

Strong theories of ordered Abelian groups

by

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Abstract. We consider strong expansions of the theory of ordered Abelian groups. We show that the assumption of strength has a multitude of desirable consequences for the structure of definable sets in such theories, in particular as relates to definable infinite discrete sets. We also provide a range of examples of strong expansions of ordered Abelian groups which demonstrate the great variety of such theories.

1. Introduction. In this article we establish some tameness properties for discrete sets which are first-order definable in ordered Abelian groups, in languages extending $L_{\text{oag}} = \{0, +, <\}$, whose complete theories are strong. Below we will recall the definitions of “strong”, “strongly dependent”, and the other important properties of theories that we will consider, and explain the motivation for their study.

Around 2005, Shelah proposed several possible definitions of what it should mean for a complete first-order theory to be “strongly dependent”, with corresponding ordinal-valued ranks, in an attempt to generalize the useful superstable / strictly stable dichotomy to the class of dependent (or NIP) theories (see [She]). However, it should be emphasized that “strongly dependent” really generalizes the idea that types have bounded weight rather than superstability, and there are stable, nonsuperstable theories which are strongly dependent. Then Adler [Ad] gave the formulations of *strongly dependent* and *inp-pattern* which are now standard (and which we follow), as well as introducing the more general class of strong theories (generalizing “strongly dependent”).

The logical relations among some of the properties of theories relevant to this paper are summarized below, where “ $X \Rightarrow Y$ ” signifies that any theory

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which is X is also Y , and all implications are nonreversible.

$$\begin{array}{ccccccc}
 \text{inp-minimal} & \Rightarrow & \text{finite burden} & \Rightarrow & \text{strong} & \Rightarrow & \text{NTP}_2 \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \text{dp-minimal} & \Rightarrow & \text{finite dp-rank} & \Rightarrow & \text{strongly NIP} & \Rightarrow & \text{NIP}
 \end{array}$$

There is a significant body of previous and concurrent work on algebraic structures (groups, rings, fields, valued fields, etc.) which have theories satisfying the various properties mentioned above. For example, a motivating conjecture is that any NIP field is either algebraically closed, real-closed, or admits a definable Henselian valuation; this is still open even for the class of stable fields, although something very close to this was proved very recently by Will Johnson [Joh]. Similar results were independently obtained by Jahnke, Simon, and Walsberg [JSW], who also characterized dp-minimal ordered Abelian groups and dp-minimal ordered fields in the pure languages of ordered groups and rings.

Other related recent work includes the study of dp- and inp-minimal ordered structures ([Goo] and [Si1]), and the existence of Abelian, solvable and nilpotent definable envelopes of groups definable in NTP_2 theories by Hempel and Onshuus [HO]. The new work by Simon and Walsberg [SW] on dp-minimal structures with tame topology also explores similar ideas, and it would be interesting to see how some of their results might be generalizable to the strong or finite dp-rank context.

We obtain the following general results:

- (1) Fields with finite burden satisfy uniform finiteness (Corollary 2.2).
- (2) In an ordered Abelian group $\langle R; +, <, \dots \rangle$ with a strong theory, no definable discrete $X \subseteq R$ has accumulation points (Corollary 2.13(1)).
- (3) In an ordered field $\langle R; <, +, \dots \rangle$ with a strong theory, there are no infinite definable discrete sets $X \subseteq R$ (Corollary 2.13(2)).
- (4) In a definably complete ordered field $\langle R; <, +, \dots \rangle$ with a strong theory, any nowhere dense definable $X \subseteq R$ is finite (Corollary 2.13(2)–(3)).
- (5) In a densely ordered, strong, definably complete Abelian group $\langle R; <, +, \dots \rangle$, the image of a discrete set under a definable function is always nowhere dense (Corollary 2.17).
- (6) In a strong Archimedean ordered Abelian group $\langle R; <, +, \dots \rangle$, any discrete definable subset $P \subseteq R$ must be a finite union of arithmetic sequences (Theorem 2.18).

We also give (in Section 3) a series of new examples illustrating the richness of the class of ordered Abelian groups with strong theories. Here are some of the main motivating examples:

- (a) Any o-minimal group (which is necessarily Abelian and divisible) is dp-minimal, hence strongly dependent, as is any weakly o-minimal group (see [DGL]).
- (b) An expansion of an o-minimal group by a generic predicate is inp-minimal, hence strong, but the theory has the independence property (see Proposition 3.15 below). Thus dense, codense definable sets are possible in the context of this paper.
- (c) The complete theory of the ordered additive group $\langle \mathbb{R}; +, < \rangle$ expanded with predicates for both \mathbb{Z} and \mathbb{Q} has finite dp-rank (see Proposition 3.1), hence is strongly dependent, so proper discrete subgroups may be definable.
- (d) There are finite dp-rank (hence strongly dependent) expansions of $\langle \mathbb{R}; +, < \rangle$ with definable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ whose graphs are dense in \mathbb{R}^2 (see Proposition 3.11).
- (e) Additive reducts of tame pairs of o-minimal fields have finite dp-rank (see Proposition 3.14). This provides examples in which there are definable discrete subsets P of the universe but no element of P has an immediate successor or predecessor.
- (f) The complete theory of $\langle \mathbb{Z}; +, < \rangle$ (that is, Presburger arithmetic) is dp-minimal. However, if we add a unary predicate P for the set $\{2^k : k \in \mathbb{N}\}$, then the complete theory of the structure is no longer strong. In fact, our results below imply that the only case in which an expansion of this structure by a unary predicate P is strong is when P is already Presburger definable.

1.1. Basic definitions and notation. “Definable” will include sets that are definable over parameters, unless otherwise specified. We always work with a complete first-order, 1-sorted theory T and a large “monster” model $\mathfrak{C} \models T$ from which we pick all of our parameters, where \mathfrak{C} is assumed to be κ -saturated for some κ larger than $|T|^+$ and any set of parameters that we happen to be working over.

The following definitions (due to Adler [Ad]) are fundamental to all that follows.

DEFINITION 1.1. For any cardinal κ , an *inp-pattern of depth κ* (“independent partition”) for a partial type $p(\bar{x})$ is a sequence of pairs $\langle (\varphi_\alpha(\bar{x}; \bar{y}_\alpha), k_\alpha) : \alpha < \kappa \rangle$ where φ_α is a formula, k_α is a positive integer, and there is an array of tuples $\langle \bar{b}_i^\alpha : \alpha < \kappa, i < \omega \rangle$ witnessing it; that is, for each $\alpha < \kappa$, the row $\{\varphi_\alpha(\bar{x}; \bar{b}_i^\alpha) : i < \omega\}$ is k_α -inconsistent, while for every function $\eta : \kappa \rightarrow \omega$, the path $\{\varphi_\alpha(\bar{x}; \bar{b}_{\eta(\alpha)}^\alpha) : \alpha < \kappa\}$ is consistent with $p(\bar{x})$.

DEFINITION 1.2. For any cardinal κ , an *ict-pattern of depth κ* (“independent contradictory types”) for a partial type $p(\bar{x})$ is a sequence of formulae

$\langle \varphi_\alpha(\bar{x}; \bar{y}_\alpha) : \alpha < \kappa \rangle$ such that there is an array of tuples $\langle \bar{b}_i^\alpha : \alpha < \kappa, i < \omega \rangle$ witnessing it; that is, for every function $\eta : \kappa \rightarrow \omega$, the partial type

$$\{\varphi_\alpha(\bar{x}; \bar{b}_{\eta(\alpha)}^\alpha) : \alpha < \kappa\} \cup \{\neg\varphi_\alpha(\bar{x}; \bar{b}_i^\alpha) : i \neq \eta(\alpha)\}$$

is consistent with $p(\bar{x})$.

The maximal cardinal κ (if it exists) such that there is an inp-pattern of depth κ for $p(\bar{x})$ can be thought of as an analogue of the concept of “weight” in stable theories (in [Ad] it is called the *burden* of p). The arrays of parameters witnessing inp- and ict-patterns can always be chosen so that each row is indiscernible over the union of the remaining rows. See [Ad] for this and other basic facts.

DEFINITION 1.3.

- (1) The *burden* (or *inp-rank*) of a theory T is the minimal cardinal κ (if it exists) such that there is no inp-pattern in a single free variable x of depth κ^+ . A theory has *finite burden* if it has burden $\leq n$ for some $n \in \mathbb{N}$, and it is *inp-minimal* if its burden is 1. We say that a structure has finite burden (is inp-minimal, etc.) if its complete theory does.
- (2) The *dp-rank* of a theory T is the minimal cardinal κ (if it exists) such that there is no ict-pattern in a single free variable x of depth κ^+ . A theory has *finite dp-rank* if it has dp-rank $\leq n$ for some $n \in \mathbb{N}$, and it is *dp-minimal* if its dp-rank is 1.
- (3) A theory is *strongly dependent* if every ict-pattern in finitely many variables is finite.
- (4) A theory is *strong* if every inp-pattern in finitely many variables is finite.

Note that “strongly dependent” is equivalent to “strong and NIP”, and every strong theory is NTP_2 . See [KOU] for more basic results, such as the subadditivity of dp-rank. Also note that our definitions of inp-rank and dp-rank differ slightly from those in [Ad]. This difference only manifests itself in the situations where the inp- or dp-rank is infinite, and as in this paper we are concerned with the finite rank cases, the difference is immaterial.

2. General results

2.1. Finite inp-rank fields. We begin with some basic results on fields of finite inp-rank, in particular in this subsection we do not assume that the structures in which we are working are ordered. Let $\mathfrak{F} = \langle F; +, \cdot, \dots \rangle$ be a field F whose theory has finite inp-rank. Notice that we allow for the situation where \mathfrak{F} is a structure in a language richer than the usual language of rings.

LEMMA 2.1. *If $X \subseteq F$ is definable and infinite and $\langle X \rangle$ is the subfield of F generated by X , then F is a finite algebraic extension of $\langle X \rangle$.*

Proof. Suppose not. Then for any $n < \omega$, we can find a collection $\{a_1, \dots, a_n\}$ of elements from F which are $\langle X \rangle$ -linearly independent. We can define a function $g : X^n \rightarrow F$ by $g(x_1, \dots, x_n) = \sum_{i=1}^n x_i a_i$, and by linear independence, g is injective. The existence of such functions g implies that the inp-rank of F is at least n for any n , a contradiction. ■

This leads to the next corollary, which was proved in the special case of dp-minimal fields by Johnson [Joh]:

COROLLARY 2.2. *The structure \mathfrak{F} has uniform finiteness (UF): for any formula $\varphi(x; \bar{y})$, there is a number $n < \omega$ such that whenever $|\varphi(F; \bar{b})| > n$, then $\varphi(F; \bar{b})$ is infinite.*

Proof. If $X_a := \varphi(F; \bar{b})$ is finite, then clearly there is a collection $\{a_i : i < \omega\}$ of elements of F which are linearly independent over the field generated by X_a . If UF failed, then we could use compactness to find an a such that X_a is infinite and a collection of $\{a_i : i < \omega\}$ linearly independent over the field generated by X_a , contradicting Lemma 2.1. ■

We can consider the special case where \mathfrak{F} is an ordered field. Recall the following basic definition (for a discussion see [Mi]):

DEFINITION 2.3. A structure $\langle R; <, \dots \rangle$ modeling the theory of dense linear orderings is called *definably complete* if any definable $X \subseteq R$ which is bounded above [below] has a least upper bound [greatest lower bound].

