BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 64, No. 1, 2016

MANIFOLDS AND CELL COMPLEXES

SL(2, 5) Has No Smooth Effective One-fixed-point Action on S^8

by

Agnieszka BOROWIECKA

Presented by Andrzej BIAŁYNICKI-BIRULA

Dedicated to Professor Masaharu Morimoto
on the occasion of his 60th birthday

Summary. We prove that there is no effective smooth one-fixed-point action of $\mathrm{SL}(2,5)$, the special linear group of 2×2 matrices over $\mathbb{Z}_5$, on the 8-dimensional sphere. The method of proof involves the intersection form.

1. Introduction. It is well known that there is no *linear* action of any finite group on the n-dimensional sphere S^n with exactly one fixed point. In the middle forties Montgomery and Samelson [MS] asked if there is a *smooth* one-fixed-point action on S^n. A first example of such an action was given by E. Stein [S] in 1977. He showed that there exists a smooth $\mathrm{SL}(2,5)$-action with one fixed point on S^7, moreover he constructed such actions as well for groups $G = \mathrm{SL}(2,5) \times \mathbb{Z}_r$ where $(120, r) = 1$. If the n-dimensional sphere S^n has an effective topological one-fixed-point G-action then by taking the reduced suspension, S^{n+1} also has such an exotic topological G-action. Hence there are effective topological one-fixed-point $\mathrm{SL}(2,5)$-actions on S^n whenever $n \geq 7$. Next, T. Petrie [P] proved in 1982 that a finite, odd order abelian group having at least three noncyclic Sylow subgroups, e.g. $\mathbb{Z}_{pqr}\times\mathbb{Z}_{pqr}$, where p, q and r are distinct odd primes, has a smooth one-fixed-point action on some sphere. Petrie also established existence of smooth one-fixed-point ac-

2010 *Mathematics Subject Classification*: 57S17, 57S25.
Key words and phrases: smooth effective one-fixed-point actions on spheres, special linear group of 2×2 matrices over $\mathbb{Z}_5$, 8-dimensional sphere, intersection form.
Received 17 October 2015.
Published online 16 May 2016.

DOI: 10.4064/ba8040-3-2016

tions for nonsolvable groups $\mathrm{SL}(2,q)$ and $\mathrm{PSL}(2,q)$, where $q \geq 5$ is an odd prime power. In 1995, M. Morimoto, E. Laitinen and K. Pawałowski [LMP] described smooth one-fixed-point G-actions on spheres in case when G is any nonsolvable group. Later, M. Morimoto and E. Laitinen [LM] constructed smooth G-actions on spheres with exactly one fixed point for any Oliver group G, proving that a finite group G acts smoothly on a sphere with exactly one fixed point if and only if G is an Oliver group. On the other hand M. Furuta [F], Buchdahl–Kwasik–Schultz [BKS] and DeMichelis [DM] showed that S^n does not have a smooth one-fixed-point action of any finite group when $0 \leq n \leq 5$. For $n \geq 6$ it has been known for about a decade that the standard n-dimensional sphere S^n has a smooth one-fixed-point action of some finite group if $n \neq 8$. Lately, A. Bak and M. Morimoto [BM] showed that there are smooth one-fixed-point actions of the alternating group of degree 5, A_5, on the 8-dimensional sphere, thus completing research stretching over several decades to determine which spheres admit this kind of action. For the survey on actions of A_5 on spheres see for example [M].

The method used for constructing actions on spheres was equivariant surgery theory. To prove that there is no smooth action with exactly one fixed point various *ad hoc* methods were used. In this paper, the intersection form is a tool for obtaining the following theorems:

Theorem 1.1. *There is no effective smooth* $\mathrm{SL}(2,5)$*-action on any* 8*-dimensional* $\mathbb{Z}$*-homology sphere with exactly one fixed point.*

The next theorem gives us some information on submodules of the tangent module if the action of $\mathrm{SL}(2,5)$ has at least three fixed points.

Theorem 1.2. *Let* $G = \mathrm{SL}(2,5)$*, let* X *be a* $\mathbb{Z}$*-homology sphere of dimension* n *with a smooth* G*-action, and let* $x \in X^G$*.*

(i) *If* $3 \leq |X^G| < \infty$ *then* $T_x(X)$ *contains a* G*-submodule isomorphic to* U_4 *or* U_5*.*
(ii) *If* $|X^G|$ *is an odd integer* ≥ 3 *then* $T_x(X)$ *contains a* G*-submodule isomorphic to* $U_{3,1}$ *or* $U_{3,2}$*.*

The modules U_i and $U_{i,j}$ are described in Section 3.

