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K3 fibrations on rigid double octic Calabi–Yau threefolds

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Abstract. We give a description of the Picard group of double octic Calabi–Yau threefolds using a K3 fibration defined by a singular line of the branch octic. In particular, we show that the group is generated by the Picard group of a generic fibre and the subgroup generated by the components of the reducible fibres.

1. Introduction. The main goal of this paper is to study K3 fibrations on rigid double octic Calabi–Yau threefolds. Fibred Calabi–Yau threefolds are of particular importance because of their arithmetic and geometric properties. On Calabi–Yau threefolds, there can exist three types of fibrations: with elliptic, abelian or K3 generic fibres. The exact type is characterised by the nef cone and the cubic form on the Picard group (see [O]).

A rich class of examples with elliptic and abelian fibrations is provided by desingularised fibred products of rational elliptic surfaces with sections introduced by C. Schoen [S]. Another class of examples, specially suitable for direct computations, are so called double octics, i.e. resolutions of double covers of $\mathbb{P}^3$ branched along an octic hypersurface. When the hypersurface is smooth, the threefold obtained is Calabi–Yau. S. Cynk and T. Szemberg [CS] considered hypersurfaces with some special singularities and devised a method of 'admissible blowing-ups' to construct a crepant resolution of singularities of a branched octic and obtain a smooth Calabi–Yau threefold. A convenient class of examples are arrangements of eight planes, because every double or triple line of the arrangement induces a K3 fibration and every quadruple and quintuple point induces an elliptic fibration. For a fibre product of rational elliptic surfaces we can construct a fibrewise Kummer fibration. Under mild conditions on the singular fibres, it admits a Calabi–Yau smooth model giving a connection between the above two classes of examples (see [KK]).

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In this paper, we will consider arrangements of eight planes that give rigid Calabi–Yau threefolds. Arrangements of eight planes in $\mathbb{P}^3$ defining Calabi–Yau threefolds were studied by C. Meyer [M]. Using extensive computer search, he identified 11 rigid examples with rational coefficients. Computations of Picard–Fuchs operators for one-parameter families of double octics from [M] yield another three rigid examples with coefficients in $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{-3})$ [CvS].

For every example we choose a double line of the arrangement. A pencil of planes containing the line will define a fibration of the threefold. Resolution of singularities of a double octic induces a K3 smooth model of a generic fibre; singular fibres correspond to the planes containing special singular points or lines of the arrangement. We compute the Picard group of the Calabi–Yau threefold and compare it with the Picard group of the generic fibre and the group spanned by components of the singular fibres. Our main result can be summarised as follows:

Theorem 1. *In Table 2 we gather equations of double covers of $\mathbb{P}^3$ branched along arrangements of eight planes which give rigid Calabi–Yau threefolds V. In each case, we have a K3 fibration and the following equality holds:*

$$\operatorname{rank}(\operatorname{Pic}(V)) = 1 + \sum_S (r(S) - 1) + \operatorname{rank}(\operatorname{Pic}(S_{\mathrm{gen}})),$$

where $r(S)$ is the number of irreducible components of a fibre S and S_{gen} is a generic fibre which is a K3 surface.

The above formula is similar to the one obtained by Oguiso [O2] for abelian fibrations. The main difference is that in Oguiso's formula a general fibre is treated as a scheme over the base whereas here we refer to a geometric fibre. We expect that the formula holds for all double octics, as rigid examples are most complicated.

The paper is organised in the following way. In Section 2, we give a detailed description of a resolution of singularities of a double octic that give smooth Calabi–Yau threefolds and describe K3 fibrations on them. Section 3 focuses on 14 examples of rigid CY threefolds. We provide explicit equations of the arrangements in $\mathbb{P}^3$. For all the fourteen examples, we give an explicit description of K3 fibrations on them. In Table 3 we collect the ranks of the Picard groups of a variety, of a generic fibre, description of the singular fibres in coordinates and the number of their irreducible components. The main result follows by a direct application of Table 3.

2. Preliminaries. Let us recall the definition of a Calabi–Yau manifold.

Definition 1. A smooth n-dimensional complex projective variety V is called a *Calabi–Yau manifold* if:

- $H^i(V, \mathcal{O}_V) = 0$, $0 < i < n$,
- $K_V = 0$.