When \mathfrak{F} is ordered, Corollary 2.2 has a strong consequence (for o-minimal open core see [DMS]):

COROLLARY 2.4. *If \mathfrak{F} is ordered and definably complete, then any model of $\text{Th}(\mathfrak{F})$ has o-minimal open core.*

QUESTION 2.5. *Does UF hold for strongly dependent fields?*

QUESTION 2.6. *Is there a finite dp-rank ordered field with a definable dense and codense subset? In other words, is there a theory T all of whose models have o-minimal open core and such that T has finite dp-rank ?*

By way of contrast, note that an expansion of a real-closed field by a generic unary predicate (as discussed in Section 3.5 below) is an inp-minimal ordered field with a dense codense predicate.

2.2. Ordered Abelian groups. *Here and below, $\mathcal{R} = \langle R; <, +, \dots \rangle$ always denotes an ordered Abelian group.* We consider R as a topological space with the order topology, and R^n as a topological space with the corresponding product topology. Note that some of the topological results below will be trivial in the case of a discretely ordered group such as $\mathcal{R} = \langle \mathbb{Z}; <, + \rangle$, but

everything holds for any strong ordered Abelian group, unless we specifically say that it is dense.

For $x \in R$, we let $|x| = \max(x, -x)$ and define the “distance” $d(x, y)$ between two elements x and y as $|x - y|$. Note that this gives an R -valued metric on R satisfying the usual triangle inequality.

PROPOSITION 2.7. *Suppose that $S \subseteq R$ is definable and infinite, $n, k \in \mathbb{N}$, and for every $g \in R$ there are at most k distinct ways to express g as a sum $s_1 + \dots + s_n$ where each s_i is in S . Then the burden of $x = x$ is at least n .*

Proof. Let $\{a_{i,j} : i \in \omega, 1 \leq j \leq n\}$ be any collection of pairwise distinct elements from S , and let $\varphi(x; a)$ be a formula expressing “ $x - a$ is a sum of $n - 1$ elements from S ”. Then the array $\{\varphi(x; a_{i,j}) : i \in \omega, 1 \leq j \leq n\}$ witnesses that $x = x$ has burden at least n . ■

EXAMPLE 2.8. Any ordered Abelian group with a \mathbb{Q} -independent definable set $S \subseteq R$ does not have finite burden.

EXAMPLE 2.9. Consider the complete theory of $\langle \mathbb{R}; <, +, P \rangle$ where $P = \{2^k : k \in \mathbb{N}\}$. Then P is not \mathbb{Q} -independent, but it satisfies the hypotheses about S in Proposition 2.7 with $k = 1$ (by the uniqueness of representations with binary digits). Thus the theory does not have finite burden.

Our next lemma is a fundamental technical statement we will use to derive several subsequent results.

LEMMA 2.10. *Suppose that there is a family of infinite definable discrete sets D_i and $\varepsilon_i > 0$ for $i \in \mathbb{N}$ such that:*

- (1) $3 \cdot D_{i+1} \subseteq (0, \varepsilon_i)$, where $3 \cdot D_{i+1}$ refers to the set $\{3x : x \in D_{i+1}\}$;
- (2) if $x \in D_i$ then $(x - \varepsilon_i, x + \varepsilon_i) \cap D_i = \{x\}$.

Then \mathcal{R} is not strong.

Proof. Let $Y_0 = D_0$ and inductively define sets Y_n for $n > 0$ as follows:

$$Y_{n+1} = \{x : x = y + d \text{ so that } y \in Y_n \text{ and } d \in D_{n+1}\}.$$

For each $j \in \mathbb{N}$ pick pairwise disjoint open intervals I_i^j for $i \in \mathbb{N}$ so that $I_i^j \cap D_j$ is infinite for all i . Let l_i^j and r_i^j be respectively the left and right endpoints of I_i^j .

We claim that if $a, b \in Y_n$ are distinct then $|a - b| > \varepsilon_n$. We verify this by induction on n . If $n = 0$, this is trivial by construction. Now suppose that $a, b \in Y_{n+1}$. By definition $a = y + d$ and $b = z + e$ where $y, z \in Y_n$ and $d, e \in D_{n+1}$. Suppose that $y \neq z$. By induction $3|y - z| > 3\varepsilon_n$, and by construction $3|d| < \varepsilon_n$ and $3|e| < \varepsilon_n$, so

$$3|a - b| = |3(y - z) + (3d - 3e)| > \varepsilon_n > 3\varepsilon_{n+1}.$$

Hence $y = z$, but then $|a - b| = |d - e| > \varepsilon_{n+1}$ by construction.

Now let $\varphi_j(x; y, z)$ for $j \geq 1$ be the formula

$$\exists w (w \in Y_{j-1} \wedge w < x \wedge y < x - w < z),$$

and consider the formulae $\varphi_j(x; l_i^j, r_i^j)$ for $i \in \mathbb{N}$. We claim that $\varphi_j(x; l_i^j, r_i^j) \wedge \varphi_j(x; l_k^j, r_k^j)$ is inconsistent if $i \neq k$. Otherwise suppose that a realizes the conjunction. We find $y_i, y_k \in Y_{j-1}$ such that $y_i < a$, $a - y_i \in I_i^j$, $y_k < a$, and $a - y_k \in I_k^j$. First notice that we must have $y_i = y_k$: if $y_i < y_k$, then $a - y_i > \varepsilon_{j-1}$, hence $a - y_i \notin I_i^j$ since $\exists x \in (0, \varepsilon_{j-1})$ for every $x \in I_i^j$. Thus $y_i = y_k$, but then $a - y_i$ must lie in both I_i^j and I_k^j , a contradiction.

Now let $\eta : \mathbb{N} \setminus \{0\} \rightarrow \mathbb{N}$. We claim that the type

$$\Gamma(x) = \bigwedge_{j \in \mathbb{N}} \varphi_j(x; l_{\eta(j)}^j, r_{\eta(j)}^j)$$

is consistent. Let $y_0 \in Y_0$ and let J_0 be the open interval $(y_0 + l_{\eta(1)}^1, y_0 + r_{\eta(1)}^1)$. Choose $d \in D_1 \cap I_{\eta(1)}^1$ so that d is neither the smallest nor the largest element in this intersection (recall that the intersection is assumed to be infinite), and set $y_1 = y_0 + d$. Hence $y_1 \in Y_1 \cap J_0$. Now let $J_1 = (y_1 + l_{\eta(2)}^2, y_1 + r_{\eta(2)}^2)$. We claim that $J_1 \subseteq J_0$. As d was neither the smallest nor the largest element in $D_1 \cap I_{\eta(1)}^1$, we can find $d_0 < d$ and $d_1 > d$ with $d_i \in D_1$ such that $y_0 + d_0$ and $y_0 + d_1$ both lie in $Y_1 \cap J_0$. Hence $(y_0 + d_0, y_0 + d_1) \subseteq J_0$. But both $3l_{\eta(2)}^2$ and $3r_{\eta(2)}^2$ are less than ε_1 , and so $J_1 \subseteq (y_0 + d_0, y_0 + d_1) \subseteq J_0$, as desired. Now choose $d \in D_2 \cap I_{\eta(2)}^2$ and set $y_2 = y_1 + d$. Continuing in this manner we construct a sequence of open intervals J_i for $i \in \mathbb{N}$ such that $J_{i+1} \subseteq J_i$. Finally, notice that if $a \in \bigcap_{i \in \mathbb{N}} J_i$ then a realizes $\Gamma(x)$.

Thus we have constructed an inp-pattern of depth \aleph_0 , and so the theory is not strong. ■

Our previous lemma allows us to derive the following theorem which severely constrains the type of discrete sets that are definable in a strong structure \mathcal{R} .

THEOREM 2.11. *Suppose that $\mathcal{R} = \langle R; +, <, \dots \rangle$ is a densely ordered Abelian group. Let $\mathfrak{C} \models \text{Th}(\mathcal{R})$ be a saturated model and suppose that for every $\varepsilon > 0$ there is an infinite definable discrete set X such that $X \subseteq (0, \varepsilon)$. Then $\text{Th}(\mathcal{R})$ is not strong.*

Proof. Given \mathcal{R} as in the hypothesis, we simply have to construct a family of definable sets D_i as in Lemma 2.10. We do so by induction on \mathbb{N} . Let D be any definable infinite discrete set and without loss of generality assume that $D \subseteq (0, \infty)$. Let $\varepsilon > 0$ be such that there are infinitely many $x \in D$ with $(x - \varepsilon, x + \varepsilon) \cap D = \{x\}$. Let $D_0 = \{x \in D : (x - \varepsilon, x + \varepsilon) \cap D = \{x\}\}$. Let $\varepsilon_0 = \varepsilon$.

Suppose we have constructed D_n and ε_n . Let $\varepsilon' \in R$ be such that $0 < 3\varepsilon' < \varepsilon_n$ and let D be an infinite definable discrete subset of $(0, \varepsilon')$. Pick $\varepsilon \in (0, \varepsilon')$ so that there are infinitely many $x \in D$ with $(x - \varepsilon, x + \varepsilon) \cap D = \{x\}$. Let $D_{n+1} = \{x \in D : (x - \varepsilon, x + \varepsilon) \cap D = \{x\}\}$ and $\varepsilon_{n+1} = \varepsilon$. ■

Next we will prove some topological tameness properties of unary sets definable in strong densely ordered Abelian groups ⁽¹⁾.

If $S \subseteq R$, define S' to be the usual Cantor–Bendixson derivative, that is,

$$S' = \{s \in R : \text{whenever } a < s < b, |(a, b) \cap S| > 1\}.$$

Note that if S is not closed then S' is not a subset of S . Define the set of *limit points of S from the left* to be

$$S'_L = \{x \in R : \text{for every } a < x, (a, x) \cap S \neq \emptyset\},$$

and similarly define the set S'_R of *limit points of S from the right*. A point x is *left-isolated* (or *right-isolated*) if $x \in S \setminus S'_L$ (respectively, $x \in S \setminus S'_R$).

