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MIXING ACTIONS OF ZERO ENTROPY FOR COUNTABLE AMENABLE GROUPS

BY

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Abstract. It is shown that each discrete countable infinite amenable group admits a 0-entropy mixing free action on a standard probability space.

0. Introduction. Let G be an amenable discrete infinite countable group. We recall that a measure preserving action $T = (T_g)_{g\in G}$ of G on a standard probability space $(X, \mathfrak{B}, \mu)$ is called *mixing* if $\mu(T_g A \cap B) \to \mu(A)\mu(B)$ as $g \to \infty$ for all measurable subsets $A, B \subset X$. The Kolmogorov–Sinai entropy for measure preserving transformations (i.e. $\mathbb{Z}$-actions) was extended to the actions of locally compact amenable groups in [OrWe]. Since 0-entropy actions are considered as "deterministic" while mixing actions are considered as "chaotic", it is natural to ask: are there actions which enjoy the two properties simultaneously? Such examples for $\mathbb{Z}$-actions were found first in the class of Gaussian transformations (see [New]), and later in the class of rank-one transformations (see [Or], [Ad], [CrSi] and references therein). The rank-one analogues of the latter family were constructed for $G = \mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}$ in [DaSi], for G being a countable direct sum of finite groups in [Da2] and, more generally, for G being a locally normal countable group in [Da3]. Dan Rudolph asked $(^1)$ whether *each* amenable countable group G has a mixing action of zero entropy. The purpose of this work is to answer this question affirmatively.

THEOREM 1. *There is a* 0*-entropy mixing free* $(^2)$ *probability preserving action of* G.

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$(^1)$ At an AMS meeting in the early 90's.

$(^2)$ We stick to the free actions to avoid degeneracy and triviality like the following: if $(T_g)_{g\in G}$ is a 0-entropy mixing action of G, F is a finite group and $\widetilde{T}_{g,f} := T_g$ for $(g, f) \in G \times F$, then $(\widetilde{T}_{g,f})_{(g,f)\in G\times F}$ is a 0-entropy mixing action of $G \times F$. We consider this nonfree action to be degenerate and uninteresting.

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Since the proof of the theorem is based on the Poisson suspensions of infinite measure preserving actions, we introduce the necessary definitions in the next section (see [Ne], [Roy], [Ja–Ru] for more details). We note that the mixing property follows essentially from the construction elaborated in [Da4]. Thus we have only to show here how to modify that construction to achieve the freeness (3) and 0-entropy of the action in question.

1. Poisson suspensions. Let X be a locally compact noncompact Cantor space (i.e. 0-dimensional, without isolated points). Denote by $C_{00}(X)$ the vector space of real-valued functions on X with compact support. This space is endowed with the usual locally convex topology, i.e. the topology of uniform convergence on the compact subsets of X. The dual $C_{00}(X)'$ is called the space of (real) Radon measures on X. We are interested in the cone $X^* \subset C_{00}(X)'$ of nonnegative Radon measures on X. Furnish X^* with the Borel σ-algebra $\mathfrak{B}^*$ generated by the $*$-weak topology related to the duality $\langle C_{00}(X), C_{00}(X)'\rangle$. We note that $\mathfrak{B}^*$ is also the Borel σ-algebra generated by the strong topology related to this duality. Since the strong topology is Polish and X^* is closed in $C_{00}(X)'$, it follows that $(X^*, \mathfrak{B}^*)$ is a standard Borel space.

Denote by $\mathcal{K}$ the set of all compact open subsets of X. Of course, $\mathcal{K}$ is infinite but countable. We also note that $\mathfrak{B}^*$ is the smallest σ-algebra on X^* such that for each $K \in \mathcal{K}$, the mapping $N_K : X^* \ni x^* \mapsto x^*(K) \in \mathbb{R}_+$ is measurable. Let μ^* be the only measure on $(X^*, \mathfrak{B}^*)$ such that

- $\mu^* \circ N_K^{-1}$ is the Poisson distribution with parameter $\mu(K)$, and
- the random variables $N_{K_1}, \ldots, N_{K_n}$ on (X^*, μ^*) are independent for each countable collection of mutually disjoint subsets $K_1, \ldots, K_m \in \mathcal{K}$.