One of standard constructions that gives a Calabi–Yau manifold is a double cover of $\mathbb{P}^3$ branched along a smooth octic surface. The name *double octic* refers to such a construction and its generalisations. In particular, when the octic is singular we need to resolve the singularities, and to obtain a Calabi–Yau manifold we have to control the canonical divisor.

In this paper we consider double octics defined by arrangements of eight planes. Then the only singularities are multiple points and lines. The aim of this section is to prove the following theorem.

THEOREM 2. *Let $S \subset \mathbb{P}^3$ be an arrangement of eight planes in which the only singularities are double or triple lines and quadruple or quintuple points. Then there exists a sequence of blowing-ups $\sigma = \sigma_1 \circ \cdots \circ \sigma_s : \mathbb{P}_3^* \to \mathbb{P}^3$ and a smooth even divisor $S^* \subset \mathbb{P}_3^*$ such that $\sigma_*(S^*) = S$. Furthermore a double cover of $\mathbb{P}_3^*$ branched along the surface S^* is a Calabi–Yau threefold.*

REMARK 1. This theorem is proved in [CS] and [CM] with weaker conditions on S. The idea and notation used in the proof are almost the same.

Let $S^{(i)}$ be the locus of points of an arrangement S of multiplicity $i \geq 2$. Let l^i be the number of line components of $S^{(i)}$. Let p^i be the number of point components of $S^{(i)}$. Let p^i_j be the number of point components of $S^{(i)}$ that are contained in exactly j triple lines.

Notice that there cannot exist point components of $S^{(2)}$, and the point components of $S^{(3)}$ can occur only in the intersection of three double lines. If we blow up those lines we also resolve those points, so we need not treat those type of singularities.

Let V be a smooth threefold and $D \subset V$ an even, reduced divisor. Let $W \subset D$ be a smooth irreducible proper subvariety and let $\sigma\colon \operatorname{Bl}_W V \to V$ be a blowing-up of V in W with exceptional divisor E. We denote by $\operatorname{mult}_{W/D}$ the generic multiplicity of D at W.

DEFINITION 2. We define a divisor $D^* \subset \operatorname{Bl}_W V$ by setting

$$D^* := \begin{cases} \tilde{D} & \text{if } \operatorname{mult}_{W/D} \text{ is even,} \\ \tilde{D} + E & \text{if } \operatorname{mult}_{W/D} \text{ is odd,} \end{cases}$$

where $\tilde{D}$ is the proper transform of D.

REMARK 2. Notice that D^* is the only reduced and even divisor satisfying

$$\tilde{D} \leq D^* \leq \sigma^* D.$$

DEFINITION 3. Let $W \subset D \subset V$ be as above. A blowing-up is called *admissible* if

$$K_{\mathrm{Bl}_W V} + \tfrac{1}{2}D^* \cong \sigma^*\big(K_V + \tfrac{1}{2}D\big).$$

LEMMA 1. *There are exactly four types of admissible blowing-ups on a smooth threefold:*

- *blowing-up of a curve of multiplicity* 2 *or* 3*,*
- *blowing-up of a quadruple or quintuple point.*

Proof. Let r be the codimension of the blown-up subvariety W, and m the multiplicity of D in W. Let

$$\epsilon = \begin{cases} 0 & \text{if } m \text{ is even,} \\ 1 & \text{if } m \text{ is odd.} \end{cases}$$

Then

$$K_{\mathrm{Bl}_W V} \cong \sigma^*(K_V) + (r-1)E, \qquad D^* = \sigma^*(D) - (m-\epsilon)E.$$

So

$$K_{\mathrm{Bl}_W V} + \frac{1}{2}D^* \cong \sigma^*\bigg(K_V + \frac{1}{2}D\bigg) + \bigg(r - 1 - \frac{m-\epsilon}{2}\bigg)E.$$

For σ to be admissible, we need $m - \epsilon = 2(r-1)$. Setting $r = 2$ and $r = 3$ ends the proof. ∎

REMARK 3. Notice that in the last proof we only use the codimension of W, so for a surface, admissible blowing-ups are blowing-ups of double or triple points.

