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A. Dziwisz and W. Szczotka (Wrocław)

FUNCTIONAL CENTRAL LIMIT THEOREMS FOR SEEDS IN A LINEAR BIRTH AND GROWTH MODEL

Abstract. A problem of heredity of mixing properties (α-mixing, β-mixing and ρ-mixing) from a stationary point process on $\mathbb{R}\times\mathbb{R}_+$ to a sequence of some of its points called 'seeds' is considered. Next, using the mixing properties, several versions of functional central limit theorems for the distances between seeds and the process of the number of seeds are obtained.

1. Introduction. The problem considered in the paper has a practical motivation and it can be illustrated in the following way. The points (for example drops of rain) land in a random fashion on the interval $[0, L]$, which is 'uncovered' (dry) initially, and a point (seed) landing on an uncovered (dry) section starts to spill over into the interval in a uniform rate in both directions. To this phenomenon, called in Quine & Szczotka 2000 (later referred to as Q-S) a linear birth and growth model, two problems are related. The first one is to characterize the number $N(L)$ of seeds on $[0, L]$, and the second one is to characterize the asymptotic of the distribution of the time to complete coverage of the interval $[0, L]$ as $L \to \infty$. In this paper we consider the first problem. The above set-up may have applications in a number of diverse fields.

The analysis of the problem is facilitated by considering the points as a bivariate point process Ξ in $(-\infty,\infty) \times [0,\infty)$, with the vertical axis representing arrival times and the horizontal axis representing location on the line. Points in Ξ whose positions represent arrivals to the uncovered part of the line are the "seeds" and the other points of Ξ are "thinned". The way of thinning can be described as follows. With each point of the process Ξ on

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$\mathbb{R} \times \mathbb{R}_+$ two branches (half-lines) are associated and they go upwards with slopes (rates) $+1$ and -1, respectively. This gives a set of zigzag functions on the upper half-plane. Take the minimum of all zigzag functions. Then the bottom vertices of that zigzag function are called the *seeds* of the process Ξ.

The problems formulated above were considered by many authors; the references can be found in Quine & Robinson (1990), Holst et al. (1996) and Q-S. Except for Q-S, in all other papers it was assumed that the point process Ξ on $\mathbb{R} \times \mathbb{R}_+$ or on $\mathbb{R} \times \mathbb{R}_+^d$ is a Poisson point process. We start from the assumption that Ξ on $\mathbb{R} \times \mathbb{R}_+$ is only stationary. This requires a different approach to the problem. Here we use the approach from Section 2 of Q-S, where for the stationary point process Ξ a stationary simple marked point process $\{(X_i, t_i), i \in \mathbb{Z}\}$ was defined (for definition a stationary simple marked point process see Brandt et al. (1990)). Theorem 2.1 of Q-S shows that the process of seeds of Ξ is the same as the process of seeds of $\{(X_i, t_i),\ i \in \mathbb{Z}\}$ obtained in the same way as above for Ξ. The seeds process of $\{(X_i, t_i),\ i \in \mathbb{Z}\}$ is denoted by $\{(X_i^*, t_i^*), i \in \mathbb{Z}\}$. Let $\{(\hat{X}_i, \hat{t}_i),\ i \in \mathbb{Z}\}$ and $\{(\tilde{X}_i^*, \tilde{t}_i^*),\ i \in \mathbb{Z}\}$ be the Palm versions of $\{(X_i, t_i),\ i \in \mathbb{Z}\}$ and $\{(X_i^*, t_i^*), i \in \mathbb{Z}\}$, respectively. This means that the distributions of $\{(\hat{X}_i, \hat{t}_i),\ i \in \mathbb{Z}\}$ and $\{(\tilde{X}_i^*, \tilde{t}_i^*),\ i \in \mathbb{Z}\}$ are the conditional distributions of $\{(X_i, t_i),\ i \in \mathbb{Z}\}$ given $X_0 = 0$ and of $\{(X_i^*, t_i^*), i \in \mathbb{Z}\}$ given $X_0^* = 0$, respectively. Hence $\{(\hat{u}_i, \hat{t}_i),\ i \in \mathbb{Z}\}$ and $\{(\tilde{u}_i^*, \tilde{t}_i^*),\ i \in \mathbb{Z}\}$ are stationary sequences, where $\hat{u}_i := \hat{X}_{i+1} - \hat{X}_i$ and $\tilde{u}_i^* := \tilde{X}_{i+1}^* - \tilde{X}_i^*$, $i \in \mathbb{Z}$. From now on, to study stochastic properties of the process of seeds we start from $\{(\hat{X}_i, \hat{t}_i),\ i \in \mathbb{Z}\}$ and from the stationary sequence $\{(\hat{u}_i, \hat{t}_i),\ i \in \mathbb{Z}\}$, but for simplicity we drop the hats over X_i, t_i and u_i.