Then $(X^*, \mathfrak{B}^*, \mu^*)$ is a standard probability space. To define μ^* rigorously we denote by $\mathfrak{F}$ the set of all finite collections of mutually disjoint nonempty compact open subsets of X. For $\mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{F}$, we write $\mathcal{F}_2 \succ \mathcal{F}_1$ if each element of $\mathcal{F}_1$ is a union of some elements from $\mathcal{F}_2$. Then $(\mathfrak{F}, \succ)$ is a directed partially ordered set. Given $K \in \mathcal{K}$, we define a probability measure ν_K on $\mathbb{R}_+$ by setting

$$\nu_K(A) := \sum_{i \in A \cap \mathbb{Z}_+} \frac{e^{-\mu(K)}\mu(K)^i}{i!}$$

for each Borel subset $A \subset \mathbb{R}_+$. We see, in particular, that ν_K is supported on $\mathbb{Z}_+$. Since $\mathbb{R}_+$ is an additive semigroup, the convolution of probability measures on $\mathbb{R}^*$ is well defined. It is easy to verify that if $K_1, K_2 \in \mathcal{K}$ and $K_1 \cap K_2 = \emptyset$ then $\nu_{K_1} * \nu_{K_2} = \nu_{K_1 \sqcup K_2}$.

(3) The freeness of the $H_3(\mathbb{R})$-actions considered in [Da4] follows from some properties of group actions that are specific to actions of connected nilpotent Lie groups.

Given $\mathcal{F} \in \mathfrak{F}$, we denote by $\mathbb{R}_+^{\mathcal{F}}$ the set of all mappings $x : \mathcal{F} \ni K \mapsto x(K) \in \mathbb{R}_+$. We define a measure $\nu^{\mathcal{F}}$ on $\mathbb{R}_+^{\mathcal{F}}$ as the direct product $\nu^{\mathcal{F}} := \bigotimes_{K\in\mathcal{F}} \nu_K$. If $\mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{F}$ and $\mathcal{F}_2 \succ \mathcal{F}_1$, we define a mapping $\pi_{\mathcal{F}_1}^{\mathcal{F}_2} : \mathbb{R}_+^{\mathcal{F}_2} \to \mathbb{R}_+^{\mathcal{F}_1}$ by setting

$$(\pi_{\mathcal{F}_1}^{\mathcal{F}_2}(x))(K) = \Big(\sum_{\{R\in\mathcal{F}_2 \mid R\subset K\}} x(R)\Big) \quad \text{for each } K \in \mathcal{F}_1 \text{ and } x \in \mathbb{R}_+^{\mathcal{F}_2}.$$

The aforementioned convolution property of ν_K implies that $\nu^{\mathcal{F}_2} \circ (\pi_{\mathcal{F}_1}^{\mathcal{F}_2})^{-1} = \nu^{\mathcal{F}_1}$. Thus $\{(\mathbb{R}^{\mathcal{F}_1}, \nu^{\mathcal{F}_2}, \pi_{\mathcal{F}_1}^{\mathcal{F}_2})_{\mathcal{F}_2\succ\mathcal{F}_1} \mid \mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{F}\}$ is a projective system of probability spaces. Hence the projective limit $(Y, \kappa) := \operatorname{proj\,lim}_{(\mathfrak{F},\succ)}(\mathbb{R}_+^{\mathcal{F}}, \nu^{\mathcal{F}})$ is well defined as a standard probability space.

We now define a Borel map $\Phi : X^* \to Y$ by setting

$$\Phi(x^*) := (\Phi(x^*)_{\mathcal{F}})_{\mathcal{F}\in\mathfrak{F}}, \qquad \text{where } \Phi(x^*)_{\mathcal{F}} := (N_K(x^*))_{K\in\mathcal{F}}.$$

Then Φ is one-to-one and onto. Indeed, if two nonnegative Radon measures take the same values on every element of $\mathcal{K}$ then these measures are equal. This proves that Φ is one-to-one. On the other hand, each element y of Y can be interpreted as a finitely additive nonnegative measure on $\mathcal{K}$, i.e. as a map $y : \mathcal{K} \to \mathbb{R}_+$ such that $y(K_1 \sqcup \cdots \sqcup K_s) = y(K_1) + \cdots + y(K_s)$ for every collection $\{K_1, \ldots, K_s\} \in \mathfrak{F}$. Then y extends uniquely to a σ-finite (non-negative) measure on X. Of course, the extension is a Radon measure. Hence Φ is onto. Thus Φ is a Borel isomorphism of X^* onto Y. It remains to export κ to X^* via Φ^{-1} and denote this measure by μ^*. It follows, in particular, that for each $K \in \mathcal{K}$ and $j \in \mathbb{Z}_+$,

$$\mu^*(\{x^* \in X^* \mid x^*(K) = j\}) = \frac{\mu(K)^j e^{-\mu(K)}}{j!}. \tag{1}$$