In Section 2 we will recall some helpful information on $\mathrm{SL}(2,5)$ and its properties, the character table and subgroups. In Section 3 we give the tables of fixed point dimensions over $\mathbb{C}$ and $\mathbb{R}$; these results base on information from Section 2. In Section 4 we recall the Slice Theorem and some helpful lemmas. Section 5 states basic facts about the intersection number. In Section 6 we prove Theorems 1.1 and 1.2.

One can expect that the intersection form and its modification may be helpful for answering further questions of determining dimensions of the spheres on which a given finite group can not act in an effective and smooth way with exactly one fixed point.

2. Basic properties of $\mathrm{SL}(2,5)$**.** In this section we assume throughout that $G = \mathrm{SL}(2,5)$. Denote in G,

$$\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

$$\mathbf{d} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix}.$$

Then G has nine conjugacy classes $(\mathbf{1})$, $(\mathbf{z})$, $(\mathbf{c})$, $(\mathbf{d})$, $(\mathbf{zc})$, $(\mathbf{zd})$, $(\mathbf{a})$, $(\mathbf{b})$ and $(\mathbf{b}^2)$, as listed in Table 1; Table 2 is the complex character table of G.

Table 1. The sizes of the conjugacy classes of $\mathrm{SL}(2,5)$ and orders of elements in $\mathrm{SL}(2,5)$

x	$\mathbf{1}$	$\mathbf{z}$	$\mathbf{c}$	$\mathbf{d}$	$\mathbf{zc}$	$\mathbf{zd}$	$\mathbf{a}$	$\mathbf{b}$	$\mathbf{b}^2$
$\lvert(x)\rvert$	1	1	12	12	12	12	30	20	20
$\lvert x\rvert$	1	2	5	5	10	10	4	6	3

Table 2. The irreducible characters of $\mathrm{SL}(2,5)$

χ_V	V	$\mathbf{1}$	$\mathbf{z}$	$\mathbf{c}$	$\mathbf{d}$	$\mathbf{zc}$	$\mathbf{zd}$	$\mathbf{a}$	$\mathbf{b}$	$\mathbf{b}^2$
1_G	$\mathbb{C}$	1	1	1	1	1	1	1	1	1
η_1	$V_{2,1}$	2	-2	$-\frac{1-\sqrt{5}}{2}$	$-\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	0	1	-1
η_2	$V_{2,2}$	2	-2	$-\frac{1+\sqrt{5}}{2}$	$-\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	0	1	-1
ξ_1	$V_{3,1}$	3	3	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	-1	0	0
ξ_2	$V_{3,2}$	3	3	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	-1	0	0
θ_1	$V_{4,1}$	4	-4	-1	-1	1	1	0	-1	1
θ_2	$V_{4,2}$	4	4	-1	-1	-1	-1	0	1	1
ψ	V_5	5	5	0	0	0	0	1	-1	-1
χ	V_6	6	-6	1	1	-1	-1	0	0	0

A more general case is considered in [D].

The centre of G consists of $\mathbf{1}$ and $-\mathbf{1}$. The special projective group $\mathrm{PSL}(2,5)$ is isomorphic to A_5. The normal subgroups of G are $\{\mathbf{1}\}$, $\{\mathbf{1},\mathbf{z}\}$ and G. Hence G is not a simple group but it is a perfect group. Next we will recall the conjugacy classes of subgroups of G.

Let $Q_{4m} = \{\langle x, y\rangle \mid y^m = x^2,\, y^{2m} = 1,\, x^{-1}yx = y^{-1}\}$ be the generalized quaternion group of order $4m$, and C_n the cyclic group of order n. We denote, in G,

$$\mathbf{a}' = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

Following [H] one can directly check that $\mathrm{SL}(2,5)$ has the following twelve subgroups up to conjugacy:

$$\mathrm{SL}(2,3) \cong \langle \mathbf{a}, \mathbf{a}', \mathbf{x} \,|\, \mathbf{a}^2 = \mathbf{a}'^2,\ \mathbf{a}^4 = 1,\ \mathbf{a}'^{-1}\mathbf{a}\mathbf{a}' = \mathbf{a}^{-1},\ \mathbf{x}\mathbf{a}\mathbf{x}^{-1} = \mathbf{a}',\ \mathbf{x}\mathbf{a}'\mathbf{x}^{-1} = \mathbf{a}\mathbf{a}' \rangle,$$