To study the properties of the process of seeds we first investigate heredity of mixing properties (α-mixing, β-mixing and ρ-mixing) from the stationary sequence $\{(u_i, t_i), i \in \mathbb{Z}\}$ to the sequences $\{(u_i^*, t_i^*), i \in \mathbb{Z}\}$ and $\{(\tilde{u}_i^*, \tilde{t}_i^*),\ i \in \mathbb{Z}\}$, where

$$u_i^* = X_{i+1}^* - X_i^* \quad \text{and} \quad \tilde{u}_i^* = \tilde{X}_{i+1}^* - \tilde{X}_i^*, \qquad i \in \mathbb{Z}$$

(Theorems 2.1–2.5). Next we use the mixing conditions to give several versions of the functional central limit theorem (FCLT) for $\{u_i^*\}$, $\{\tilde{u}_i^*\}$ and also for the process of seeds (Theorems 3.1–3.6).

The problem of heredity of α-mixing was also considered in Q-S and FCLT for $\{\tilde{u}_i^*\}$ was obtained in Theorem 6.1 there by using Theorem 1.7 of Peligrad (1986, p. 202) for stationary α-mixing sequences of random variables. Here, we use the method from Q-S with some modifications, giving a better rate of convergence to zero of the α-mixing function (compare Theorem 2.1 here with Theorem 5.1 of Q-S). That in turn allows us to get a stronger version of Theorem 6.1 of Q-S (FCLT for $\{\tilde{u}_i^*\}$ in the α-mixing case), which we formulate in Theorem 3.4.

FCLT for $\{\tilde{u}_i^*, i \geq 1\}$ in the ρ-mixing case is obtained here in Theorem 3.6. It is based on our conditions for ρ-mixing of $\{\tilde{u}_i^*\}$ (Theorems 2.3–2.5) and on Herrndorf's (1984) version of FCLT for ρ-mixing sequences.

From now on, doubly infinite sequences, for example $\{a_i, -\infty < i < \infty\}$, where the a_i have an arbitrary nature, are written just as $\{a_i\}$. Therefore we write $\{(X_i, t_i)\}$ instead of $\{(X_i, t_i), i \in \mathbb{Z}\}$.

2. Heredity of mixing conditions

2.1. General notation for mixing. In this subsection we recall definitions and notation for α-mixing, β-mixing and ρ-mixing. All the definitions can be found in Bradley (1986).