If X is partitioned into a union of mutually disjoint open noncompact subsets $X_1, \ldots, X_m$ then the mapping

$$x^* \mapsto (x^* \restriction X_1, \ldots, x^* \restriction X_m)$$

is a measure preserving Borel isomorphism of (X^*, μ^*) onto the direct product space $(X_1^* \times \cdots \times X_m^*, (\mu\restriction X_1)^* \times \cdots \times (\mu\restriction X_m)^*)$.

Given a Borel σ-algebra $\mathfrak{F} \subset \mathfrak{B}$ generated by a topology which is weaker than the original topology on X, we denote by $\mathfrak{F}^*$ the smallest sub-σ-algebra of $\mathfrak{B}^*$ such that the mapping N_K is measurable for each $K \in \mathfrak{F} \cap \mathcal{K}$. For an increasing sequence of topologies $\tau_1 \prec \tau_2 \prec \cdots$ on X which are weaker than the original topology on X, we denote by $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \cdots$ the increasing sequence of Borel sub-σ-algebras generated by these topologies. Since every compact open subset from $\bigvee_{n=1}^{\infty} \mathfrak{F}_n$ is contained in $\mathfrak{F}_m$ for some $m > 0$, it follows that

$$\Big(\bigvee_{n=1}^{\infty} \mathfrak{F}_n\Big)^* = \bigvee_{n=1}^{\infty} \mathfrak{F}_n^*. \tag{2}$$

Given an action $T = (T_g)_{g\in G}$ of G on X by μ-preserving homeomorphisms T_g, we associate a Borel action $T^* := (T_g^*)_{g\in G}$ on X^* by setting $T_g^* x^* := x^* \circ T_g^{-1}$, $g \in G$. It follows from (1) that T^* preserves μ^*. The dynamical system $(X^*, \mathfrak{B}^*, \mu^*, T^*)$ is called the *Poisson suspension* of $(X, \mathfrak{B}, \mu, T)$.

2. Proof of Theorem 1. We will proceed in several steps.

STEP 1. First we construct a free strictly ergodic infinite measure preserving G-action on a locally compact noncompact Cantor space. For that we will utilize the (C,F)-construction (see [Da1], [Da5]). Let $(F_n, C_{n+1})_{n=0}^\infty$ be a sequence of finite subsets in G such that $(F_n)_{n=0}^\infty$ is a Følner sequence in G, $F_0 = \{1\}$, and for each $n \ge 1$ the following three basic conditions are satisfied:

- $1 \in F_n$ and $\#C_n > 1$,
- $F_n^{-1} F_n F_n C_{n+1} \subsetneq F_{n+1}$,
- $F_n c \cap F_n c' = \emptyset$ for all $c \neq c' \in C_{n+1}$.

We let $X_n := F_n \times C_{n+1} \times C_{n+2} \times \cdots$. Then X_n endowed with the infinite product of the discrete topologies on F_n and C_j, $j > n$, is a compact Cantor set. Moreover, the mapping

$$X_n \ni (f_n, c_{n+1}, c_{n+2}, \dots) \mapsto (f_n c_{n+1}, c_{n+2}, \dots) \in X_{n+1}$$

is a topological embedding of X_n into X_{n+1}.

We now consider the union $X = \bigcup_{n\ge 1} X_n$ and endow it with the topology of inductive limit. Then X is a locally compact noncompact Cantor set and X_n is a compact open subset of X for each n. Given $g \in G$ and $n > 0$, we define a homeomorphism $T_g^{(n)}$ from a clopen subset $(g^{-1}F_n \cap F_n) \times C_{n+1} \times C_{n+2} \times \cdots$ of X_n to a clopen subset $(F_n \cap gF_n) \times C_{n+1} \times C_{n+2} \times \cdots$ of X_n by setting

$$T_g^{(n)}(f_n, c_{n+1}, c_{n+2}, \dots) \mapsto (gf_n, c_{n+1}, c_{n+2}, \dots).$$

It is easy to verify that the sequence $(T_g^{(n)})_{n\ge 1}$ uniquely determines a homeomorphism T_g of X such that $T_g \restriction X_n = T_g^{(n)}$ for each n.