$Q_{20} = \langle \mathbf{a}, \mathbf{zc} \rangle$, $Q_{12} = \langle \mathbf{a}, \mathbf{b} \rangle$, $Q_8 = \langle \mathbf{a}, \mathbf{a}' \rangle$, $C_{10} = \langle \mathbf{zc} \rangle$, $C_6 = \langle \mathbf{b} \rangle$, $C_5 = \langle \mathbf{c} \rangle$, $C_4 = \langle \mathbf{a} \rangle$, $C_3 = \langle \mathbf{b}^2 \rangle$, $C_2 = \langle \mathbf{z} \rangle$, $\{e\} = \{\mathbf{1}\}$.

In the above presentations we list only generators, skipping the obvious relations. The subgroups are shown in the Hasse diagram below, with A denoting $\mathrm{SL}(2,3)$.

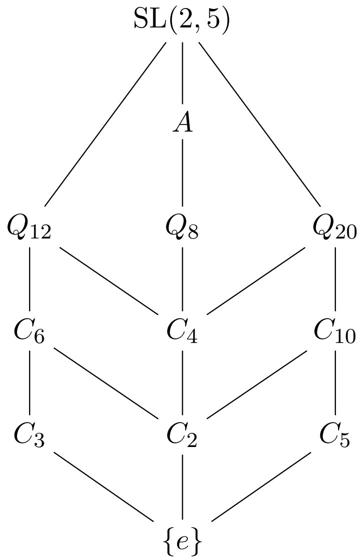

Fig. 1. The subgroups of $\mathrm{SL}(2,5)$ up to isomorphism

Let p be a prime. A finite group K is called a *p-group* if $|K| = p^s$ for some integer $s \geq 0$. The set of all p-subgroups of a group G is denoted by $\mathcal{P}(G)$. A finite group K is called a *mod-p-cyclic group* if K contains a normal subgroup P such that P is a p-group and K/P is cyclic. The set of all mod-p-cyclic subgroups of a group G is denoted by $\mathcal{PC}(G)$.

For $G = \mathrm{SL}(2,5)$, p-subgroups and mod-p-cyclic subgroups up to conjugacy classes are as follows:

$$\begin{aligned}\mathcal{P}(G) &= \big\{\{e\}, \langle \mathbf{z} \rangle, \langle \mathbf{a} \rangle, \langle \mathbf{b}^2 \rangle, \langle \mathbf{c} \rangle, \langle \mathbf{a}, \mathbf{a}' \rangle\big\} = \big\{\{e\}, C_2, C_4, C_3, C_5, Q_8\big\},\\ \mathcal{PC}(G) &= P(G) \cup \{\langle \mathbf{b} \rangle, \langle \mathbf{zc} \rangle, \langle \mathbf{a}, \mathbf{b} \rangle, \langle \mathbf{a}, \mathbf{zc} \rangle, \langle \mathbf{a}, \mathbf{a}', \mathbf{x} \rangle\}\\ &= P(G) \cup \{C_6, C_{10}, Q_{12}, Q_{20}, A\}.\end{aligned}$$

3. Fixed point sets over $\mathbb{C}$ and $\mathbb{R}$. The following theorems and definitions are well known facts [D], [CR].

THEOREM 3.1. *If U_i is an irreducible $\mathbb{C}G$-module, κ_i an irreducible character associated with U_i, and H_j a representative of the conjugacy class of a subgroup of G. Then*

$$\dim U_i^{H_j} = \frac{1}{|H_j|} \sum_{g \in H_j} \kappa_i(g)$$

where $U_i^{H_j}$ is the H_j-fixed point set of U_i.

Applying the complex character table of $\mathrm{SL}(2,5)$ to the formula in Theorem 3.1, one can calculate the fixed point dimension. The results of this straightforward computation are gathered in Table 3.