COROLLARY 2.12. *Suppose that $\mathcal{R} = \langle R; +, <, \dots \rangle$ is a densely ordered Abelian group with a strong theory and $S \subseteq R$ is definable. Then:*

- (1) *No $x \in R$ is a limit point of $S \setminus S'$.*
- (2) *No $x \in R$ is a limit point of $S \setminus S'_R$ from the right.*
- (3) *No $x \in R$ is a limit point of $S \setminus S'_L$ from the left.*

Proof. We only write the proof of (2), since (1) follows by a similar reasoning, and (3) follows from (2) applied to $-S$. Suppose the contrary: $x \in R$ is a limit point of $S \setminus S'_R$ from the right. Without loss of generality, $x = 0$ (by translating S). Now for any $\epsilon > 0$, there are infinitely many points of $S \setminus S'_R$ in $(0, \epsilon)$, and so by compactness there must be some $\delta \in R$ with $\delta > 0$ such that

$$\{a \in S : (a, a + \delta) = \emptyset\} \cap (0, \epsilon)$$

is infinite. But the set above is discrete, so we have contradicted Theorem 2.11. ■

In particular, the above corollary implies that in a structure \mathcal{R} with strong theory, for no definable set S can S' be finite and nonempty, since otherwise each point in S' would be a limit point of $S \setminus S'$.

We summarize several further corollaries of our previous results.

COROLLARY 2.13. *Let $\mathcal{R} = \langle R; +, <, \dots \rangle$ be a densely ordered Abelian group and suppose that $\text{Th}(\mathcal{R})$ is strong. Let \mathfrak{C} be a big model of $\text{Th}(\mathcal{R})$. Then:*

⁽¹⁾ Technically speaking, these results are true in any strong ordered Abelian group, but when the ordering is discrete they are completely trivial since every point is isolated.

- (1) If X is an infinite definable discrete set in \mathfrak{C} then X has no accumulation point in \mathfrak{C} . In particular X must be closed.
- (2) If \mathcal{R} is an expansion of an ordered field then in \mathfrak{C} there are no infinite definable discrete sets.
- (3) If \mathcal{R} is definably complete and $X \subseteq \mathfrak{C}$ is definable and nowhere dense then X is discrete.

Proof. (1) is an immediate consequence of Lemma 2.10. For (2), let \mathfrak{C} have field structure and suppose that $D \subseteq C$ is discrete and definable. Without loss of generality assume that $D \subseteq C^{>0}$. If D is unbounded then the set $D^{-1} = \{x^{-1} : x \in D\}$ is discrete and has 0 as an accumulation point, violating (1). Hence D is bounded. Say that $\Delta > d$ for all $d \in D$. But then for any $\varepsilon > 0$ the set $D_\varepsilon := \{(\varepsilon/\Delta)d : d \in D\}$ is a discrete subset of $(0, \varepsilon)$. Hence by Theorem 2.11, D must be finite.

For (3), suppose that $X \subset C$ is infinite, nowhere dense, and not discrete. Without loss of generality, X is closed. Thus $C \setminus X$ is open, and so by definable completeness consists of an infinite family of open intervals. Let Y be the set of all midpoints of intervals in $C \setminus X$, which we easily see is definable. But then Y is discrete and infinite; as moreover X is not discrete, Y must have an accumulation point, violating (1). ■

The following corollary provides a counterpoint to Theorem 2.11, namely we cannot have definable discrete sets which are too “spread out”.

COROLLARY 2.14. *Suppose that $\mathcal{R} = \langle R; +, <, \dots \rangle$ is any ordered Abelian group and that $T = \text{Th}(\mathcal{R})$ is strong. Let $D \subseteq R$ be an infinite definable discrete set which is not bounded above. For any $\delta > 0$, let*

$$D(\delta) := \{x \in D : (x - \delta, x + \delta) \cap D = \{x\}\}.$$

Then there is some positive $\delta \in R$ such that $D(\delta) = \emptyset$.

Proof. Otherwise for any $\delta \in R$ the set $D(\delta)$ is infinite. We will show that, in some nonstandard model, there is a family of sets of the form $D_i = D(\varepsilon_i) \cap I_i$ which satisfy the hypotheses of Lemma 2.10, where I_i is the open interval (ℓ_i, r_i) .

We do so by choosing all of the parameters ℓ_i , r_i , and ε_i using compactness. It is enough to show that the following partial type Γ is consistent:

- (i) $(3\ell_{i+1}, 3r_{i+1}) \subseteq (0, \varepsilon_i)$;
- (ii) $0 < 3\varepsilon_{i+1} < \varepsilon_i$; and
- (iii) for every $n \in \mathbb{N}$, $|D(\varepsilon_i) \cap (\ell_i, r_i)| \geq n$.

We claim that any finite $\Gamma_0 \subseteq \Gamma$ is satisfied by values of ℓ_i , r_i , and ε_i in the model \mathcal{R} , and conclude by compactness. To see this, first pick a sufficiently large $N \in \mathbb{N}$ such that all the ℓ_i , r_i , and ε_i mentioned in Γ_0 satisfy $i \in \{1, \dots, N\}$ and also N bounds any n from a formula of type (iii) in Γ_0 . Pick

$\varepsilon_N > 0$ arbitrarily, then pick ℓ_N, r_N positive and sufficiently far apart that $|D(\varepsilon_N) \cap (\ell_N, r_N)| \geq N$. Then continue by reverse induction on i : once we have picked ℓ_{i+1}, r_{i+1} , and ε_{i+1} (for $i \geq 0$), first pick $\varepsilon_i > \max(3\varepsilon_{i+1}, 3r_{i+1})$, then pick ℓ_i and r_i far enough apart that $|D(\varepsilon_i) \cap (\ell_i, r_i)| \geq N$. ■

COROLLARY 2.15. *The theory of the structure $\langle \mathbb{R}; +, <, 2^{\mathbb{Z}} \rangle$ is not strong, and the theory of the structure $\langle \mathbb{Z}; +, <, 2^{\mathbb{N}} \rangle$ is not strong.*

The following proposition can be thought of as a version of the definable Baire property (for which see [FS]).

PROPOSITION 2.16. *Suppose that $\mathcal{R} = \langle R; +, <, \dots \rangle$ is a densely ordered Abelian group such that $\text{Th}(\mathcal{R})$ is strong. One cannot find an interval I and a definable family $D(\delta)$ for $\delta \in I$ such that each $D(\delta)$ is discrete, if $\delta_1 < \delta_2 \in I$ then $D(\delta_1) \subseteq D(\delta_2)$, and $D = \bigcup_{\delta \in I} D(\delta)$ is somewhere dense.*

Proof. This follows readily from Lemma 2.10. Suppose that there were a counterexample $D(\delta)$ with $\delta \in I$ such that D is dense in an interval J . We build a family D_n, ε_n violating Lemma 2.10. By compactness choose $\delta_0 \in I$ such that $D(\delta_0) \cap J$ is infinite. Pick ε_0 such that

$$D_0 := \{x \in D(\delta_0) \cap J : (x - \varepsilon_0, x + \varepsilon_0) \cap D(\delta_0) = \{x\}\}$$

is infinite. Thus we have D_0 and ε_0 . Next pick $a \in D_0 \cap J$ so that $(a, a + \varepsilon_0) \subseteq J$. As D is dense in J , we may find $\delta_1 \in I$ such that $3 \cdot D(\delta_1) \cap (a, a + \varepsilon_0)$ is infinite. Pick ε_1 so that

$$D' := \{x \in 3 \cdot D(\delta_1) \cap (a, a + \varepsilon_0) : (x - \varepsilon_1, x + \varepsilon_1) \cap D(\delta_1) = \{x\}\}$$

is infinite. Let $D_1 = \{x - a : x \in D'\}$. Continue in this manner to construct a sequence D_n, ε_n for $n \in \omega$ violating Lemma 2.10. ■

This form of the Baire property has the following corollary, indicating that the image of a discrete set under a definable function must be nowhere dense. This should be compared to results from, for example, [Hie] where having a discrete set whose image under a definable function is somewhere dense strongly indicates the structure in question is “wild”.

COROLLARY 2.17. *Let $\mathcal{R} = \langle R; +, <, \dots \rangle$. Suppose that $\text{Th}(\mathcal{R})$ is strong, densely ordered, and satisfies DC. If P is a definable discrete set and $f : R^n \rightarrow R$ is a definable function then $f[P^n]$ is discrete.*

Proof. By Corollary 2.13(3) it suffices to show that $f[P^n]$ is nowhere dense. Suppose this fails. By Corollary 2.13(1), P is closed. Without loss of generality we may assume that $f[P^n] \subseteq I$ for an interval I and that $f[P^n]$ is dense in I . For convenience, we may assume that $0 \in P$.

We proceed by induction on n . Suppose that $n = 1$. For $\delta \in R^{>0}$ let $P(\delta) = P \cap (-\delta, \delta)$. Thus $f[P] = \bigcup_{\delta \in R^{>0}} f[P(\delta)]$. As P is closed and discrete with least element 0, and $\text{Th}(\mathcal{R})$ is definably complete, $P(\delta)$ must be finite

and nonempty for small enough δ . In particular $f[P(\delta)]$ is discrete for all small enough δ . By Proposition 2.16 and Corollary 2.13, $f[P(\delta)]$ must be dense in I for all sufficiently large δ . Thus by definable completeness we may find $\Delta \in R$, the supremum of all $\delta \in R$, such that $f[P(\delta)]$ is discrete. Note that as P is closed, $\Delta \in P$ or $-\Delta \in P$. Hence $f[P(\Delta)]$ is discrete but $f[P(\Delta)] \cup \{f(\Delta)\} \cup \{f(-\Delta)\}$ is somewhere dense in I , which is clearly absurd.