Proof of Theorem 2. First we blow up quintuple points, then triple lines, quadruple points and double lines. Every time we replace the divisor so that the blowing-ups are admissible. Note that in the process, some new singularities will appear, but not as bad as the already resolved ones. The explicit numbers and types of the resulting singularities are put together in Table 1 and Proposition 1. In this way we obtain S^*. Then there exists a double cover V of $\mathbb{P}_3^*$ branched along S^*. The canonical divisor is

$$K_V \cong \pi^*\big(K_{\mathbb{P}_3^*} + \tfrac{1}{2}S^*\big) \cong \pi^*\big(\sigma^*\big(K_{\mathbb{P}^3} + \tfrac{1}{2}S\big)\big) = 0.$$

The cohomology groups of the structure sheaf are

$$\begin{aligned} h^1(V, \mathcal{O}_V) &= h^1\big(\mathbb{P}_3^*, \mathcal{O}_{\mathbb{P}_3^*} \oplus \mathcal{O}_{\mathbb{P}_3^*}\big(-\tfrac{1}{2}S^*\big)\big) \\ &= h^1(\mathbb{P}_3^*, \mathcal{O}_{\mathbb{P}_3^*}) + h^1\big(\mathbb{P}_3^*, \mathcal{O}_{\mathbb{P}_3^*}\big(-\tfrac{1}{2}S^*\big)\big) \\ &= h^1(\mathbb{P}_3^*, \mathcal{O}_{\mathbb{P}_3^*}) + h^2\big(\mathbb{P}_3^*, \mathcal{O}_{\mathbb{P}_3^*}\big(K_{\mathbb{P}_3^*} + \tfrac{1}{2}S^*\big)\big) \\ &= h^1(\mathbb{P}_3^*, \mathcal{O}_{\mathbb{P}_3^*}) + h^2(\mathbb{P}_3^*, \mathcal{O}_{\mathbb{P}_3^*}(0)) \\ &= h^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) + h^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = 0, \\ h^2(V, \mathcal{O}_V) &= h^1(V, \Omega_V^3) = h^1(V, \mathcal{O}_V) = 0, \qquad \text{since } K_V = 0. \ ∎ \end{aligned}$$

Remark 4. Exactly the same method shows that we can obtain a K3 surface as a double cover of $\mathbb{P}^2$ branched along an arrangement of six lines if its only singularities are double or triple points.

A blowing-up of a singular point or a line resolves the singularity but sometimes introduces new singularities. The following proposition summarizes the change of the numbers of singular points of various types under blowing-ups.

Proposition 1. *A blowing-up of*

- *a point of type p_0^5 introduces five double lines,*
- *a point of type p_1^5 introduces five double lines and one point of type p_1^4,*
- *a point of type p_2^5 introduces five double lines and two points of type p_1^4,*
- *a triple line introduces three double lines, and if the line contains a point of type p_1^4 then the blowing-up resolves that point and introduces one more double line,*
- *a point of type p_0^4 or a double line does not introduce new singularities.*

Table 1 shows what type and how many singularities are left after blowing-ups. The first row contains the original number of singularities of various types; the second, third and fourth rows contain the number of singularities after blowing-up of fivefold points, triple lines and fourfold points, respectively.

In columns there is the number and the type of singularities. The rows with arrows are blowing-ups.

Table 1. Types of singularities and number of blowing-ups needed to resolve them

	p_2^5	p_1^5	p_0^5	l^3	p_1^4	p_0^4	l^2
$p_2^5+p_1^5+p_0^5\downarrow$							
	0	0	0	l^3	$p_1^4+2p_2^5+p_1^5$	p_0^4	$l^2+5p_2^5+5p_1^5+5p_0^5$
$l^3\downarrow$							
	0	0	0	0	0	p_0^4	$l^2+7p_2^5+6p_1^5+5p_0^5+3l^3+p_1^4$
$p_0^4\downarrow$							
	0	0	0	0	0	0	$l^2+7p_2^5+6p_1^5+5p_0^5+3l^3+p_1^4$

At the end we blow up all double lines. So we need exactly $8p_2^5+7p_1^5+6p_0^5+l^3+p_0^4+p_1^4+28$ *blowing-ups to obtain* $\mathbb{P}_3^*$.