Table 3. The fixed point dimension over $\mathbb{C}$ for $\mathrm{SL}(2,5)$

H	$\{e\}$	C_2	C_3	C_4	C_5	C_6	C_{10}	Q_8	Q_{12}	Q_{20}	A	G
$\dim \mathbb{C}$	1	1	1	1	1	1	1	1	1	1	1	1
$\dim V_{2,1}$	2	0	0	0	0	0	0	0	0	0	0	0
$\dim V_{2,2}$	2	0	0	0	0	0	0	0	0	0	0	0
$\dim V_{3,1}$	3	3	1	1	1	1	1	0	0	0	0	0
$\dim V_{3,2}$	3	3	1	1	1	1	1	0	0	0	0	0
$\dim V_{4,1}$	4	0	2	0	0	0	0	0	0	0	0	0
$\dim V_{4,2}$	4	4	2	2	0	2	0	1	1	0	1	0
$\dim V_5$	5	5	1	3	1	1	1	2	1	1	0	0
$\dim V_6$	6	0	2	0	2	0	0	0	0	0	0	0

To get the fixed point dimension over $\mathbb{R}$ the Frobenius–Schur theorem is needed.

THEOREM 3.2 (Frobenius–Schur theorem). *Let G be a finite group and V an irreducible $\mathbb{C}G$-module. There exists an irreducible $\mathbb{R}G$-module U such that $V \cong \mathbb{C} \otimes_{\mathbb{R}} U$ if and only if*

$$B_V := \frac{1}{|G|} \sum_{g \in G} \chi_V(g^2) = 1.$$

If $B_V = 1$ then the $\mathbb{C}G$-module V is called *of real type*, if $B_V = 0$ *of complex type*, and if $B_V = -1$ *of quaternionic type*. Therefore the irreducible $\mathbb{C}G$-modules with character 1_G, ψ, θ_2, ξ_1 or ξ_2 are of real type, and those with character χ, θ_1, η_1 or η_2 are of quaternionic type.

Let V be an $\mathbb{R}G$-module. Then $\mathbb{C} \otimes_{\mathbb{R}} V$ is a $\mathbb{C}G$-module. We call $c(V) := \mathbb{C} \otimes_{\mathbb{R}} V$ the *complexification* of V.

Let V be a $\mathbb{C}G$-module. The vector space V over $\mathbb{C}$ can be considered as a vector space over $\mathbb{R}$, and it is denoted by $V_{\mathbb{R}}$. If $\{v_1, \ldots, v_m\}$ is a basis of V over $\mathbb{C}$, then $\{v_1, iv_1, \ldots, v_m, iv_m\}$ is a basis of $V_{\mathbb{R}}$ over $\mathbb{R}$ and $\dim_{\mathbb{R}} V_{\mathbb{R}} = 2\dim_{\mathbb{C}} V$. We call $r(V) := V_{\mathbb{R}}$ the *realification* of V.

Hence, $\dim_{\mathbb{C}} V = \dim_{\mathbb{R}} U$ when V is of real type and $V = c(U)$, while $\dim_{\mathbb{C}} V = \frac{1}{2}\dim_{\mathbb{R}} U$ when V is of complex or quaternionic type and $U = V_{\mathbb{R}}$.

Table 4 shows $\dim_{\mathbb{R}} \bar{U}^H$ for irreducible $\mathbb{R}G$-modules $\bar{U}$; the $\mathbb{R}$ in the table stands for the trivial module.

Table 4. The fixed point dimension over $\mathbb{R}$ for $\mathrm{SL}(2,5)$

H	$\{e\}$	C_2	C_3	C_4	C_5	C_6	C_{10}	Q_8	Q_{12}	Q_{20}	A	G
$\dim \mathbb{R}$	1	1	1	1	1	1	1	1	1	1	1	1
$\dim U_{3,1}$	3	3	1	1	1	1	1	0	0	0	0	0
$\dim U_{3,2}$	3	3	1	1	1	1	1	0	0	0	0	0
$\dim U_4$	4	4	2	2	0	2	0	1	1	0	1	0
$\dim U_5$	5	5	1	3	1	1	1	2	1	1	0	0
$\dim W_{4,1}$	4	0	0	0	0	0	0	0	0	0	0	0
$\dim W_{4,2}$	4	0	0	0	0	0	0	0	0	0	0	0
$\dim W_8$	8	0	4	0	0	0	0	0	0	0	0	0
$\dim W_{12}$	12	0	4	0	4	0	0	0	0	0	0	0

4. Euler–Poincaré characteristic restrictions. In order to apply the intersection form to prove Theorem 1.1 we need the following results:

DEFINITION 4.1. Let G be a finite group. For a G-fixed point x of a smooth G-manifold X, let $T_x(X)$ denote the tangent space of X at x. Then $T_x(X)$ inherits a linear G-action from the G-action on X. This linear G-action will be referred to as the *tangential G-representation of X at x* or the *tangent G-module*.