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{A}$ and $\mathcal{B}$ be some subsigma fields of $\mathcal{F}$. The dependencies α, β and ρ between σ-fields $\mathcal{A}$ and $\mathcal{B}$ are defined as follows:

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup_{A,B} |P(AB) - P(A)P(B)|,$$

$$\beta(\mathcal{A}, \mathcal{B}) = \sup_{A_i, B_j, 1\leq i\leq I, 1\leq j\leq J} \frac{1}{2} \sum_{i=1}^{I} \sum_{j=1}^{J} |P(A_i B_j) - P(A_i)P(B_j)|,$$

$$\rho(\mathcal{A}, \mathcal{B}) = \sup_{A,B} \frac{|P(AB) - P(A)P(B)|}{P^{1/2}(A)P^{1/2}(B)},$$

where the supremum is taken over all $A \in \mathcal{A}$, $B \in \mathcal{B}$ for the α- and ρ-dependencies, and over all partitions $\{A_1, \ldots, A_I\}$ and $\{B_1, \ldots, B_J\}$ of Ω, where $A_i \in \mathcal{A}$ and $B_j \in \mathcal{B}$, for the β-dependency. Here $P^{1/2}(D) = (P(D))^{1/2}$ for any event D.

The above measures of dependency allow us to define the corresponding measures of dependency between random variables in a sequence. Namely, let $\{Z_i, i \in \mathbb{Z}\} \equiv \{Z_i\}$ be a sequence of r.v.'s on $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_k^m$ the σ-field generated by the random variables Z_i with $k \leq i \leq m$, $-\infty \leq k \leq m \leq \infty$, written $\mathcal{F}_k^m = \sigma\{Z_i, k \leq i \leq m\}$. Furthermore let

$$\alpha(n) := \sup_{k\in\mathbb{Z}} \alpha(\mathcal{F}_{-\infty}^k, \quad \mathcal{F}_{k+n}^\infty), \quad \beta(n) := \sup_{k\in\mathbb{Z}} \beta(\mathcal{F}_{-\infty}^k, \mathcal{F}_{k+n}^\infty),$$
$$\rho(n) := \sup_{k\in\mathbb{Z}} \rho(\mathcal{F}_{-\infty}^k, \mathcal{F}_{k+n}^\infty).$$

If $\{Z_i, i \in \mathbb{Z}\}$ is a stationary sequence of r.v.'s then

$$\alpha(n) = \alpha(\mathcal{F}_{-\infty}^0, \mathcal{F}_n^\infty) = \alpha(\mathcal{F}_{-\infty}^{-n}, \mathcal{F}_0^\infty),$$

and similar relations hold for β- and ρ-mixing. The sequence $\{Z_i\}$ is called *α-mixing*, *β-mixing* or *ρ-mixing* if $\alpha(n) \to 0$, $\beta(n) \to 0$ or $\rho(n) \to 0$, respectively.

2.2. Mixing properties for a point process. There are at least two approaches to defining a mixing property for a stationary point process on $\mathbb{R}\times\mathbb{R}_+$. Consider a stationary point process on $\mathbb{R}$. Let $\{\eta_i,\, i\in\mathbb{Z}\}\equiv\{\eta_i\}$ be a sequence of points in the intervals $[i,i+1)$, $i\in\mathbb{Z}$, respectively, of the point process, and $\{u_i,\ i\in\mathbb{Z}\}\equiv\{u_i\}$ the lengths of the intervals between the ith and $(i+1)$th points of the point process. Then the point process is said to be *α-mixing in the first sense* if the sequence $\{\eta_i\}$ is α-mixing, and *α-mixing in the second sense* if $\{u_i\}$ is α-mixing. In a similar way we can define two approaches to β-mixing and ρ-mixing of a point process on $\mathbb{R}$, but we omit it here. It is obvious that if the random variables η_i are pairwise independent, then the u_i need not be, and vice versa. However, below we show that if the sequence $\{u_i\}$ is α-mixing, then so is $\{\eta_i\}$ (with a different mixing function). This enables a comparison between the approaches to FCLT for the process of seeds presented in Chiu & Quine (1999), and in Q-S and here.

LEMMA 2.1. *If the stationary sequence $\{u_i\}$ is α-mixing with mixing function $\alpha_u=\{\alpha_u(n)\}$ then the stationary sequence $\{\eta_i\}$ is α-mixing with mixing function $\alpha_\eta=\{\alpha_\eta(n)\}$, where*

$$\alpha_\eta(n)\le\alpha_u(k_n)+4P(C_n^c),$$

with $C_n=\{\sum_{j=1}^{k_n}u_j<n\}$ and $n/k_n\to 2Eu_1$.