Proof. For a quintuple point we add the exceptional divisor which gives us five double lines and as many quadruple points of type p_1^4, as there are triple lines going through that point. While blowing up triple lines, we also blow up points of type p_1^4. Here we also add the exceptional divisor, which gives us an additional three double lines for every triple line and one for a quadruple point. Summing up, we obtain $3l^3+p_1^4+2p_2^5+p_1^5$ additional double lines. When we blow up a quadruple point or a double line, we do not add the exceptional divisor and do not have other singularities. Adding the above numbers and taking into account the fact that $28 = l^2 + 3l^3$, we obtain the formula for the number of blowing-ups. ∎

2.1. The description of the K3 fibration. To give a good description of the fibration on a Calabi–Yau manifold we need to introduce the following notation. We denote by $S_1, \ldots, S_8$ the eight planes of the arrangement S given by the equations $F_i = 0$, $i = 1, \ldots, 8$. Let $\lambda_{i,j}$ be the line of intersection of S_i and S_j. It can happen that $\lambda_{i,j} = \lambda_{i,k} = \lambda_{j,k}$. Then we obtain a triple line $\lambda_{i,j,k}$. Consider the pencil of planes in $\mathbb{P}^3$ which contain the double line $\lambda_{7,8}$. Every such plane, denoted by $\Pi_{(\alpha:\beta)}$, is given by the equation

$$\alpha F_7 + \beta F_8 = 0.$$

We denote by Π_{gen} the generic element of the pencil. The intersection of the octic with Π_{gen} is the double line $\lambda_{7,8}$ and six additional lines given by the equations

$$F_i = \alpha F_7 + \beta F_8 = 0.$$

We denote them by λ_i.

Lemma 2. *Possible singularities on Π_{gen} are double or triple points.*

- *A double point comes from a double line or a quadruple point on $\lambda_{7,8}$.*
- *A triple point comes from a triple line or a quintuple point on $\lambda_{7,8}$.*

Proof. If a point belongs to $\lambda_{7,8}$, then it must be quadruple or quintuple. If the point is not on $\lambda_{7,8}$, it belongs to two lines. We can assume that they are λ_1 and λ_2. Then the point belongs to $\lambda_{1,2}$. For a triple point which does not belong to $\lambda_{7,8}$, from the genericity we deduce that it is not isolated, so belongs to a triple line. There are no quadruple points or double lines on Π_{gen}, since there are no quadruple lines, sextuple points or double planes on the octic S. ∎

If one looks carefully, one sees that by blowing up points and lines on the octic we blow up the induced singularities on Π_{gen} in the admissible way. This proves that generically we obtain a K3 surface, so what we have is really a K3 fibration.

2.1.1. *The characterisation of singular fibres of the K3 fibration.* The aim of this section is to prove the following:

Proposition 2. *The singular fibres are either*

- *a singular K3 surface, that is, a surface with a singularity which after resolving will become a K3 surface, or*
- *a reducible surface Π^*, with the number of irreducible components denoted by $r(\Pi^*)$.*

The proof consists of four lemmas which give the full description of types of singular fibres depending on the singularities of the octic.

Lemma 3. *A blowing-up of a double line on a plane Π, a quintuple point or a quadruple point of type p_0^4 not belonging to $\lambda_{7,8}$ increases the number $r(\Pi^*)$ of irreducible components by* 1.

Proof. When blowing up both a line and a point, the only plane which intersects the exceptional divisor is Π, so Π^* must contain it. Because it is of dimension 2, it increases the number of irreducible components by 1. In the case of a quintuple point we have to notice that after blowing up we obtain some other singularities which can also increase $r(\Pi^*)$. ∎

Lemma 4. *If on a plane Π there exists a quadruple point P of type p_1^4 not belonging to $\lambda_{7,8}$, then after resolving it we obtain an additional irreducible component.*

Proof. Because P is contained in a triple line, we blow up the line first. The exceptional divisor is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Because the multiplicity of the line is odd, we add the exceptional divisor E. The proper transform of the fourth plane which goes through P contains a line $\sigma^{-1}(P)$, so $\sigma^{-1}(P)$ becomes a double line, and is contained in the preimage of Π. From Lemma 3 after blowing this line up we obtain an additional component. ∎

Lemma 5. *If on a plane Π there exists a triple line with k quadruple points of type p_1^4, then after blowing it up we obtain an additional component and $k+3$ double lines.*

Proof. Because a triple line is contained in Π the exceptional divisor is too. Furthermore, by Table 1, we obtain an additional $k+3$ lines which come from k quadruple points and three planes. Because we add the exceptional divisor, all of them become double lines. ∎

Lemma 6. *If there is a triple point P on a plane Π, which comes from the intersection of three distinct lines, say $\lambda_1 = \Pi \cap S_1$, $\lambda_2 = \Pi \cap S_2$, $\lambda_3 = \Pi \cap S_3$, then after a sequence of blowing-ups we obtain a singular K3 surface or a reducible surface.*

Proof. Without loss of generality, we may assume that we do not have any other types of singularities, i.e. after blowing-ups the resulting surface is irreducible.