THEOREM 4.2 (Slice Theorem). *Let G be a finite group, M a smooth G-manifold, and $x_0 \in M^G$. Then there exists a G-invariant neighbourhood U_{x_0} of x_0 in M and a G-diffeomorphism from the G-module $T_{x_0}(M)$ onto U_{x_0}.*

According to the above theorem, the G-action on the manifold M around the point x_0 is equivalent to the linear action defined by the G-module $T_{x_0}(M)$.

LEMMA 4.3. *If a finite mod-p-cyclic group H acts on a $\mathbb{Z}_p$-acyclic finite complex X, then $\chi(X^H) = 1$.*

Now we are ready to prove the following lemma:

LEMMA 4.4. *If a mod-p-cyclic group H acts smoothly on a $\mathbb{Z}_p$-homology sphere X with $x_1 \in X^H$, then $\chi(X^H) = 1 + (-1)^{\dim T_{x_1}(X)^H}$.*

Proof. By the Slice Theorem we can take an H-linear-disc neighbourhood N of x_1 in X. Set

$$Y = X/(X \setminus \operatorname{Interior}(N)) \quad (= N/\partial N).$$

Then $Y \cong S(\mathbb{R} \oplus T_{x_1}(X))$. Let $f\colon X \to Y$ denote the pinching map. Since the induced map $f_*\colon H_n(X,\mathbb{Z}_p) \to H_n(Y,\mathbb{Z}_p)$ is an isomorphism, where $n = \dim X$, f is a $\mathbb{Z}_p$-homology equivalence. Thus the mapping cone

$$C_f := \{(X \times [0,1]) \cup Y/\langle (x,1) \sim f(x)\rangle\}/(X \times \{0\})$$

is a $\mathbb{Z}_p$-acyclic finite complex with an H-action. Hence by Lemma 4.3 we get $\chi((C_f)^H) = 1$. The equalities

$$\chi((C_f)^H) = \chi(Y^H) - \chi(X^H) + 1,$$
$$\chi(Y^H) = 1 + (-1)^{\dim Y^H}$$

yield $\chi(X^H) = 1 + (-1)^{\dim T_{x_1}(X)^H}$. ■

COROLLARY 4.5. *Let X and H be as in the lemma above. Then for arbitrary $x_1, x_2 \in X$,*

$$\dim T_{x_1}(X)^H \equiv \dim T_{x_2}(X)^H \bmod 2.$$

5. Intersection number. Following Davis–Kirk [DK], we recall basic information on the intersection number that is needed in this paper.

Recall that an *orientation* of a real finite-dimensional vector space V is an equivalence class of bases of V where two bases are considered equivalent if the determinant of the change of basis matrix is positive.

Suppose that V and W are oriented subspaces of an oriented vector space Z, and that $\dim(V) + \dim(W) = \dim(Z)$. Suppose V and W are transverse, that is, $V \cap W = \{0\}$. Let $\mathcal{B}_V$, $\mathcal{B}_W$ and $\mathcal{B}_Z$ denote bases in the given equivalence classes. Then the *intersection number of V and W* is -1 or 1, depending on the sign of the determinant of the change of basis matrix from the basis $\{\mathcal{B}_V, \mathcal{B}_W\}$ of Z to $\mathcal{B}_Z$.

Suppose that A and B are smooth, compact, connected, oriented submanifolds of dimensions a and b of a compact oriented manifold M of dimension m, where $a + b = m$. Assume that A and B are transverse, i.e. at each point $p \in A \cap B$, the tangent subspaces T_pA and T_pB span T_pM. Moreover assume that the boundary of A is embeded in the boundary of M, the boundary of B is empty, and B is contained in the interior of M. Since A and B are transverse and compact, their intersection consists of a finite number of points. We define the *intersection number of A and B* to be the integer

$$A \cdot B = \sum_{p \in A \cap B} \eta_p$$

where η_p is the intersection number of the oriented subspaces T_pA and T_pB in T_pM.

