorem on non-abelian H^2 and local-global principles*, J. Amer. Math. Soc. 11, (1998), 731–750.

[Gro65] A. Grothendieck, *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Seconde partie*, Publ. IHES 24 (1965), 231 pp.

[Har96] D. Harari, *Obstructions de Manin transcendantes*, in: Number Theory (Paris, 1993–1994), London Math. Soc. Lecture Note Ser. 235, Cambridge Univ. Press, 1996, 75–87.

[Har00] D. Harari, *Weak approximation and non-abelian fundamental groups*, Ann. Sci. École Norm. Sup. 33 (2000), 467–484.

[Har02] D. Harari, *Groupes algébriques et points rationnels*, Math. Ann. 322 (2002), 811–826.

[HS02] D. Harari and A. N. Skorobogatov, *Non-abelian cohomology and rational points*, Compos. Math. 130 (2002), 241–273.

[HS13] Y. Harpaz and T. M. Schlank, *Homotopy obstructions to rational points*, in: Torsors, Étale Homotopy and Applications to Rational Points, London Math. Soc. Lecture Note Ser. 405, Cambridge Univ. Press, 2013, 280–413.

[HVA13] B. Hassett and A. Várilly-Alvarado, *Failure of the Hasse principle on general K3 surfaces*, J. Inst. Math. Jussieu 4 (2013), 853–877.

[Man71] Yu. I. Manin, *Le groupe de Brauer–Grothendieck en géométrie diophantienne*, in: Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1, 1971, 401–411.

[Mey10] A. N. Meyer, *Essential dimension of algebraic groups*, PhD Thesis, 2010.

[Mil12] J. S. Milne, *Reductive groups (v1.00)*, www.jmilne.org/math/ (2012).

[Poo10] B. Poonen, *Insufficiency of the Brauer–Manin obstruction applied to étale covers*, Ann. of Math. 171 (2010), 2157–2169.

[Ser01] J.-P. Serre, *Galois Cohomology*, Springer, 2001.

[Sko99] A. N. Skorobogatov, *Beyond the Manin obstruction*, Invent. Math. 135 (1999), 399–424.

[Sko01] A. N. Skorobogatov, *Torsors and Rational Points*, Cambridge Tracts in Math. 144, Cambridge Univ. Press, Cambridge, 2001.

[Sko09] A. N. Skorobogatov, *Descent obstruction is equivalent to étale Brauer–Manin obstruction*, Math. Ann. 344 (2009), 501–510.

[Spr98] T. A. Springer, *Linear Algebraic Groups*, 2nd ed., Progr. Math. 9, Springer, 1998.

[Sto07] M. Stoll, *Finite descent obstructions and rational points on curves*, Algebra Number Theory 1 (2007), 349–391.

[VAV11] A. Várilly-Alvarado and B. Viray, *Failure of the Hasse principle for Enriques surfaces*, Adv. Math. 226 (2011), 4884–4901.

[Vos11] V. E. Voskresenskiĭ, *Algebraic Groups and Their Birational Invariants*, Transl. Math. Monogr. 179, Amer. Math. Soc., 2011.

Francesca Balestrieri
Mathematical Institute
Oxford, OX2 6HD, United Kingdom
E-mail: balestrieri@maths.ox.ac.uk