From the assumption, a point P is an isolated triple point on the octic. We resolve it by three blowing-ups of the double lines $\lambda_{1,2}$, $\lambda_{1,3}$ and $\lambda_{2,3}$, say in this order. The exceptional divisor of the first blowing-up is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, which means that the intersection of the proper transforms of the planes S_3 and Π is equal to $\sigma^{-1}(P) \cup \sigma^{-1}(\lambda_3)$, where σ is a blowing-up.

Therefore, on Π it looks like blowing up the point P with adding the exceptional divisor, so we have an admissible blowing-up. Then, on the exceptional divisor we have three double points which come from $\lambda_1, \lambda_2, \lambda_3$, but we are left with only two blowing-ups. That is why we obtain a singular K3 surface. ∎

In the case of a double cover, it may happen that the preimage of an irreducible surface is reducible.

Definition 4. A plane whose intersection with an octic gives four double lines is said to be a *contact plane.*

We obtain the following:

Proposition 3. *Let H be a plane in $\mathbb{P}^3$. Let $\widetilde{H}$ be the proper transform of H after all blowing-ups. Then the double cover of $\widetilde{H}$ is reducible and contains exactly two irreducible components if and only if H is a contact plane.*

Proof. If H is not a contact plane, then its intersection with the octic consists of a line with odd multiplicity. Hence, the proper transform $\widetilde{H}$ intersects the branch divisor, so its double cover is irreducible.

If H is a contact plane, then its proper transform is disjoint from the branch divisor. This is caused by the fact that the exceptional divisor of a blowing-up of a line is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The proper transform of H is parallel to the branch divisor. Therefore, the preimage of $\widetilde{H}$ consists of exactly two disjoint copies of $\widetilde{H}$. ∎

Remark 5. Notice that the number of contact planes is finite.

To finish this section we will prove a formula which connects the Picard number of $\mathbb{P}_3^*$ with the Picard number of the generic fibre and the number of irreducible components of singular fibres.

Proposition 4. *The following equalities hold:*

$$\operatorname{rank}(\operatorname{Pic}(\mathbb{P}_3^*)) - 1 = \#\{\textit{blowing-ups}\} = \operatorname{rank}(\operatorname{Pic}(\Pi^*_{\mathrm{gen}})) + \sum_{\Pi^*} (r(\Pi^*) - 1).$$

Denoting fibres by $S := \pi^{-1}(\Pi^*)$, we obtain

$$\operatorname{rank}(\operatorname{Pic}(V)) \geq 1 + \operatorname{rank}(\operatorname{Pic}(S_{\mathrm{gen}})) + \sum_{S} (r(S) - 1).$$

Moreover, if k is the number of contact planes in a pencil then

$$\sum_{\Pi^*} (r(\Pi^*) - 1) + k = \sum_{S} (r(S) - 1).$$

Remark 6. Notice that a generic surface Π^* is irreducible and does not come from a contact plane. This means that generically $r(\Pi^*) = r(S) = 1$, so we add only finitely many nonzero elements in both equalities.

Proof of Proposition 4. The first equality is trivial since $\rho_{\mathbb{P}^3} = 1$.

To prove the second, notice that every blowing-up except the blowing-up of the line $\lambda_{7,8}$ gives us an additional linearly independent divisor in the Picard group of the generic fibre or an additional irreducible component in a singular fibre. Moreover, we have $\rho_{\Pi_{\mathrm{gen}}} = 1$, because Π_{gen} is isomorphic to $\mathbb{P}^2$. Hence, the number of blowing-ups is the right-hand side of the second equality.

The inequality is true due to the counting of the linearly independent divisors.

The last equality is an application of Proposition 3. ∎

3. Singular fibres of K3 fibrations on rigid double octics. In his doctoral thesis C. Meyer [M] gave a complete classification over $\mathbb{Q}$ of arrangements of eight planes which give rigid Calabi–Yau manifolds.