Since A and B are oriented manifolds, they have fundamental classes $[A, \partial A] \in H_a(A, \partial A)$, $[B] \in H_b(B)$. Let $i_B \colon A \subset M$ and $i_B \colon B \subset M$ denote the inclusion maps. Then $i_A([A, \partial A]) \in H_a(M, \partial M)$ and $i_B([B]) \in H_b(M)$. We have the following theorem:

THEOREM 5.1. *Let $\alpha \in H^b(M)$ be the Poincaré dual to $i_A([A, \partial A])$, and $\beta \in H^a([M, \partial M])$ the Poincaré dual to $i_B([B])$, i.e.*

$$\alpha \cap [M, \partial M] = i_A([A, \partial A]) \quad \textit{and} \quad \beta \cap [M, \partial M] = i_B([B]).$$

Then

$$A \cdot B = \langle \alpha \cup \beta, [M, \partial M] \rangle$$

where $\langle\ ,\ \rangle$ denotes the Kronecker pairing.

We call the cup product

$$H^b(M) \times H^a(M, \partial M) \xrightarrow{\cup} H^m(M, \partial M)$$

the *intersection pairing*. The *intersection form* is the cup product in the middle dimensional cohomology.

6. Proofs of Theorems 1.1 and 1.2. Henceforth, G denotes $\mathrm{SL}(2,5)$, and A is the subgroup of G isomorphic to $\mathrm{SL}(2,3)$.

Proof of Theorem 1.1. By the character table, any faithful real G-module of dimension 8 is isomorphic to $W_{4,i} \oplus W_{4,j}$, $W_{4,i} \oplus U_4$, or W_8, where $i, j = 1, 2$. Suppose Σ is an 8-dimensional $\mathbb{Z}$-homology sphere with a smooth one-fixed-point action of G. Let $T_{x_0}(\Sigma)$ denote the tangential G-module at the fixed point x_0 of Σ.

First consider the case where $T_{x_0}(\Sigma) \cong W_{4,i} \oplus W_{4,j}$. Let P be any nontrivial p-subgroup of G. According to Table 4, Σ^P is of dimension 0. Then by the Smith theorem, Σ^P is a $\mathbb{Z}_p$-homology sphere and therefore consists of exactly two points. Let H be an arbitrary proper subgroup of G, that is, $H \neq \{e\}, G$. Note that H is mod-p-cyclic. For all $\{e\} \neq P \subset H$ we have $\Sigma^G \subsetneq \Sigma^H \subseteq \Sigma^P$, therefore $\dim \Sigma^H = 0$, and by Lemma 4.4, $|\Sigma^H| = \chi(\Sigma^H) = 0$ or 2. Since Σ^G consists of one point, $|\Sigma^H| = 2$. Thus $\Sigma^H = \Sigma^P$. Since $\langle \mathbf{z} \rangle \subset A$ and $\langle \mathbf{z} \rangle \subset Q_{20}$, we have

$$\Sigma^A = \Sigma^{\mathbf{z}} = \Sigma^{Q_{20}},$$

and consequently $\Sigma^G = \Sigma^{\mathbf{z}}$. This is a contradiction.

Second, consider the case $T_{x_0}(\Sigma) \cong W_{4,i} \oplus U_4$. Then $\Sigma^{\mathbf{z}}$ is a $\mathbb{Z}_2$-homology sphere of dimension 4 and the intersection form on $H_2(\Sigma_{\mathbf{z}}, \mathbb{Z}_2)$ is trivial. There exist $a, b \in A_5$ of order 3 such that $A_5 = \langle a, b \rangle$. Since $(\Sigma^{\mathbf{z}})^{A_5}$ consists of exactly one point, the intersection number in $\mathbb{Z}_2$ of $(\Sigma^{\mathbf{z}})^a$ and $(\Sigma^{\mathbf{z}})^b$ is equal to 1. This is a contradiction.

Third, consider the case $T_{x_0}(\Sigma) \cong W_8$. Then the intersection form on $H_4(\Sigma, \mathbb{Z}_2)$ is also trivial. There exist $a, b \in \mathrm{SL}(2,5)$ of order 3 such that $\mathrm{SL}(2,5)$ is generated by these elements. Then the intersection number in $\mathbb{Z}_2$ of Σ^a and Σ^b is 1, a contradiction. ∎

We fix an epimorphism $\pi\colon \mathrm{SL}(2,5) \to A_5$. We tabulate the fixed point dimension of irreducible A_5-representations [BM].