Now suppose we have $f[P^{n+1}]$ dense in I . For $\delta \in R^{>0}$ let $P(\delta) = P \cap (-\delta, \delta)^{n+1}$. As in the previous case we find $\Delta \in R^{>0}$ which is the supremum of all δ such that $f[P(\delta)]$ is discrete, and note that $\Delta \in P$ or $-\Delta \in P$. For $1 \leq i \leq n + 1$ let P_i^+ be the set

$$\{\bar{a} \in P^{n+1} : -\Delta \leq a_j \leq \Delta \text{ for all } 1 \leq j \leq n + 1 \text{ and } a_i = \Delta\}$$

and let P_i^- be defined analogously with Δ replaced by $-\Delta$. Thus $f[P(\Delta)^{n+1}]$ is discrete, but

$$f[P(\Delta)^{n+1}] \cup \bigcup_{1 \leq i \leq n+1} f[P_i^+] \cup \bigcup_{1 \leq i \leq n+1} f[P_i^-]$$

is somewhere dense in I . Thus $f[P_{i^*}^\circ]$ is somewhere dense for some i^* and $\circ \in \{+, -\}$; for simplicity assume that $i^* = 1$ and $\circ = +$. But then setting $\tilde{f} : R^n \rightarrow R$ to be the function $\tilde{f}(x_1, \dots, x_n) = f(\Delta, x_1, \dots, x_n)$ we conclude that $\tilde{f}[P^n]$ is somewhere dense, and we are done by induction. ■

2.3. Discrete definable sets in Archimedean ordered groups. We continue the study of definable discrete unary sets in strong ordered Abelian groups $\mathcal{R} = \langle R; +, <, \dots \rangle$, focusing on the case where there is a model R of the theory which is *Archimedean*: that is, for any two $a, b \in R^{>0}$, there is some $n \in \mathbb{N}$ such that $na > b$ and $nb > a$. The goal will be the following theorem, which will be proved by a series of lemmas:

THEOREM 2.18. *Suppose that $\mathcal{R} = \langle R; +, <, \dots \rangle$ is an ordered Abelian group whose complete theory is strong and whose universe R is Archimedean, and let $P \subseteq R$ be definable, infinite, and discrete. Then:*

- (1) *The set ΔP of all differences $x - y$ such that $x, y \in P$ and x is the immediate successor of y in P is finite.*
- (2) *$P = F \cup X_1 \cup \dots \cup X_k$ where F is finite and each X_i is an infinite \mathbb{N} -indexed arithmetic progression (i.e. a set of the form $\{a + b \cdot i : i \in \mathbb{N}\}$ for some fixed $a, b \in R$).*
- (3) *The X_i above may be taken to be “commensurable” in the sense that there exists a single \mathbb{Z} -indexed arithmetic progression Y containing each X_i as a subsequence.*
- (4) *If $f : P \rightarrow R$ is definable, then the image of f is discrete.*

REMARK 2.19. Part (1) is an easy corollary of part (2), but we give it special mention because it will still be true even in “nonstandard” models of $\text{Th}(\mathcal{R})$ in which the universe is not Archimedean, whereas parts (2) and (3) which mention arithmetic sequences do not have clear analogues for non-Archimedean structures. Part (4) is also true in nonstandard models.

Note that some of the progressions X_i in part (2) may tend to $-\infty$ while others tend to ∞ , which is why in part (3) we require a progression of the form $Y = \{a + b \cdot i : i \in \mathbb{Z}\}$.

In the special case when $R = \mathbb{Z}$, Theorem 2.18 is reminiscent of Szemerédi’s Theorem from combinatorics (see [Sz]). Indeed, if $P \subseteq \mathbb{Z}$ is infinite and definable in a strong theory, then Corollary 2.14 above implies that P has positive upper density, and so for every $k \in \mathbb{N}$, the set P must contain an arithmetic sequence of length k . However, conclusion (2) of Theorem 2.18 is much stronger than this (and derived from stronger hypotheses), and our proof is independent of Szemerédi’s result.

We also note in passing the following generalization of [AD⁺, Proposition 6.6], which states that no proper expansion of $\langle \mathbb{Z}; <, + \rangle$ is dp-minimal.

COROLLARY 2.20. *If $m \in \mathbb{N}$, $R \subseteq \mathbb{Z}^m$, and $\text{Th}(\mathbb{Z}; <, +, R)$ is strong, then R is definable in $\text{Th}(\mathbb{Z}; <, +)$.*

Proof. By [MV, Theorem 5.1], if $R \subseteq \mathbb{Z}^m$ is not definable in $\langle \mathbb{Z}; <, + \rangle$, then there is some $R' \subseteq \mathbb{Z}$ which is definable in $\langle \mathbb{Z}; <, +, R \rangle$ but not in $\langle \mathbb{Z}; <, + \rangle$. Now apply Theorem 2.18. ■

QUESTION 2.21. *Suppose that we omit the hypothesis in Theorem 2.18 that R is Archimedean, but add the hypothesis that every element of P has both an immediate predecessor and an immediate successor (unless it is the minimum or the maximum of P). Are conclusions (1) and (4) still true?*

Note that it is sufficient to prove Theorem 2.18 for the case in which every element of P is positive (by dividing P into its positive and negative parts). So from now until the end of the proof of Theorem 2.18, we fix some infinite discrete $P \subseteq R$ which is definable in some strong structure on R which is an Archimedean ordered Abelian group.

We will prove Theorem 2.18 in a series of lemmas: first part (1) is Corollary 2.29, then (2) and (3) follow from Corollary 2.30 and Proposition 2.35, and finally (4) is Proposition 2.36.

From now until the end of this section, we always work in the fixed Archimedean model \mathcal{R} . This means that we use “ P ” to mean the interpretation of P in the universe R (not in some saturated extension), and likewise all other definable sets mentioned below are, by abuse of notation, identified with their interpretations in \mathcal{R} .

We record an elementary fact which will be used later:

FACT 2.22. If $F \subseteq R^{>0}$ is finite, then the set of all finite sums of the form

$$n_1 f_1 + \cdots + n_k f_k$$

where $n_i \in \mathbb{N}$ and $f_i \in F$ is discrete.

LEMMA 2.23. P has order type ω .

Proof. If \mathcal{R} is discretely ordered, then this follows immediately from the fact that \mathcal{R} is Archimedean. So assume \mathcal{R} is densely ordered. Since \mathcal{R} is Archimedean, we may assume that $(R, <, +)$ is embedded in $(\mathbb{R}, <, +)$ as an ordered subgroup.

Since $P \subseteq \mathbb{R}^{>0}$ is infinite, there is a subset $P_0 \subseteq P$ which is order-isomorphic to ω or to ω^* , the reverse order type of ω . Suppose that $(P_0, <) \cong \omega^*$. Then since P_0 is bounded below by 0, we have $\alpha = \inf(P_0) \in \mathbb{R}$. For any $\epsilon > 0$ and $a \in R$ we can define

$$X_{a,\epsilon} := \{|a - x| : x \in P \text{ and } |a - x| < \epsilon\}.$$

Then $X_{a,\epsilon} \subseteq (0, \epsilon)$, and by choosing $a \in R$ sufficiently close to α we can ensure that $X_{a,\epsilon}$ is infinite; but this contradicts Theorem 2.11.

So there is some $P_0 \subseteq P$ which is order-isomorphic to ω . If P_0 were not cofinal in R , then repeating the same argument as above with $\alpha = \sup(P_0) \in \mathbb{R}$, we would obtain a contradiction, so P_0 is cofinal in R and hence in P . Similarly, there can only be finitely many elements of P between two consecutive elements of P_0 (or between the first element of P_0 and 0), since otherwise we would have a bounded subset $P_{00} \subseteq P$ order-isomorphic to ω or ω^* , yielding the same contradiction as before. ■

DEFINITION 2.24. Let $\{a_i : i \in \mathbb{N}\}$ be an enumeration of P in increasing order.

- (1) Let $s_P : P \rightarrow P$ be the function $s_P(a_i) = a_{i+1}$.
- (2) Given any $f : P \rightarrow R$, let $\Delta f : P \rightarrow R$ be the function $(\Delta f)(x) = f(s_P(x)) - f(x)$.
- (3) $\Delta P = \{s_P(x) - x : x \in P\}$.

In what follows, we will apply the same notation $s_{P'}$ and $\Delta_{P'}$ to other infinite discrete definable sets $P' \subseteq R^{>0}$, which is warranted in light of Lemma 2.23.

LEMMA 2.25. ΔP is bounded.

Proof. Assume that ΔP is unbounded.

CLAIM 2.26. There is some infinite definable $P' \subseteq P$ such that

$$\lim_{n \rightarrow \infty} \Delta_{s_{P'}(n)} = \infty.$$

Proof. Let

$$P' = \{x \in P : \forall y \in P [y < x \Rightarrow s_P(y) - y < s_P(x) - x]\}.$$

Then $\Delta s_{P'}(x) \geq \max\{\Delta s_P(y) : y \leq x\}$ and the desired property follows. ■

Replacing the original P with P' as in the claim, without loss of generality we have $\lim_{n \rightarrow \infty} \Delta s_P(n) = \infty$. But this immediately contradicts Corollary 2.14 above. ■

LEMMA 2.27. *Suppose that $\{P_a : a \in X\}$ is an infinite definable family of subsets P_a of P , where $X \subseteq R^m$. Then there is a $K \in R$ such that for every $a \in X$, if P_a is infinite then $\limsup \Delta P_a \leq K$.*

Proof. Suppose towards a contradiction that there is no such K . Then for any $K \in R$, there is always some $a \in X$ such that the set

$$P_{a,K} := \{x \in P_a : s_{P_a}(x) - x > K\}$$

is infinite, and clearly every element of $\Delta P_{a,K}$ is greater than K .

Then we will contradict Lemma 2.10 by finding, in some elementary extension of \mathcal{R} , elements $\{a_i \in X : i \in \mathbb{N}\}$ and $K_i > 0$ such that:

- (i) for every $n \in \mathbb{N}$, $|P_{a_0, K_0}| \geq n$; and
- (ii) for every $n \in \mathbb{N}$, $|(0, K_i) \cap 3 \cdot P_{a_{i+1}, K_{i+1}}| \geq n$.

(Note that for $D_i = P_{a_i, K_i}$ and $\varepsilon_i = K_i$, condition (2) of Lemma 2.10 is automatic.)

Let Γ be the type expressing these properties of the parameters a_i, K_i , and let $\Gamma_0 \subseteq \Gamma$ be a finite subset which only mentions parameters with $i \leq N$ and such that N also bounds all the n from formulae of type (ii) in Γ_0 . Start by picking $K_N > 0$ arbitrarily and then $a_N \in X$ such that P_{a_N, K_N} is infinite. Given a_{i+1} and K_{i+1} for $i > 0$ such that $P_{a_{i+1}, K_{i+1}}$ is infinite, first pick K_i large enough that $|(0, K_i) \cap 3 \cdot P_{a_{i+1}, K_{i+1}}| \geq N$, then pick $a_i \in X$ such that P_{a_i, K_i} is infinite. ■

LEMMA 2.28. ΔP has finitely many limit points.

Proof. Suppose not, and let L be the set of all limit points of ΔP . If 0 is the unique limit point of L , then we immediately contradict Corollary 2.12.

Otherwise, either L has no limit points and is unbounded in R , or else L has a limit point greater than 0; in either case, we can choose an $\epsilon > 0$ in R and an infinite collection $S = \{a_i : i \in \mathbb{N}\}$ of limit points of ΔP such that $\inf(S) \geq \epsilon$ (S need not be definable). Then we pick a pairwise disjoint collection $\{(\ell_i, r_i) : i \in \mathbb{N}\}$ of intervals such that $a_i \in (\ell_i, r_i)$ and $\ell_i > \epsilon/2$ for every i . Let

$$P_{\ell_i, r_i} = \{x \in P : \ell_i < s_P(x) - x < r_i\}.$$

The sets P_{ℓ_i, r_i} are definable, pairwise disjoint, and infinite.