S. Cynk and D. van Straten [CvS] searched for rigid examples in one-parameter families given by C. Meyer and found three more arrangements that are not defined over $\mathbb{Q}$. We will call them a, b, c.

In Table 2, we found an explicit equation of the octic in every case. We also give the Picard number of $\mathbb{P}_3^*$ and of the resulting threefolds Y. In the description of a fibration we use Meyer's results, so we use his notation.

Table 2. Equations of rigid double octics

No.	ρ_Y	$\rho_{\mathbb{P}_3^*}$	Equation $0 = (x-t)(x+t)(y-t)(y+t)(z-t)(z+t)\cdots$
1	70	70	$(x+y)(y+z)$
3	62	62	$(x+y)(x+y+z+t)$
19	54	54	$(x+y)(x-y-z+t)$
32	50	50	$(x+z)(x+2y+z)$
69	50	50	$(x+y+z+t)(x+y-z-t)$
93	46	46	$(x+y+z+t)(x+y-2t)$
238	44	41	$(x+y+z+t)(x+y+z-t)$
239	40	39	$(x-y+z+t)(x+y-z-t)$
240	40	39	$(2x-2y+z+t)(x-y+z-t)$
241	40	39	$(x-2y+z)(x+2y+z)$
245	38	38	$(2x-2y+z+t)(2x+2y+z+t)$
a	46	46	$(x+(1+i\sqrt{3})y-i\sqrt{3}z)(x+i\sqrt{3}y-(1+i\sqrt{3})t)$
b	38	38	$(2x-(1+\sqrt{5})y+(3+\sqrt{5})z)(2x-2y+(1+\sqrt{5})z+(1+\sqrt{5})t)$
c	38	38	$(-i\sqrt{3}x+i\sqrt{3}y-z+t)(x-(1+i\sqrt{3})y-(1+i\sqrt{3})z-t)$

REMARK 7. Notice that in the open set $\{t=1\}$ octics consist of planes containing six faces of a cube and two additional planes. There are pictures

of 11 arrangements in [M]; we have not been able to draw the ones involving imaginary numbers.

As mentioned before, every double and triple line of the arrangement defines a K3 fibration of the Calabi–Yau threefold. For most of the fibrations we take the line $z+t=z-t=0$. Then the plane Π_α of the pencil is given by $z=\alpha t$, $\alpha \in \mathbb{C}\cup\{\infty\}$, where $z=\infty$ denotes the plane $t=0$.

In two cases, (32) and (69), we take the line $y=t=0$. Then the plane Π_β of the pencil is given by $y=\beta t$, $\beta\in\mathbb{C}\cup\{\infty\}$, where $y=\infty$ denotes also the plane $t=0$.

Remark 8. In the case of elliptic fibrations, any two fibrations on a Calabi–Yau threefold are equivalent. This is no longer true if we consider K3 fibrations. B. Hunt and R. Schimmrigk [HS] constructed a Calabi–Yau threefold with two inequivalent (rational) K3 fibrations.

Another such example can be seen in cases (32) and (69). In both cases we have triple lines ($x+z=x\pm t=0$ in (32) and $x+y=z+t=0$ in (69)) which in the case of the first fibration would belong to a singular member ($z=\pm t$) but in the second fibration are transversal to all members, raising the Picard number of the generic fibre by 2 and 1 respectively.

In case (238) one plane of the pencil is a contact plane.

Here, as an example, we present a description of how the fibration was achieved in case no. 1:

Generically: We have two triple lines which intersect Π_{gen}, namely $x+t=y-t=0, x-t=y+t=0$ and the quintuple point on the chosen line, namely $[1:0:0:0]$, which give three triple points on Π_{gen}. To obtain Π^*_{gen} we blow up Π_{gen} exactly 18 times. So $\rho_{\Pi^*_{\text{gen}}}=19$.