Proof. By definition we have $\alpha_\eta(n)=\sup_{A,B}|P(AB)-P(A)P(B)|$, where the supremum is taken over all $A\in\sigma(\{\eta_i,i\le 0\})$ and $B\in\sigma(\{\eta_i,i>n\})$. Notice that

$$|P(AB)-P(A)P(B)|\le|P(ABC_n)-P(A)P(BC_n)|+2P(C_n^c).$$

But for any n there exists a Borel set $\tilde{B}_n$ in $\sigma(\mathbb{R}^\infty)$ such that

$$B\cap C_n=\{\{u_j,\, j\ge k_n\}\in\tilde{B}_n\}\cap C_n=\hat{B}_n\cap C_n,$$

where $\hat{B}_n=\{\{u_j,\, j\ge k_n\}\in\tilde{B}_n\}$. Furthermore, any $A\in\sigma(\eta_i,i\le 0)$ belongs to $\sigma(u_i,i\le 0)$. Hence

$$\begin{aligned}|P(ABC_n)-P(A)P(BC_n)|&=|P(A\hat{B}_nC_n)-P(A)P(\hat{B}_nC_n)|\\&=|P(A\hat{B}_n)-P(A\hat{B}_nC_n^c)-P(A)P(\hat{B}_n)+P(A)P(\hat{B}_nC_n^c)|\\&\le|P(A\hat{B}_n)-P(A)P(\hat{B}_n)|+2P(C_n^c)\le\alpha_u(k_n)+2P(C_n^c).\end{aligned}$$

This finishes the proof. ∎

In the next section we will understand that the stationary point process Ξ on $\mathbb{R}\times\mathbb{R}_+$ is α-mixing, β-mixing or ρ-mixing if the marked point process $\Psi=\{(X_i,t_i),\, i\in\mathbb{Z}\}$ is α-mixing, β-mixing or ρ-mixing in the second sense, i.e. the sequence $\{(u_i,t_i)\}$ is mixing in the appropriate sense, where $\{u_i=X_{i+1}-X_i\}$ is stationary.

2.3. Heredity of mixing conditions from $\{(u_i, t_i)\}$ to $\{(u_i, t_i, s_i)\}$

2.3.1. *General notation.* In this subsection we introduce general notation which will be used to prove heredity of the mixing conditions from the input process $\{(u_i, t_i)\}$ to the output process $\{(u_i, t_i, s_i)\}$, where $s_i = 1$ if the ith point (X_i, t_i) is the seed, and $s_i = 0$ otherwise. First we recall from Q-S the algorithm of getting the seeds. It can be decomposed into two steps. In the first step we remove all points which are covered by left branches, i.e. with each point (X_i, t_i) is associated $l_i = 0$ if (X_i, t_i) is covered by a left branch, and $l_i = 1$ otherwise. After that operation the first output process $\{(u_i, t_i, l_i)\}$ is obtained, where

$$l_i = \mathbb{I}\Big(X_i + t_i < \min_{j>i}(X_j + t_j)\Big) \quad \text{for } i \in \mathbb{Z},$$

and $\mathbb{I}(A)$ denotes the indicator of the event A, i.e. $\mathbb{I}(A)(\omega) = 1$ if $\omega \in A$ and $= 0$ otherwise. In the second step all points covered by a right branch are removed, which transforms the sequence $\{(u_i, t_i, l_i)\}$ into the second output process $\{(u_i, t_i, l_i, r_i)\}$ with

$$r_i = \mathbb{I}\Big(X_i - t_i > \max_{j<i}(X_j - t_j)\Big).$$

Finally let $s_i = r_i l_i$, $i \in \mathbb{Z}$.