By Lemma 2.27, there is some $K \in R$ such that $\limsup \Delta P_{\ell_i, r_i} < K$ for every i . Pick $n \in \mathbb{N}$ large enough that $n \cdot (\epsilon/2) > K$ (here we are using the assumption that we are working in a fixed Archimedean model). By our assumptions, for each $i \in \mathbb{N}$,

$$P_{\ell_i, r_i} \cap [j \cdot K, (j + 1) \cdot K] \neq \emptyset$$

holds for all but finitely many $j \in \mathbb{N}$. Therefore there is some j such that more than n of the sets P_{ℓ_i, r_i} intersect the interval $[j \cdot K, (j + 1) \cdot K]$, which has length K . But any two distinct elements of $\bigcup_{i \in \mathbb{N}} P_{\ell_i, r_i}$ are a distance of at least $\epsilon/2$ apart, so this is absurd. ■

COROLLARY 2.29. ΔP is finite.

Proof. We already know by Lemma 2.25 that ΔP is bounded, so it suffices to show that it has no limit points. By Lemma 2.28, the only case to consider is when ΔP has a finite, nonzero number of limit points. But then each of these limit points would be a limit point of $\Delta P \setminus (\Delta P)'$, contradicting Corollary 2.12. ■

COROLLARY 2.30. Say X_1, X_2 are two infinite arithmetical sequences contained in P (not necessarily definable). Then X_1 and X_2 are commensurable: there is a single arithmetic sequence Y which contains both X_1 and X_2 as subsequences.

Proof. Otherwise, for every $\epsilon > 0$, the set ΔP would contain an element in $(0, \epsilon)$, contradicting Corollary 2.29. ■

DEFINITION 2.31. Given a finite sequence $\sigma = \langle c_1, \dots, c_n \rangle$ of elements of ΔP , we define

$$P_\sigma = \{a \in P : \forall i [1 \leq i < n \Rightarrow s_P^i(a) - s_P^{i-1}(a) = c_i]\}.$$

We say that σ is *infinitely recurring* if P_σ is infinite.

LEMMA 2.32. There is some fixed $m \in \mathbb{N}$ such that for any sufficiently large $n \in \mathbb{N}$, there are precisely m infinitely recurring sequences of length n .

Proof. It is enough to show that there is some fixed m such that for any $n \in \mathbb{N}$, there are at most m sequences of length n such that P_σ is infinite: this is because any $\sigma = \langle c_1, \dots, c_n \rangle$ such that P_σ is infinite extends to at least one $\sigma' = \langle c_1, \dots, c_{n+1} \rangle$ of length $n + 1$ such that $P_{\sigma'}$ is infinite (since ΔP is finite and by the pigeonhole principle).

The general idea is to apply Lemma 2.27 above to the sets P_σ , but we cannot do this directly since they may not constitute a single uniformly definable family.

So for $(b, c), (b', c') \in P^2$, say $(b, c) \sim (b', c')$ if the interval $[b, c]$ is “isomorphic via translation” to $[b', c']$, that is, $c' - b' = c - b$ and

$$\forall x \in [b, c] (x \in P \Leftrightarrow x + b' - b \in P).$$

Then let

$$P_{(b,c)} = \{a \in P : (b, c) \sim (a, a + c - b)\}.$$

Note that for any finite sequence σ from ΔP , the set P_σ equals $P_{(b,c)}$ for some $(b, c) \in P^2$.

By Lemma 2.27, there is a $K \in R$ such that for every $(b, c) \in P^2$, if $P_{(b,c)}$ is infinite then

$$\limsup \Delta P_{(b,c)} \leq K.$$

If $m \in \mathbb{N}$ and there are at least m distinct sequences $\sigma_1, \dots, \sigma_m$ of length n such that the sets P_{σ_i} are all infinite, then the sets $P_{\sigma_1}, \dots, P_{\sigma_m}$ are pairwise disjoint, and it follows that if $k = \min \Delta P$, then there must be some σ_i such that $\limsup \Delta P_{\sigma_i} \geq m \cdot k$. Therefore $k \cdot m \leq K$, yielding a finite bound on m since k and K are from the Archimedean structure \mathcal{R} . ■

Now use the previous lemma to fix $n, m \in \mathbb{N}$ such that for every $n' \geq n$, there are precisely m distinct infinitely recurring sequences of length n' . Also, let $\sigma_1, \dots, \sigma_m$ list all of the infinitely recurring sequences of length n .

LEMMA 2.33. *For any infinitely recurring sequence σ from ΔP of length n , there is a unique infinitely recurring sequence σ' of length $n + 1$ such that σ' extends σ .*

Proof. The existence of some such infinitely recurring extension σ' is immediate from the finiteness of ΔP and the pigeonhole principle. If we choose σ'_i to be some infinitely recurring extension of σ_i of length $n + 1$, then $\sigma'_1, \dots, \sigma'_m$ are m distinct infinitely recurring sequences of length $n + 1$, so by the choice of m and n , they must list every infinitely recurring sequence of length $n + 1$. This establishes the uniqueness of σ'_i . ■

DEFINITION 2.34. Given an infinitely recurring sequence $\sigma = \langle c_1, \dots, c_n \rangle$ of length n , let $\sigma' = \langle c_1, \dots, c_{n+1} \rangle$ be the extension provided by Lemma 2.33. Then we define

$$\sigma^+ = \langle c_2, \dots, c_{n+1} \rangle,$$

which is another infinitely recurring sequence of length n .

Now if we recursively define a sequence τ_1, τ_2, \dots such that $\tau_1 = \sigma_1$ and $\tau_{i+1} = \tau_i^+$, then the fact that there are only finitely many infinitely recurring sequences of length n implies that $\{\tau_i : i \in \omega\}$ is *eventually periodic*: there is some ℓ (the period) and some N such that $d_{i+\ell} = d_i$ for every $i \geq N$. If $\tau_i, \dots, \tau_{i+\ell} = \tau_i$ is a cycle of length ℓ , then starting from any sufficiently large a realizing P_{τ_i} , a routine inductive argument implies that any $a' \in P$ with $a' \geq a$ must realize one of the predicates $P_{\tau_i}, \dots, P_{\tau_{i+\ell-1}}$. It follows from the definition of m that $\ell = m$ and that the sequence τ_1, τ_2, \dots is in fact m -periodic.

By the previous paragraph, we may assume that $\sigma_i^+ = \sigma_{i+1}$ if $i < m$ and $\sigma_m^+ = \sigma_1$.

PROPOSITION 2.35. *If $d_i = a_{i+1} - a_i$, then the sequence d_0, d_1, \dots is eventually m -periodic. Thus P is the union of some finite set and finitely many infinite arithmetic sequences, establishing part (2) of Theorem 2.18.*

Proof. As observed in the proof of Lemma 2.33, $\sigma'_1, \dots, \sigma'_m$ must enumerate all the infinitely recurring sequences of length $n + 1$. This plus the fact that there are only finitely many sequences from ΔP of length n or $n + 1$ implies that we can choose an $N \in \omega$ such that for every $i \geq N$,

- (1) a_i realizes one of the predicates $P_{\sigma_1}, \dots, P_{\sigma_m}$, and
- (2) a_i realizes one of the predicates $P_{\sigma'_1}, \dots, P_{\sigma'_m}$.

So if $i \geq N$ and a_i realizes P_{σ_j} , then a_{i+1} realizes $P_{\sigma_j^+}$ and σ_j^+ is either σ_{j+1} (if $j < m$) or σ_1 (if $j = m$). This implies that $\{d_i : i \in \omega\}$ is m -periodic for $i \geq N$. ■

Finally, we establish part (4) of Theorem 2.18:

PROPOSITION 2.36. *If $P \subseteq R^{>0}$ is definable, infinite, and discrete, then for any definable $f : P \rightarrow R$, the image $f(P)$ is also discrete.*

Proof. Suppose $f(P)$ is not discrete. Then the derivative $(f(P))'$ is non-empty, and by Corollary 2.12, $(f(P))'$ must be infinite, since otherwise each of its finitely many points would be a limit point of $f(P) \setminus (f(P))'$. Therefore we can choose an infinite collection $\{I_i : i \in \mathbb{N}\}$ of pairwise disjoint intervals such that $f(P) \cap I_i$ is infinite for each i . Let $P_i = f^{-1}[I_i]$, which is a definable family of infinite, pairwise disjoint subsets of P .

Let $P = \{a_i : i < \omega\}$ be the enumeration of P in increasing order, and set $\tilde{P}_j = \{i \in \mathbb{N} : a_i \in P_j\}$. For every positive integer n , all but finitely many of the sets \tilde{P}_j have upper density less than $1/n$. But if $k = \min \Delta P$ (which exists by Corollary 2.29), then this implies that $\limsup \Delta P_j \geq n \cdot k$ for all but finitely many $j \in \mathbb{N}$, contradicting Lemma 2.27. ■

REMARK 2.37. The proof of Theorem 2.18(2) can clearly be applied to any unary definable set $P \subseteq R$ in a strong ordered Abelian group $\langle R; <, +, \dots \rangle$ such that $P \cap R^{>0}$ has an initial segment $P_0 = \{a_i : i < \omega\}$ of order type ω , and such that there is a single Archimedean class C which contains every a_i and every $a_{i+1} - a_i$; in this case, the conclusion is that P_0 is a finite union of arithmetic progressions. In fact, it is enough just to assume that every $a_{i+1} - a_i$ is in the same Archimedean class C , since (working via translations) we may as well assume that every a_i is in C .

We note in passing that in algebra there is a theorem by Conrad (see [Co]) which says that any Archimedean left-ordered group (that is, a group equipped with an ordering which is invariant under left translations) embeds as an ordered group into $\langle \mathbb{R}; +, < \rangle$, and hence is Abelian. Therefore Theorem 2.18

applies to any Archimedean left-ordered group with a strong theory. There do seem to be examples of dp-rank 2 left-ordered non-Abelian groups, such as the Klein bottle group, which we may discuss in a future paper, and it may be interesting to investigate whether Theorem 2.18 can be generalized to left-ordered groups with strong theories.

3. Examples. In this section we provide a series of examples that illustrate the large variety of possibilities for structures of finite inp-rank or dp-rank.