$\alpha=-1$: It is the plane of the arrangement. A blowing-up of a quintuple point $[1:0:0:0]$ gives a double line at infinity and a quadruple point of type p^4_1. A blowing-up of another quintuple point $[-1:1:-1:1]$ of type p^5_2 gives us, due to Lemma 3 and Table 1, an irreducible component, two quadruple points of type p^4_1 and five double lines. The triple line $y-t=z+t=0$ contains one of those points, a point at infinity and $[1:1:-1:1]$. Due to Lemma 5 and Table 1 we obtain an additional component and six double lines. From Lemma 4 after blowing-up the remaining quadruple points of type p^4_1 — the one got from the blowing-up and $[1:-1:-1:1]$, we obtain two components. We have exactly 16 double lines: 12 from blowing-ups and 4 from the intersection of the planes $x=t, x=-t, y=-t, x=-y$. By adding all irreducible components we conclude that the blown-up surface has exactly 21 irreducible components.

$\alpha = 1$: This is symmetric to the case $\alpha = -1$. Only coordinates have changed. The quintuple points are $[1:0:0:0]$ and $[1:-1:1:1]$, the quadruple points are $[-1:-1:1:1]$ and $[-1:1:1:1]$, a triple line is $y+t = z-t = 0$. We also have 21 irreducible components.

$\alpha = \infty$: We have two double lines $x = t = 0, y = t = 0$ and a quintuple point $[0:0:1:0]$ of type p_2^5. When we blow up a point we will get an additional component, two points of type p_1^4 and five double lines. Blowing up the quadruple points gives us two components. Seven double lines give another seven components. The blown-up surface consists of 11 irreducible components.

Now, we have to prove that we estimated the Picard number of the generic fibre $\rho_{S_{\text{gen}}}$ correctly.

It equals at least $\operatorname{rank}\operatorname{Pic}(\Pi^*_{\text{gen}})$. From Proposition 4 we have an upper bound.

For cases (1), (3), (19), (32), (69), (93), (245) we do not have contact planes. We have the equality $\operatorname{rank}(\operatorname{Pic}(\mathbb{P}_3^*)) = \operatorname{rank}(\operatorname{Pic}(V))$, which gives, due to Proposition 4,

$$\operatorname{rank}(\operatorname{Pic}(\Pi^*_{\text{gen}})) = \operatorname{rank}(\operatorname{Pic}(\mathbb{P}_3^*)) - 1 - \sum_{\Pi^*}(r(\Pi^*) - 1)$$
$$= \operatorname{rank}(\operatorname{Pic}(V)) - 1 - \sum_{S}(r(S) - 1) \geq \operatorname{rank}(\operatorname{Pic}(S_{\text{gen}})) \geq \operatorname{rank}(\operatorname{Pic}(\Pi^*_{\text{gen}})).$$

Thus every inequality is an equality, so in particular

$$\operatorname{rank}(\operatorname{Pic}(S_{\text{gen}})) = \operatorname{rank}(\operatorname{Pic}(\Pi^*_{\text{gen}})).$$

In case (239) we have the contact plane $y = z$, which does not belong to the pencil. On the generic plane Π_{gen} we have the line $y - z = z - \alpha t = 0$ which goes through three double points, $[1:\alpha:\alpha:1]$, $[-1:\alpha:\alpha:1]$ and $[1:0:0:0]$. Its proper transform is disjoint from the branch divisor so its double cover consists of two disjoint lines. In particular this means that $\operatorname{rank}(\operatorname{Pic}(S_{\text{gen}})) \geq \operatorname{rank}(\operatorname{Pic}(\Pi^*_{\text{gen}}))+1$ because each of these lines is a divisor which does not come from Π_{gen}, thus is linearly independent.

In cases (240) and (241) the problem is analogous because we have the lines $x - y = z - \alpha t = 0$ and $x + z = z - \alpha t = 0$.

In case (238) we have three lines which go through double points: $x+y = z-\alpha t = 0, x+z = z-\alpha t = 0, y+z = z-\alpha t = 0$.They increase the rank of the Picard group only by two. They are divisors which do not come from Π_{gen}, hence are linearly independent with those from Π_{gen}. If we choose two of those and the exceptional divisor of a double point we can show that two of those are linearly independent. All three cannot be linearly independent because we also have a contact plane in the pencil which increases the number