It is straightforward to check that $T := (T_g)_{g\in G}$ is a free topological action of G on X. This action is minimal and uniquely ergodic, i.e. there is a unique T-invariant σ-finite Radon measure μ on X such that $\mu(X_0) = 1$. To define μ explicitly we consider, for each $n \ge 0$ and $f \in F_n$, the subset $[f]_n := \{(f, c_{n+1}, c_{n+2}, \dots) \in X_n \mid c_i \in C_i \text{ for all } i > n\}$. Then $[f]_n$ is compact and open. We call it a *cylinder*. The family of all cylinders is a base for the topology on X. Every compact open subset of X is a union of finitely many mutually disjoint cylinders. Hence every Radon measure on X

is determined uniquely by its values on the cylinders. It remains to note that $\mu([f]_n) = 1/(\#C_1 \cdots \#C_n)$ for each $n \geq 0$ and $f \in F_n$.

In addition to the above three basic conditions on $(F_n, C_{n+1})_{n\geq 0}$, we will assume that

$$\lim_{n\to\infty} \frac{\#F_n}{\#C_1 \cdots \#C_n} = \infty.$$

This is equivalent to the fact that $\mu(X) = \infty$.

STEP 2. In this step we obtain a free probability preserving G-action. For that we need two more conditions on $(F_n, C_{n+1})_{n=0}^{\infty}$:

- ($\triangle$) for each element g of infinite order in G, there are infinitely many n such that $g^{l_n} F_n C_{n+1} \subset F_{n+1} \setminus (F_n C_{n+1})$ for some $l_n > 0$, and
- ($\square$) for each element g of finite order in G, there are infinitely many n such that $gF_n = F_n$.

It is easy to see that if ($\triangle$) is satisfied then

$$T_{g^{l_n}} X_n = T_{g^{l_n}} \bigsqcup_{f\in F_n} [f]_n = T_{g^{l_n}} \bigsqcup_{f\in F_n} \bigsqcup_{c\in C_{n+1}} [fc]_{n+1} = \bigsqcup_{f\in F_n} \bigsqcup_{c\in C_{n+1}} [g^{l_n} fc]_{n+1}.$$

We used the fact that $X_n = \bigsqcup_{f\in F_n} [f]_n$ and $T_s[f]_n = [sf]_n$ whenever $f, sf \in F_n$ and $s \in G$. Hence $T_{g^{l_n}} X_n \subset X_{n+1}$ and $T_{g^{l_n}} X_n \cap X_n = \emptyset$.

Let (X^*, μ^*, T^*) be the Poisson suspension of (X, μ, T). Then μ^* is T^*-invariant and $\mu^*(X^*) = 1$. We now verify that T^* is free. For that we will check that for each $g \in G \setminus \{1\}$, the subset of fixed points of the transformation T^*_g is μ^*-null. Consider two cases.

If g is of infinite order and A is a T_g-invariant subset of positive finite measure in X then in view of ($\triangle$) there is $n > 0$ such that $\mu(A \cap X_n) > 0.9\mu(A)$ and $T_{g^{l_n}} X_n \cap X_n = \emptyset$. Hence

$$\mu(A \cap X_n) = \mu(T_{g^{l_n}} A \cap T_{g^{l_n}} X_n) = \mu(A \cap T_{g^{l_n}} X_n) \leq \mu(A \setminus X_n) \leq 0.1\mu(A),$$

a contradiction. Thus the transformation T_g has no invariant subsets of finite positive measure. Therefore the Poisson suspension T^*_g of T_g is weakly mixing [Roy]. This yields $\mu^*(\{x^* \in X^* \mid T^*_g x^* = x^*\}) = 0$.