3.1. $\text{Th}(\langle \mathbb{R}; +, <, 0, 1, \mathbb{Z}, \mathbb{Q} \rangle)$. Corollary 2.13 states that in a strong densely ordered expansion of an Abelian group there are no definable infinite discrete sets with an accumulation point. In this subsection we show by example that it is possible to have such a structure with an infinite definable discrete set, and furthermore that this is possible in a definably complete structure. Also we would like to have an example of a finite dp-rank structure with a definable dense and codense subset. We show that in fact we may have both types of sets simultaneously in a finite dp-rank structure.

We show:

PROPOSITION 3.1. $\text{Th}(\langle \mathbb{R}; +, <, \mathbb{Z}, \mathbb{Q} \rangle)$ has dp-rank 3.

Let $\mathcal{L} = \{+, <, 0, 1, Z, Q, \lfloor \cdot \rfloor, \lambda\}_{\lambda \in \mathbb{Q}}$ where Z, Q are unary relations and $\lfloor \cdot \rfloor$ and the λ 's are unary functions. Consider the \mathcal{L} -structure $\mathcal{R} = \langle \mathbb{R}; +, <, 0, 1, \mathbb{Z}, \mathbb{Q}, \lfloor \cdot \rfloor, \lambda \rangle_{\lambda \in \mathbb{Q}}$ where the λ are unary functions for multiplication by a rational and $\lfloor \cdot \rfloor$ is the integer part function ($\lfloor x \rfloor$ is the greatest integer less than or equal to x). Let $T = \text{Th}(\mathcal{R})$. Let \mathcal{L}_0 be \mathcal{L} with the symbol Q omitted. We need to establish that $\text{Th}(\mathcal{R})$ has quantifier elimination. Our proof will rely heavily on the fact that the \mathcal{L}_0 -structure $\mathcal{R}_0 = \langle \mathbb{R}; +, <, 0, 1, Z, \lfloor \cdot \rfloor, \lambda \rangle_{\lambda \in \mathbb{Q}}$ has quantifier elimination and is universally axiomatizable [Mi]. Let $T_0 = \text{Th}(\mathcal{R}_0)$. Before proving quantifier elimination we collect some basic facts, which we state in a form necessary for use in the ensuing results, on terms in models of T_0 . The proofs are by straightforward induction on terms. In the remainder of this section we write A for the universe of \mathfrak{A} and similarly for other Fraktur letters denoting models.

LEMMA 3.2. Let $\mathfrak{B} \models T_0$ and let $t(x, \bar{y})$ be a term. Let $\mathfrak{A} \subseteq \mathfrak{B}$ and $l \in Z(A)$. Then:

(1) $t(x, \bar{y})$ is of the form

$$t(x, \bar{y}) = \sum_{i=1}^m \lambda_i \lfloor s_i(x, \bar{y}) \rfloor + \lambda x + s(\bar{y}),$$

where the s_i and s are terms. In particular if $\mathfrak{B} \models Z(t(b, \bar{a}))$ with $\bar{a} \subseteq A$ then $\mathfrak{B} \models Z(\mu b + a)$ for some $\mu \in \mathbb{Q}$ and some $a \in A$.

- (2) Suppose that $\bar{a} \subseteq A$ and let $t = t(x, \bar{a})$. There are intervals $I_1, \dots, I_n \subseteq (l, l + 1)$ definable from A such that $t \upharpoonright I_i$ is equal to $\lambda_i x + a_i$ for each $i \in \{1, \dots, n\}$ for some $\lambda_i \in \mathbb{Q}$ and $a_i \in A$ and such that $[t] \upharpoonright I_i$ is constant for each $i \in \{1, \dots, n\}$.
- (3) Let $\bar{a} \subseteq A$ and set $t = t(x, \bar{a})$. There is a partition of $Z(B)$ into sets X_1, \dots, X_n definable in the structure $\langle Z(B), +, < \rangle$ such that $t \upharpoonright X_i = \lambda_i x + b_i$ for some $\lambda_i \in \mathbb{Q}$ and $b_i \in B$, for all $i \in \{1, \dots, n\}$.

These facts have useful consequences for definable sets in models of T_0 . The following lemma is an easy consequence of Lemma 3.2 and quantifier elimination for T_0 .

LEMMA 3.3. Let $\mathfrak{B} \models T_0$ and let $X \subseteq B$ be definable over $\mathfrak{A} \subseteq \mathfrak{B}$.

- (1) If $n \in Z(A)$ then $X \cap (n, n + 1)$ is a finite union of points and intervals definable from A .
- (2) If $X \subseteq Z(B)$ then X is definable in the structure $\langle Z(B); +, < \rangle$.

PROPOSITION 3.4. $T = \text{Th}(\mathcal{R})$ has quantifier elimination.

Proof. We write $\text{qftp}(a)$ for the quantifier free type of a in \mathcal{L} , and $\text{qftp}_0(a)$ for the quantifier free \mathcal{L}_0 -type.

We use a standard embedding test. Let \mathfrak{B} and \mathfrak{B}' be models of T such that \mathfrak{B}' is ω -saturated and suppose that \mathfrak{A} is a substructure of both \mathfrak{B} and \mathfrak{B}' which is finitely generated by \bar{a} . As T_0 eliminates quantifiers and is universally axiomatizable, we have $\mathfrak{A} \upharpoonright_{\mathcal{L}_0} \preceq \mathfrak{B} \upharpoonright_{\mathcal{L}_0}$ and $\mathfrak{A} \upharpoonright_{\mathcal{L}_0} \preceq \mathfrak{B}' \upharpoonright_{\mathcal{L}_0}$. For $b \in B \setminus A$ we must find $b' \in B'$ such that $\text{qftp}(b'/\bar{a}) = \text{qftp}(b/\bar{a})$.

First suppose that $b \in Z(B)$. As T_0 eliminates quantifiers, we may find $b' \in \mathfrak{B}'$ such that $\text{qftp}_0(b/\bar{a}) = \text{qftp}_0(b'/\bar{a})$. We claim that in fact $\text{qftp}(b/\bar{a}) = \text{qftp}(b'/\bar{a})$. To verify this we need only check that if $t(x, \bar{y})$ is a term then $\mathfrak{B} \models Q(t(b, \bar{a}))$ if and only if $\mathfrak{B}' \models Q(t(b', \bar{a}))$. By Lemma 3.2, $t(x, \bar{y})$ is of the form

$$t(x, \bar{y}) = \sum_{i=1}^m \lambda_i [s_i(x, \bar{y})] + \lambda x + s(\bar{y}).$$

Thus $\mathfrak{B} \models Q(t(b, \bar{a}))$ if and only if $b + \frac{1}{\lambda} s(\bar{a}) \in Q(B)$. But this holds if and only if $\frac{1}{\lambda} s(\bar{a}) \in Q(B)$ if and only if $\frac{1}{\lambda} s(\bar{a}) \in Q(\mathfrak{B}')$, which finally holds if and only if $\mathfrak{B}' \models Q(t(b', \bar{a}))$.

Hence we may assume that $b \notin Z(B)$, and furthermore that $[b] \in A$. Thus by Lemma 3.3, if $\varphi(x) \in \text{qftp}_0(b/\bar{a})$ then $\varphi(B)$ has nonempty interior. In particular if $a \in A$ then the coset $a + Q(B)$ has nontrivial intersection with $\varphi(B)$, and the same holds in \mathfrak{B}' .

Thus if for some $a \in A$ it is the case that $\mathfrak{B} \models Q(b + a)$ then we may find $b' \in B'$ such that b' realizes $\text{qftp}_0(b/\bar{a})$ and that $\mathfrak{B}' \models Q(b' + a)$. We claim that $\text{qftp}(b/\bar{a}) = \text{qftp}(b'/\bar{a})$. As above we are reduced to checking

that $\mathfrak{B} \models Q(\lambda b + c)$ if and only if $\mathfrak{B}' \models Q(\lambda b' + c)$ where $c \in A$. Now, $\lambda b + c \in Q(B)$ if and only if $b \in Q(B) - \frac{1}{\lambda}c$, but as $b \in Q(B) - a$ this holds if and only if $\frac{1}{\lambda}c - a \in Q(B)$. The previous clause holds if and only if $\frac{1}{\lambda}c - a \in Q(B')$, which holds if and only if $\mathfrak{B}' \models Q(\lambda b + c)$.

The final case is when $b \notin Q(B) + a$ for any $a \in A$. The proof is much the same as the previous case simply using the fact that the union of finitely many complements of cosets of $Q(B)$ (or $(Q(B'))$) is codense in the line. ■

So far in considering models of T we have focused on their \mathcal{L}_0 reducts. Now we need to consider the structure

$$\mathcal{R}_1 = \langle \mathbb{R}; +, <, 0, 1, \mathbb{Q}, \lambda \rangle_{q \in \mathbb{Q}}.$$

Let \mathcal{L}_1 be its language and T_1 its theory. Note that by [DMS], T_1 has quantifier elimination. In particular if $\mathfrak{B} \models T_1$ then any definable subset of B is a finite union of points, intervals, intervals intersected with a coset of $Q(B)$, or intervals intersected with the union of the complements of finitely many cosets of $Q(B)$. For convenience let us call these basic definable sets *cells*. Note also that by results from [vdD], if $\mathfrak{B} \models T_1$ then the induced structure on $Q(B)$, and hence on any coset of $Q(B)$, is weakly o-minimal. We have:

LEMMA 3.5. T_1 has dp-rank 2.

Proof. Note that by results from [BDO], T_1 does not have the independence property, thus by [Ad] it suffices to show that T_1 has burden 2. Fix a saturated model \mathfrak{C} of T_1 . It is trivial to see that the burden of T_1 is at least 2 by simply considering disjoint intervals and cosets of $Q(C)$.

Suppose for contradiction that there were an inp-pattern of depth 3. Thus we have formulae $\varphi_i(x, \bar{y})$ for $0 \leq i \leq 2$ and parameters \bar{a}_i^j with $0 \leq i \leq 2$ and $j \in \omega$ witnessing this. By results from [DGL] we may assume that each φ_i defines a cell. First suppose that one of the formulae, say φ_0 , defines an interval intersected with a coset. Say $\varphi_0(x, \bar{a}_0^0)$ defines $I \cap Q(C) + c$. It follows that the other two rows of the inp-pattern would witness that the induced structure on $Q(C) + c$ would have burden at least 2, contradicting that weakly o-minimal structures are dp-minimal (see [DGL]).

Since not all of the formulae can define intervals, we may assume that φ_0 defines an interval intersected with the complement of finitely many cosets. For each $j \in \omega$ let I^j be the interior of the closure of $\varphi_0(C, \bar{a}_0^j)$. Note that in order for the collection $\{\varphi_0(x, \bar{a}_0^j) : j \in \omega\}$ to be inconsistent, the collection $\{I^j : j \in \omega\}$ must be inconsistent. Hence in our inp-pattern we can replace the top row with one consisting entirely of intervals. We can then do the same for the following two rows, obtaining an inp-pattern consisting only of intervals, a contradiction. ■

Via quantifier elimination we also have an easy analogue of Lemma 3.3(1).