Table 3. All necessary facts about all fibrations

No.	$\rho_{\Pi^*_{\mathrm{gen}}}$	$\rho_{S_{\mathrm{gen}}}$	$\rho_{\mathbb{P}^*_3}$	ρ_Y	Number of irreducible components of a singular fibre					
1	19	19	70	70	-1	1	∞			
					21	21	11			
3	18	18	62	62	-3	-1	1	∞		
					1	25	10	11		
19	18	18	54	54	-1	1	3	∞		
					15	11	2	11		
32	18	18	50	50	-1	0	1	∞		
					10	4	10	11		
69	17	17	50	50	-1	1	∞			
					15	15	5			
93	17	17	46	46	-3	-1	1	∞		
					3	10	9	10		
238	16	18	41	44	-3	-1	1	3	∞	
					1	10	10	1	8	
239	16	17	39	40	-3	-2	-1	0	1	∞
					1	1	11	1	9	5
240	16	17	39	40	-5	-1	1	3	∞	
					1	10	9	3	4	
241	16	17	39	40	-3	-1	1	3	∞	
					1	10	10	1	5	
245	16	16	38	38	-3	-1	1	5	∞	
					1	11	8	1	5	
a	17	17	46	46	-1	1	$1-\frac{2}{\sqrt{3}}i$		∞	
					10	10	2		10	
b	16	16	38	38	-1	1	∞			
					10	9	5			
c	16	16	38	38	-1	1	∞			
					9	10	5			

of irreducible components of the singular fibre $t=0$ by one. Thus in case (238) the last equality of Proposition 4 is

$$\sum_{\Pi^*}(r(\Pi^*)-1)+1=\sum_{S}(r(S)-1).$$

Recall that in cases (239)–(241), $\operatorname{rank}(\operatorname{Pic}(\mathbb{P}^*_3))+1=\operatorname{rank}(\operatorname{Pic}(V))$ and in case (238) $\operatorname{rank}(\operatorname{Pic}(\mathbb{P}^*_3))+3=\operatorname{rank}(\operatorname{Pic}(V))$. Thus, if we fix $i=1$ or $i=2$

according to the case, we can show the equality

$$\operatorname{rank}(\operatorname{Pic}(S_{\mathrm{gen}})) = \operatorname{rank}(\operatorname{Pic}(\Pi^*_{\mathrm{gen}})) + i$$

using the following inequalities:

$$\operatorname{rank}(\operatorname{Pic}(\Pi^*_{\mathrm{gen}})) + i = \operatorname{rank}(\operatorname{Pic}(\mathbb{P}^*_3)) + i - 1 - \sum_{\Pi^*}(r(\Pi^*) - 1)$$

$$= \operatorname{rank}(\operatorname{Pic}(V)) - 1 - \sum_{S}(r(S) - 1) \geq \operatorname{rank}(\operatorname{Pic}(S_{\mathrm{gen}})) \geq \operatorname{rank}(\operatorname{Pic}(\Pi^*_{\mathrm{gen}})) + i.$$

Summing up, we have proved that all the data in Table 3 are correct and thus we have proved Theorem 1.

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References

[CM] S. Cynk and C. Meyer, *Geometry and arithmetic of certain double octic Calabi–Yau manifolds*, Canad. Math. Bull. 48 (2005), 180–194.

[CvS] S. Cynk and D. van Straten, *Calabi–Yau conifold expansions*, in: Arithmetic and Geometry of K3 Surfaces and Calabi–Yau Threefolds, Fields Inst. Comm. 67, Springer, New York, 2013, 499–515.

[CS] S. Cynk and T. Szemberg, *Double covers and Calabi–Yau varieties*, in: Banach Center Publ. 44, Inst. Math., Polish Acad. Sci., Warszawa, 1998, 93–101.

[HS] B. Hunt and R. Schimmrigk, *K3-fibered Calabi–Yau threefolds, I, the twist map*, Int. J. Math. 10 (1999), 833–867.

[KK] G. Kapustka and M. Kapustka, *Fiber products of elliptic surfaces with section and associated Kummer fibrations*, Int. J. Math. 20 (2009), 401–426.

[M] C. Meyer, *Modular Calabi–Yau Threefolds*, Fields Inst. Monogr. 22, Amer. Math. Soc., Providence, RI, 2005.

[O] K. Oguiso, *On certain rigid fibered Calabi–Yau threefolds*, Math. Z. 221 (1996), 437–448

[O2] K. Oguiso, *Shioda–Tate formula for an abelian fibered variety and applications*, J. Korean Math. Soc. 46 (2009), 237–248.

[S] C. Schoen, *On fiber products of rational elliptic surfaces with section*, Math. Z. 197 (1988), 177–199.

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