Consider now the case where g is of finite order. Let $H \subset G$ denote the cyclic subgroup generated by g. It follows from ($\square$) that there is an open subset $Y \subset X$ of infinite measure such that the sets $T_h Y$, $h \in H$, form an open partition of X. Indeed, let $gF_{n_i} = F_{n_i}$ for an increasing sequence $n_1 < n_2 < \cdots$. Choose a subset $S_1 \subset F_{n_1}$ which meets each H-coset in F_{n_1} exactly once. If $i > 1$, choose a subset $S_i \subset F_{n_i}$ which meets each H-coset in $F_{n_i} \setminus (F_{n_{i-1}} C_{n_{i-1}+1} \cdots C_{n_i})$ exactly once. It remains to set $Y := \bigsqcup_{i=1}^{\infty} \bigsqcup_{s\in S_i} [s]_{n_i}$. Of course, Y is open and noncompact and $X = \bigsqcup_{h\in H} T_h Y$. Since $\infty = \mu(X) = \mu(\bigsqcup_{h\in H} T_h Y)$ and T preserves μ, it follows that $\mu(Y) = \infty$. Then the dynamical system $(X^*, \mu^*, (T^*_h)_{h\in H})$ is isomorphic to the finite direct product $(Y^*, (\mu\restriction Y)^*)^H$ endowed with the natural shiftwise action of H (see §1). Since the measure $(\mu\restriction Y)^*$ is nonatomic, this action is free.

STEP 3. We now verify that $h(T^*) = 0$. Let $\mathfrak{F}_n$ denote the σ-algebra on X generated by a single set $[1]_n$, and let $\mathfrak{B}_n$ denote the σ-algebra on X generated by the compact open sets $[f]_n$, $f \in F_n$. Then $\mathfrak{B}_1 \subset \mathfrak{B}_2 \subset \cdots$ and $\bigvee_{n=1}^\infty \mathfrak{B}_n$ is the entire Borel σ-algebra $\mathfrak{B}$ on X. Moreover, $\mathfrak{B}_n = \bigvee_{g\in F_n} T_g\mathfrak{F}_n$ and

$$\mathfrak{B}_n^* = \bigvee_{g\in F_n} T_g^*\mathfrak{F}_n^* \subset \bigvee_{g\in G} T_g^*\mathfrak{F}_n^* =: (\mathfrak{F}_n^*)^G$$

for each n. Therefore $\bigvee_{n=1}^\infty (\mathfrak{F}_n^*)^G \supset \bigvee_{n=1}^\infty \mathfrak{B}_n^* = \mathfrak{B}^*$. The latter equality follows from (2). Moreover, it is easy to verify that $(\mathfrak{F}_1^*)^G \subset (\mathfrak{F}_2^*)^G \subset \cdots$. Hence

$$h(T^*) = \lim_{n\to\infty} h(T^* \mid (\mathfrak{F}_n^*)^G) \le \limsup_{n\to\infty} H(\mathfrak{F}_n^*).$$

We recall that $\mathcal{K}$ denotes the collection of all compact open subsets of X. Since $\mathfrak{F}_n \cap \mathcal{K} = \{[1]_n\}$, the σ-algebra $\mathfrak{F}_n^*$ is generated by the countable partition of X^* into the sets $N_{[1]_n}^{-1}(r) = \{x^* \in X^* \mid x^*([1]_n) = r\}$, $r = 0, 1, \ldots$. This yields

$$\limsup_{n\to\infty} H(\mathfrak{F}_n^*) = \lim_{n\to\infty} f(\mu([1]_n)),$$

where $f(t)$ is the entropy of the Poisson distribution $(e^{-t}, e^{-t}t, e^{-t}t^2/2, \ldots)$. Since $\mu([1]_n) = 1/(\#C_1 \cdots \#C_n) \to 0$, it follows that $h(T^*) = 0$.

STEP 4. We show that some extra conditions on $(F_n, C_{n+1})_{n\ge0}$ imply mixing for the dynamical system (X^*, μ^*, T^*). Thus from now on we will assume that the following hold for each n (in addition to the conditions on $(F_n, C_{n+1})_{n\ge0}$ listed above):

(i) $F_nF_n^{-1}F_nC_{n+1} \subset F_{n+1}$,
(ii) the sets $F_nc_1c_2^{-1}F_n^{-1}$, $c_1 \ne c_2 \in C_{n+1}$, and $F_nF_n^{-1}$ are all pairwise disjoint, and
(iii) $\#C_n \to \infty$ as $n \to \infty$.