LEMMA 3.6. *Let $\mathfrak{B} \models T$ and suppose that $X \subseteq B$ is definable and $n \in Z(B)$. Then $X \cap (n, n + 1)$ is definable in $\mathfrak{B} \setminus \mathcal{L}_1$.*

Proof. As for Lemma 3.3(1). Just note that Lemma 3.2 holds in models of T as well as in models of T_0 . ■

In order to establish that T has dp-rank 3 we need a basic fact which follows from [Si2, Proposition 4.20].

FACT 3.7. *Let T be any theory and \mathfrak{C} a saturated model of T . Let $a \in C$ and $\bar{b} \in C$. The dp-rank of $\text{tp}(a)$ is bounded by the sum of the dp-rank of $\text{tp}(\bar{b})$ and the dp-rank of $\text{tp}(a/\bar{b})$.*

Proof of Proposition 3.1. Let \mathfrak{C} be a large saturated model of T . First we show that there is an inp-pattern of depth three. Let $\{a_i : i \in \omega\}$ be distinct elements of $Z(\mathfrak{C})$ and let $\varphi_0(x, a_i) := "x \in (a_i, a_i + 1/2)"$. Note that the $\varphi_0(x, a_i)$ are of course pairwise inconsistent. Next pick pairwise disjoint open intervals $\{(l_i, r_i) : i \in \omega\}$ each contained in $(0, 1/2)$. Let $\varphi_1(x, l_i r_i) := "x - \lfloor x \rfloor \in (l_i, r_i)"$. Once again the $\varphi_1(x, l_i r_i)$ are pairwise inconsistent, and if $i, j \in \omega$ then $\varphi_0(x, a_i) \wedge \varphi_1(x, l_j r_j)$ defines a nonempty open set. Finally pick $c_i \in \mathfrak{C}$ so that $Q(\mathfrak{C}) + c_i$ are distinct cosets, and let $\varphi_2(x, c_i) := "x \in Q + c_i"$. It is now easy to see that the $\varphi_0(x, a_i)$, $\varphi_1(x, l_i r_i)$, and $\varphi_2(x, c_i)$ form an inp-pattern of depth three.

Let $c \in C$. We must show that $\text{tp}(c)$ has dp-rank at most 3. First consider $\lfloor c \rfloor$: by Lemma 3.3(2) and the dp-minimality of Presburger arithmetic (see [FG]), $\text{tp}(\lfloor c \rfloor)$ has dp-rank at most 1. Now by Lemmas 3.5 and 3.6, $\text{tp}(c/\lfloor c \rfloor)$ has rank at most 2. Hence by Fact 3.7, $\text{tp}(c)$ has rank at most 3. ■

Notice that essentially an identical proof establishes the fact that T_0 has dp-rank 2.

3.2. DOAGs with dense graphs. In this subsection we show that there are definably complete structures of finite dp-rank in which there are definable unary functions whose graphs are dense in the plane. This situation was considered in [DMS] where the absence of such functions is shown to imply a good degree of tractability in structures of o-minimal open core. Thus our examples indicate that even under the assumption of finite dp-rank we cannot hope that definable functions are quite so tractable. In this subsection we also construct structures of dp-rank equal to n for any $n \geq 1$.

Let H be a Hamel basis for the reals over the rationals. For $h^* \in H$ let π_{h^*} be projection on h , that is, the function that maps $a = \sum_{h \in H} a_h \cdot h$ in \mathbb{R} to $a_h \cdot h$. Fix $h_1, \dots, h_n \in H$ and for convenience denote π_{h_n} by π_n . Let \mathfrak{M}_n be the structure $\langle \mathbb{R}; +, 0, 1, <, \pi_1, \dots, \pi_n, \lambda \rangle_{\lambda \in \mathbb{Q}}$ where the λ 's are unary functions for multiplication by λ . Let \mathcal{L} be the language in which this structure is presented, and let \mathcal{L}_0 be \mathcal{L} without the π_i 's. Let $T_n = \text{Th}(\mathfrak{M}_n)$.

We begin by noting that the functions π_h are “wild”:

LEMMA 3.8. *The graph of $\pi_h : \mathbb{R} \rightarrow \mathbb{R}$ is dense in \mathbb{R}^2 .*

Proof. This follows easily from the density of H . ■

Nonetheless the theories T_n are well-behaved.

PROPOSITION 3.9. *T_n eliminates quantifiers.*

Proof. First note that any term t considered to be in a variable x is equivalent to a term of the form:

$$\lambda_1\pi_1(x) + \cdots + \lambda_n\pi_n(x) + \mu x + s$$

where the λ_i and μ are rational numbers and s is an \mathcal{L} -term that does not involve x .

Thus we can reduce to the case where we need to eliminate the existential quantifier from a formula of the form

$$\exists x \left(\bigwedge_{i=1}^k \sum_{r=1}^n \lambda_r^i \pi_r(x) + \mu^i x < s^i \wedge \bigwedge_{j=1}^l \sum_{r=1}^n \lambda_r^j \pi_r(x) + \mu^j x = s^j \right).$$

But this is equivalent to

$$\begin{aligned} \exists x_1, \dots, x_{n+1} \left(\bigwedge_{r=1}^n (\pi_r(x_r) = x_r \wedge \pi_r(x_{n+1}) = 0) \right. \\ \wedge \bigwedge_{i=1}^k \sum_{r=1}^n \lambda_r^i x_r + \mu^i (x_1 + \cdots + x_{n+1}) < s^i \\ \left. \wedge \bigwedge_{j=1}^l \sum_{r=1}^n \lambda_r^j x_r + \mu^j (x_1 + \cdots + x_{n+1}) = s^j \right). \end{aligned}$$

Thus we are reduced to eliminating the quantifiers from formulae of the form

$$\begin{aligned} \exists x_1, \dots, x_{n+1} \left(\bigwedge_{r=1}^n (\pi_r(x_r) = x_r \wedge \pi_r(x_{n+1}) = 0) \right. \\ \left. \wedge \varphi(x_1, \dots, x_{n+1}, s_1(\bar{y}), \dots, s_l(\bar{y})) \right) \end{aligned}$$

where $\varphi(\bar{x}, \bar{z})$ is a quantifier-free formula in the language \mathcal{L}_0 and the s_i are \mathcal{L} -terms. We do so by serially eliminating the quantifiers beginning with $\exists x_{n+1}$. Thus consider the formula

$$(*) \quad \exists x_{n+1} \left(\bigwedge_{r=1}^n (\pi_r(x_r) = x_r \wedge \pi_r(x_{n+1}) = 0) \wedge \varphi(\bar{x}, \bar{s}(\bar{y})) \right).$$

(Here we write $\bar{s}(\bar{y})$ for $s_1(\bar{y}), \dots, s_l(\bar{y})$.) Let $\psi(x_1, \dots, x_n, \bar{z})$ be the quantifier-free \mathcal{L}_0 -formula expressing

“ $\varphi(\bar{x}, -, \bar{z})$ has nonempty interior”.

Note that there are $N \in \mathbb{N}$ and $\mu_i^j \in \mathbb{Q}$ for $i \in \{1, \dots, n\}$ and $j \leq N$ and \mathcal{L} -terms $k_1(\bar{z}), \dots, k_N(\bar{z})$ such that if $\mathfrak{M} \models T_n$ and $\mathfrak{M} \models \neg\psi(a_1, \dots, a_n, s_1(\bar{b}), \dots, s_l(\bar{b}))$ then $\mathfrak{M} \models a_{n+1} = \mu_1^j a_1 + \dots + \mu_n^j a_n + k_j(\bar{b})$ for some $j \leq N$.

Hence we can reduce to the case where either $\varphi(\bar{x}, \bar{s}(\bar{y}))$ implies that $\varphi(x_1, \dots, x_n, -, \bar{s}(\bar{y}))$ has interior, or $\varphi(\bar{x}, \bar{s}(\bar{y}))$ is of the form $x_{n+1} = \mu_1 x_1 + \dots + \mu_n x_n + t(\bar{y})$ where t is an \mathcal{L} -term.

In the former case, as the set of all x_{n+1} such that $\pi_i(x_{n+1}) = 0$ for all $1 \leq i \leq n$ is dense in the line, formula (*) is always consistent and hence equivalent to

$$\bigwedge_{r=1}^n (\pi_r(x_r) = x_r) \wedge \psi(\bar{x}, \bar{s}(\bar{y})).$$

In the latter case, (*) is equivalent to

$$\bigwedge_{r=1}^n (\pi_r(x_r) = x_r \wedge \mu_r x_r + \pi_r(t(\bar{y}))) = 0.$$

In either case we have reduced to eliminating the quantifiers from a formula of the form

$$\exists x_1, \dots, x_n \left(\bigwedge_{r=1}^n (\pi_r(x_r) = x_r) \wedge \sigma(x_1, \dots, x_n, t_1(\bar{y}), \dots, t_k(\bar{y})) \right),$$

where $\sigma(\bar{x}, \bar{z})$ is an \mathcal{L}_0 -formula and the t_i 's are \mathcal{L} -terms. We can now eliminate the variables x_n through x_1 serially by essentially the identical method we used to eliminate x_{n+1} , noting that for all $1 \leq j \leq n$ the set $X_j = \{a : \pi_j(b) = a \text{ for some } b\}$ is dense in the line. ■

Before we can establish the dp-rank of T_n we need a basic lemma whose proof is straightforward.

LEMMA 3.10. *Let $\mathfrak{C} \models T_n$.*

(1) *Let $a_1, \dots, a_n \in C$. The induced structure on the set*

$$F(\bar{a}) = \left\{ x : \bigwedge_{i=1}^n \pi_i(x) = a_i \right\}$$

is weakly o-minimal.

(2) *For each i the induced structure on the set*

$$V_i = \{x : \pi_i(y) = x \text{ for some } y\}$$

is weakly o-minimal.