Table 5. Fixed point dimension over $\mathbb{R}$ for A_5

H	$\{e\}$	C_2	C_3	C_5	D_4	D_6	D_{10}	A_4	A_5
$\dim U_{3,1}$	3	1	1	1	0	0	0	0	0
$\dim U_{3,2}$	3	1	1	1	0	0	0	0	0
$\dim U_4$	4	2	2	0	1	1	0	1	0
$\dim U_5$	5	3	1	1	2	1	1	0	0

One can regard the irreducible real A_5-modules $U_{3,1}$, $U_{3,2}$, U_4 and U_5 as real $\mathrm{SL}(2,5)$-modules via π. There are four irreducible complex $\mathrm{SL}(2,5)$-modules which are faithful and moreover of quaternionic type. These give four faithful irreducible real $\mathrm{SL}(2,5)$-modules, namely $W_{4,1}, W_{4,2}, W_8, W_{12}$. Values of the H-fixed-point dimension, where H is a subgroup of $\mathrm{SL}(2,5)$, are listed in Table 4.

Proof of Theorem 1.2. (i) By the Smith theorem, X^{Q_8} is a $\mathbb{Z}_2$-homology sphere. By assumption $|X^G| \geq 3$, $\dim X^{Q_8} > 0$ and hence X^{Q_8} is connected. Thus $\dim T_x(X)^{Q_8} > 0$, which implies that $T_x(X) \supset U_4$ or $T_x(X) \supset U_5$.

(ii) Suppose that $T_x(X)$ does not contain a G-submodule isomorphic to $U_{3,1}$ or $U_{3,2}$. Note that

$$\dim T_x(X)^{C_4} = \dim T_x(X)^{Q_8} + \dim T_x(X)^{Q_{12}}.$$

Since X^{C_2} is a $\mathbb{Z}_2$-homology sphere, the mod-2 intersection number of X^{Q_8} and $X^{Q_{12}}$ is 0. But this is not possible because $X^{Q_8} \cap X^{Q_{12}} = X^G$ consists of an odd number of points. ∎

Acknowledgements. This paper mostly was written during a scholarship at the Okayama University, Japan, where I attended a seminar run by Prof. Masaharu Morimoto. I would like to thank him deeply for invaluable help, advice, supervision and giving the crucial support during my stay in Japan. I wish to thank the Heiwa Nakajima Foundation for the scholarship, without which I would not be able to afford to study and live in Japan. I thank Prof. Krzysztof Pawałowski for listening and supporting in the past couple of years. His comments substantially improved the presentation of this paper.

References

[BM] A. Bak and M. Morimoto, *The dimension of spheres with smooth one-fixed-point actions*, Forum Math. 17 (2005), 199–216.

[BKS] N. P. Buchdahl, S. Kwasik and R. Schultz, *One fixed point actions on low-dimensional spheres*, Invent. Math. 102 (1990), 633–662.

[CR] Ch. W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Pure Appl. Math. 11, Interscience, 1962.

[DK] J. F. Davis and P. Kirk, *Lecture Notes in Algebraic Topology*, Grad. Stud. Math. 35, Amer. Math. Soc., Providence, RI, 2001.

[DM] S. DeMichelis, *The fixed point set of a finite group action on a homology sphere*, Enseign. Math. 35 (1989), 107–116.

[D] L. Dornhoff, *Group Representation Theory, Part A*, Pure Appl. Math. 7, Dekker, New York, 2002.

[F] M. Furuta, *A remark on a fixed point of finite group action on S^4*, Topology 28 (1989), 35–38.

[H] J. F. Humphreys, *A Course in Group Theory*, Oxford Univ. Press, Oxford, 1996.

[LM] E. Laitinen and M. Morimoto, *Finite groups with smooth one-fixed-point actions on spheres*, Forum Math. 10 (1998), 479–520.

[LMP] E. Laitinen, M. Morimoto and K. Pawałowski, *Finite groups with smooth one-fixed-point action on spheres*, Comment. Math. Helv. 70 (1995), 10–38.

[MS] D. Montgomery and H. Samelson, *Fiberings with singularities*, Duke Math. J. 13 (1946), 51–56.

[M] M. Morimoto, *Most standard spheres have one-fixed-point actions of A_5. II*, K-Theory 4 (1991), 289–302.

[P] T. Petrie, *One fixed point actions on spheres. I*, Adv. Math. 46 (1982), 3–14.

[S] E. Stein, *Surgery on products with finite fundamental group*, Topology 16 (1977), 473–493.

Agnieszka Borowiecka
E-mail: aborowiecka@wp.pl