Now we introduce notation for some σ-fields:

$$\begin{aligned}
\mathcal{B}(-\infty, n) &= \sigma((u_i, t_i),\ i \le n), & \mathcal{B}(n, \infty) &= \sigma((u_i, t_i),\ i > n),\\
\mathcal{B}^l(-\infty, n) &= \sigma((u_i, t_i, l_i),\ i \le n), & \mathcal{B}^l(n, \infty) &= \sigma((u_i, t_i, l_i),\ i > n),\\
\mathcal{B}^r(-\infty, n) &= \sigma((u_i, t_i, l_i, r_i),\ i \le n), & \mathcal{B}^r(n, \infty) &= \sigma((u_i, t_i, l_i, r_i),\ i > n).
\end{aligned}$$

Notice that $\mathcal{B}^l(n, \infty) = \mathcal{B}(n, \infty)$ and $\mathcal{B}^r(-\infty, -n) = \mathcal{B}^l(-\infty, -n)$.

Furthermore $\mathcal{B}^l(-\infty, 0)$ is generated by the events $A \cap H$ where $A \in \mathcal{B}(-\infty, 0)$ and

$$H := \{l_{k_p} = 1,\ l_{k_{p-1}} = 1, \ldots, l_{k_0} = 1,\ l_{m_q} = 0, \ldots, l_{m_1} = 0\}$$

with $k_p < k_{p-1} < \cdots < k_0 \le 0$ and $m_q < m_{q-1} < \cdots < m_1 \le 0$.

Similarly, $\mathcal{B}^r(0, \infty)$ is generated by the sets AH^r where $A \in \mathcal{B}^l(0, \infty)$ and

$$H^r := \{r_{k_0} = 1,\ r_{k_1} = 1, \ldots, r_{k_p} = 1,\ r_{m_1} = 0, \ldots, r_{m_q} = 0\}$$

with $0 < k_0 < k_1 < \cdots < k_{p-1} < k_p$ and $0 < m_1 < m_2 < \cdots < m_q$.

Denoting $Y_i = X_i + t_i$, $Y_i^r = X_i - t_i$ and using the definitions of l_i and r_i we get

$$H = \bigcap_{i=0}^{p} \Big\{Y_{k_i} < \inf_{j>k_i} Y_j\Big\} \cap \bigcap_{i=1}^{q} \Big\{Y_{m_i} \ge \inf_{j>m_i} Y_j\Big\}, \tag{2.1}$$

$$H^r = \bigcap_{i=0}^{p} \Big\{ Y^r_{k_i} > \sup_{j<k_i} Y^r_j \Big\} \cap \bigcap_{i=1}^{q} \Big\{ Y^r_{m_i} \le \sup_{j<m_i} Y^r_j \Big\}. \tag{2.2}$$

For $0 < c < 1$ and integer n we define events H_n and H^r_n by

$$H_n := \bigcap_{i=0}^{p} \Big\{ Y_{k_i} < \inf_{k_i<j\le nc} Y_j \Big\} \cap \bigcap_{i=1}^{q} \Big\{ Y_{m_i} \ge \inf_{m_i<j\le nc} Y_j \Big\},$$

$$H^r_n := \bigcap_{i=0}^{p} \Big\{ Y^r_{k_i} > \sup_{-nc\le j<k_i} Y^r_j \Big\} \cap \bigcap_{i=1}^{q} \Big\{ Y^r_{m_i} \le \sup_{-nc\le j<m_i} Y^r_j \Big\}.$$

Furthermore let

$$G_n := \Big\{ \inf_{0<j<\infty} Y_j = \inf_{0<j\le nc} Y_j \Big\},$$

$$G_{n,r} := \Big\{ \sup_{-\infty<j<0} Y^r_j = \sup_{-nc\le j<0} Y^r_j \Big\}.$$

Then

$$HG_n = H_n G_n, \tag{2.3}$$

$$H^r G_{n,r} = H^r_n G_{n,r}. \tag{2.4}$$

Of course $H_n \in \mathcal{B}(-\infty, nc)$ and $H^r_n \in \mathcal{B}^l(-nc, \infty)$.