PROPOSITION 3.11. *T_n has dp-rank $n + 1$.*

Proof. Fix $\mathfrak{C} \models T$. By Lemma 3.8, for each $1 \leq i \leq n$ the graph of π_i is dense in the plane. This fact allows us to easily construct an ict-pattern of depth $n + 1$. We must show for $c \in C$ that $\text{tp}(c)$ has dp-rank at

most $n + 1$. First note that $\text{tp}(c/\pi_1(c), \dots, \pi_n(c)) \vdash c \in F(\pi_1(c), \dots, \pi_n(c))$. Thus by Lemma 3.10 and the dp-minimality of weakly o-minimal structures, $\text{tp}(c/\pi_1(c), \dots, \pi_n(c))$ has dp-rank at most 1. Furthermore $\text{tp}(\pi_i(c))$ has dp-rank at most 1 for each $1 \leq i \leq n$ by Lemma 3.10. By the subadditivity of dp-rank (see [KOU]) we see that $\text{tp}(\pi_1(c), \dots, \pi_n(c))$ has rank at most n . Hence by Fact 3.7, $\text{tp}(c)$ has dp-rank at most $n + 1$. ■

3.3. Reducts of tame pairs. In this section we consider an example which is the reduct of a tame pair of real-closed fields (as studied in [vdDL]), and show it has dp-rank 2. Our motivation here is twofold. First, Corollary 2.13 establishes that in a strong theory there cannot be a definable infinite discrete set with an accumulation point, and the current example shows that it is possible to have an infinite definable discrete set in a theory of finite inp-rank which accumulates to a bounded cut. This also highlights the strength of the definable completeness assumption. Secondly, Theorem 2.18 applies to Archimedean structures of finite inp-rank, and shows that discrete definable sets are approximately arithmetic progressions indexed by \mathbb{N} ; our current example demonstrates that without the Archimedean assumption this fails, namely we have a discrete definable set on which the induced ordering is dense.

Let R be a real-closed field which is a proper elementary extension of \mathbb{R} . We consider the following structure:

$$\mathcal{R} = \langle R; +, <, P, V, \text{st}, 0, 1, \lambda \rangle_{\lambda \in \mathbb{Q}}$$

where P is a unary predicate for the real numbers, V is a unary predicate for the convex hull of the real numbers, st is the standard part map, and the λ 's are unary functions for multiplication by λ . For convenience we set $\text{st}(x) = x$ if $x \notin V$. Notice that this is simply the additive reduct of a tame pair of real-closed fields as studied in [vdDL]. Let $T = \text{Th}(\mathcal{R})$.

We need a quantifier elimination result, which though a priori weaker than those in [vdDL] does not appear to immediately follow from them.

PROPOSITION 3.12. *T has quantifier elimination.*

Proof. We let \mathfrak{B} and \mathfrak{B}^* be models of T with \mathfrak{B}^* countably saturated and let $\mathfrak{A} = \langle a_1, \dots, a_n \rangle$ be a finitely generated substructure of both models. We need to show that if $b \in B$ then there is $b^* \in B^*$ such that $\text{qftp}(b/a_1, \dots, a_n) = \text{qftp}(b^*/a_1, \dots, a_n)$.

First suppose that $\mathfrak{B} \models Pb$. In this case note that the quantifier-free type of b is completely determined by the order type of b over A . Thus we readily find b^* .

Thus we assume that $\mathfrak{B} \models \neg Pb$, and without loss of generality also assume that $\text{st}(b) \in A$. First assume that furthermore $\mathfrak{B} \models Vb$. In this case we readily see that if $b \notin A$ then $\text{qftp}(b/A)$ is completely determined by the

formula $\text{st}(x) = \text{st}(b)$ and the order type of b over A . Once again we readily find b^* . Finally, suppose that $\mathfrak{B} \models \neg Va$. In this case the quantifier-free type of b is determined by its order type over A together with which (if any) of the cosets of the form $V + a$ for $a \in A$ or $P + a$ for $a \in A$ contain b . We can easily find b^* . ■

LEMMA 3.13. *Let $\mathfrak{C} \models T$.*

- (1) *The induced structure on $P(C)$ is o-minimal.*
- (2) *If $a \in P(C)$ then the induced structure on $\{x : \text{st}(x) = a\}$ is o-minimal.*
- (3) *A definable subset of $\neg V(C)$ consists of a finite union of “cells” where a cell is either a convex set, a convex set intersected with a coset of P , or a convex set intersected with finitely many complements of cosets of P .*

Proof. Immediate via quantifier elimination. ■

PROPOSITION 3.14. *T has dp-rank 2.*

Proof. It is immediate from results from [Si1] that the dp-rank of T is at least 2. By [GH], T does not have the independence property, hence we must show that all types $p(x)$ have burden at most 2. First of all suppose that $Vx \in p$ and let a realize p . By Lemma 3.13 and the dp-minimality of o-minimal structures the type of $\text{st}(b)$ has dp-rank 1. For the same reasons $\text{tp}(b/\text{st}(b))$ has dp-rank 1. Hence by Fact 3.7, $p(x)$ has dp-rank (or burden) at most 2.

Now suppose that $\neg Vx \in p$. From here the proof is almost identical to that of Lemma 3.5. Suppose there were an inp-pattern in $p(x)$ of depth 3. Via Lemma 3.13 and results from [DGL] we can assume that each row of the pattern consists of cells as described in Lemma 3.13. If one row consisted of cells containing a coset of P then the remaining two rows of the pattern would induce an inp-pattern of depth 2 on this coset, but by Lemma 3.13 the induced structure on the coset is o-minimal and hence this is impossible. Hence the rows must consist of convex sets intersected with the complements of finitely many cosets of P . But in order for this to be an inp-pattern the convex sets on each row must be pairwise disjoint. Hence we would have an inp-pattern of depth 3 consisting entirely of convex sets, which is impossible. ■

3.4. Generic expansions of o-minimal theories. In this subsection we show that if T is o-minimal and T_G is the expansion of T with a new generic unary predicate (as constructed in [CP]) then T_G is inp-minimal. (See [Ch] for a similar result in the context of NTP_2 theories.) Thus, if T extends the theory of divisible ordered Abelian groups, this gives a good

example of an expansion of a divisible ordered Abelian group which is inp-minimal but not dp-minimal (recall that by a result from [CP], T_G will have the independence property). In particular in models of T_G there will be infinite definable dense and codense sets, which is in marked contrast to the dp-minimal case where, as shown in [Si1], any infinite definable set must have nonempty interior. In the specific case where T is the theory of real-closed fields we have an example of an inp-minimal expansion of a real-closed field that is not weakly o-minimal.

Let T be o-minimal in a language \mathcal{L} . Let G be a new unary predicate and let T_G be the theory where G is a generic predicate as in [CP] (see also [Fra] for the specific situation where T is o-minimal). Let T^* be any completion of T_G . We show:

PROPOSITION 3.15. *T^* has burden one.*

Proof. Fix \mathfrak{C} a large model of T^* . As T is o-minimal and by the quantifier elimination for T_G (see [CP]), any formula $\varphi(x, \bar{a})$ with parameters is equivalent to a disjunction of formulae of the form

$$x \in I(\bar{a}) \wedge \bigwedge_{l=1}^n G(f_l(x, \bar{a})) \wedge \bigwedge_{k=1}^m \neg G(g_k(x, \bar{a}))$$

where $I(\bar{a})$ is either an open interval or a point, the f 's and g 's are definable functions continuous and monotone on $I(\bar{a})$ and such that for any $c \in I(\bar{a})$ the values $\{f_1(c, \bar{a}), \dots, f_n(c, \bar{a}), g_1(c, \bar{a}), \dots, g_m(c, \bar{a})\}$ are all distinct. For convenience let us call such a formula a *cell*.

Suppose for a contradiction that there is an inp-pattern of depth 2. Hence we find formulae $\varphi_1(x, \bar{y})$ and $\varphi_2(x, \bar{y})$ and mutually indiscernible sequences $\{\bar{a}_i : i \in \mathbb{R}\}$ and $\{\bar{b}_i : i \in \mathbb{R}\}$ witnessing this. Let r be a natural number such that $\{\varphi_1(x, \bar{a}_i) : i \in \mathbb{R}\}$ and $\{\varphi_2(x, \bar{b}_i) : i \in \mathbb{R}\}$ are r -inconsistent. By results from [DGL] we may assume that each φ_i is a cell of the form

$$x \in I_i(\bar{y}) \wedge \bigwedge_{l=1}^{n_i} G(f_l^i(x, \bar{y})) \wedge \bigwedge_{k=1}^{m_i} \neg G(g_k^i(x, \bar{y}))$$

for $i \in \{1, 2\}$.

Trivially if either $I_1(\bar{a}_0)$ or $I_2(\bar{b}_0)$ is a single point, we arrive at a contradiction, thus both $I_1(\bar{a}_0)$ and $I_2(\bar{b}_0)$ are open intervals. Furthermore if both $\{I_1(\bar{a}_i) : i \in \mathbb{R}\}$ and $\{I_2(\bar{b}_i) : i \in \mathbb{R}\}$ were inconsistent, we would have an inp-pattern of depth 2 consisting exclusively of open intervals, which is also impossible. Hence without loss of generality we assume that $\{I_1(\bar{a}_i) : i \in \mathbb{R}\}$ is consistent. If $\varphi_1(x, \bar{y})$ contains no terms of the form $G(f(x, \bar{y}))$ or no terms of the form $\neg G(g(x, \bar{y}))$ then by the genericity of G we easily deduce that $\{\varphi_1(x, \bar{a}_i) : i \in \mathbb{R}\}$ is consistent. Hence we assume this is not the case.

Let I be an open interval such that $I \subseteq \bigcap_{i \in \mathbb{R}} I_1(\bar{a}_i)$. It follows that if $c \in I$ and $i_1 < \dots < i_r \in \mathbb{R}$ then for some $1 \leq l \leq n_1$ and $1 \leq s \leq r$ as well as some $1 \leq k \leq m_1$ and some $1 \leq t \leq r$ we must have $f_l^1(c, \bar{a}_{i_s}) = g_k^1(c, \bar{a}_{i_t})$ since otherwise by the continuity of all the functions this would fail in a neighborhood of c , and thus by the genericity of G we would conclude that $\{\varphi_1(x, \bar{a}_{i_j}) : 1 \leq j \leq r\}$ is consistent.

Fix $c \in I$. As T is dp-minimal, we may find an open interval $J \subseteq \mathbb{R}$ such that in the reduct of \mathfrak{C} to \mathcal{L} the sequence $\{\bar{a}_i : i \in J\}$ is indiscernible over c . Pick $i_1 < \dots < i_{r^*} \in J$ where $r^* \geq \max\{r, 3\}$. For notational convenience suppose that $f_1^1(c, \bar{a}_{i_1}) = g_1^1(c, \bar{a}_{i_2})$. Hence by indiscernibility, $g_1^1(c, \bar{a}_{i_2}) = g_1^1(c, \bar{a}_{i_3})$ and $f_1^1(c, \bar{a}_{i_2}) = g_1^1(c, \bar{a}_{i_3})$. Thus $f_1^1(c, \bar{a}_{i_2}) = g_1^1(c, \bar{a}_{i_2})$, which violates the assumption that $f_1^1(x, \bar{a}_{i_2})$ and $g_1^1(x, \bar{a}_{i_2})$ take on distinct values on all of $I_1(\bar{a}_2)$. We have arrived at a contradiction, and hence T^* has burden one. ■

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