Denote by $U_T = (U_T(g))_{g\in G}$ the associated Koopman unitary representation of G in $L^2(X,\mu)$, i.e. $U_T(g)f := f \circ T_g^{-1}$ for each $f \in L^2(X,\mu)$. As shown in [Da4, Theorem 5.1] ([4]), conditions (i)–(iii) imply that T is mixing as an infinite measure preserving action, i.e. $U_T(g) \to 0$ weakly as $g \to \infty$. Let U_{T^*} stand for the Koopman unitary representation of G in $L^2(X^*, \mu^*)$

([4]) Though we considered in [Da4] mainly the actions of the Heisenberg group $H_3(\mathbb{Z})$, the proof of Theorem 5.1 there does not use any specific property of $H_3(\mathbb{Z})$. The theorem holds for each amenable group.

associated with the Poisson suspension (X^*, μ^*, T^*). It is well known that U_{T^*} is unitarily equivalent to the Fock unitary representation of G generated by U_T in the Fock space generated by $L^2(X, \mu)$ (see [Ne]). It follows that $U_{T^*}(g)$ converges weakly to the orthogonal projection onto the constants in $L^2(X^*, \mu^*)$ as $g \to \infty$. This is equivalent to the fact that T^* is mixing.

Summarizing, T^* is a mixing free action of G on (X^*, μ^*) and $h(T^*) = 0$. Thus Theorem 1 is proved.

Added in proof. After the paper had been accepted, the author learned about the following recent result by R. D. Tucker-Drob [Proc. Amer. Math. Soc. 143 (2015), 5227–5232] which complements our footnote (^{2}): if an action of a discrete countable group G is mixing then there is a finite normal subgroup $F \subset G$ such that the stabilizer of a.e. point is F.

REFERENCES

[Ad] T. M. Adams, *Smorodinsky's conjecture on rank-one mixing*, Proc. Amer. Math. Soc. 126 (1998), 739–744.

[CrSi] D. Creutz and C. E. Silva, *Mixing on a class of rank-one transformations*, Ergodic Theory Dynam. Systems 24 (2004), 407–440.

[Da1] A. I. Danilenko, *(C, F)-actions in ergodic theory*, in: Geometry and Dynamics of Groups and Spaces, Progr. Math. 265, Birkhäuser, Basel, 2008, 325–351.

[Da2] A. I. Danilenko, *Mixing rank-one actions for infinite sums of finite groups*, Israel J. Math. 156 (2006), 341–358.

[Da3] A. I. Danilenko, *Uncountable collection of mixing rank-one actions for locally normal groups*, in: École de Théorie Ergodique, Sémin. Congr. 20, Soc. Math. France, Paris, 2010, 253–266.

[Da4] A. I. Danilenko, *Mixing actions of the Heisenberg group*, Ergodic Theory Dynam. Systems 34 (2014), 1142–1167.

[Da5] A. I. Danilenko, *Actions of finite rank: weak rational ergodicity and partial rigidity*, Ergodic Theory Dynam. Systems, online.

[DaSi] A. I. Danilenko and C. E. Silva, *Mixing rank-one actions of locally compact Abelian groups*, Ann. Inst. H. Poincaré Probab. Statist. 43 (2007), 375–398.

[Ja–Ru] É. Janvresse, T. Meyerovitch, E. Roy and T. de la Rue, *Poisson suspensions and entropy for infinite transformations*, Trans. Amer. Math. Soc. 362 (2010), 3069–3094.

[Ne] Yu. A. Neretin, *Categories of Symmetries and Infinite-Dimensional Groups*, London Math. Soc. Monogr. (N.S.) 16, Clarendon Press, New York, 1996.

[New] D. Newton, *On Gaussian processes with simple spectrum*, Z. Wahrsch. Verw. Gebiete 5 (1966), 207–209.

[Or] D. S. Ornstein, *On the root problem in ergodic theory*, in: Proc. Sixth Berkeley Sympos. Math. Statist. Probab. (Berkeley, CA, 1970/1971), Vol. II, Univ. of California Press, Berkeley, CA, 1972, 347–356.

[OrWe] D. S. Ornstein and B. Weiss, *Entropy and isomorphism theorems for actions of amenable groups*, J. Anal. Math. 48 (1987), 1–141.

[Roy] E. Roy, *Poisson suspensions and infinite ergodic theory*, Ergodic Theory Dynam. Systems 29 (2009), 667–683.

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