Let $\bar{G}_n$ and $\bar{G}_{n,r}$ denote the complements of G_n and $G_{n,r}$, respectively.

LEMMA 2.2. *If $\{(u_i, t_i)\}$ is stationary and ergodic then*

$$P(\bar{G}_n) \le P\Big(\sum_{j=1}^{nc} u_j \le t_1\Big), \qquad P(\bar{G}_{n,r}) \le P\Big(\sum_{j=1}^{nc} u_{-j} \le t_0\Big), \tag{2.5}$$

which gives

$$P(\bar{G}_n), P(\bar{G}_{n,r}) \to 0 \qquad \textit{as } n \to \infty. \tag{2.6}$$

Proof. Notice that

$$\bar{G}_n = \Big\{ \inf_{0<j<\infty} Y_j < \inf_{0<j\le nc} Y_j \Big\} = \Big\{ \inf_{j\ge nc} Y_j < \inf_{0<j\le nc} Y_j \Big\}$$
$$\subset \{X_{nc} < Y_1\} = \Big\{ \sum_{j=1}^{nc} u_j < t_1 \Big\},$$

which by stationarity of $\{(u_i, t_i)\}$ gives the first inequality in (2.5). Similarly,

$$\bar{G}_{n,r} = \Big\{ \sup_{-\infty<j<0} Y^r_j > \sup_{-nc\le j<0} Y^r_j \Big\} = \Big\{ \sup_{j<-nc} Y^r_j > \sup_{-nc\le j<0} Y^r_j \Big\}$$
$$\subset \{X_{-nc-1} > Y^r_{-1}\},$$

which by stationarity of $\{(u_i, t_i)\}$ gives the second inequality of (2.5).

The second assertion of the lemma follows immediately from (2.5) and ergodicity of $\{(u_i, t_i)\}$. ∎

2.3.2. *Heredity of α-mixing and β-mixing from $\{(u_i,t_i)\}$ to $\{(u_i,t_i,l_i,r_i)\}$*

THEOREM 2.1.

(i) *If $\{(u_i,t_i)\}$ is α-mixing with mixing function $\alpha=\{\alpha(n)\}$, then $\{(u_i,t_i,l_i)\}$ is α-mixing with mixing function $\alpha_l=\{\alpha_l(n)\}$ such that*

$$\alpha_l(n)\le\alpha(n(1-c))+4P(\bar G_n).$$

(ii) *If $\{(u_i,t_i,l_i)\}$ is α-mixing with mixing function $\alpha_l=\{\alpha_l(n)\}$, then $\{(u_i,t_i,l_i,r_i)\}$ is α-mixing with mixing function $\alpha_r=\{\alpha_r(n)\}$ such that*

$$\alpha_r(n)\le\alpha_l(n(1-c))+4P(\bar G_{n,r}).$$

Proof. Notice that

$$\alpha_l(n)=\sup_{A,H,B}|P(AHB)-P(AH)P(B)|,$$

where the supremum is taken over all $A\in\mathcal{B}(-\infty,0)$, $B\in\mathcal{B}^l(n,\infty)=\mathcal{B}(n,\infty)$ and all H of the form (2.1). But

$$\begin{aligned}
&|P(AHB)-P(AH)P(B)|\\
&\quad\le|P(AHG_nB)-P(AHG_n)P(B)|+|P(AH\bar G_nB)-P(AH\bar G_n)P(B)|\\
&\quad\le|P(AH_nG_nB)-P(AH_nG_n)P(B)|+2P(\bar G_n)\\
&\quad\le|P(AH_nB)-P(AH_n)P(B)|\\
&\qquad+|P(AH_n\bar G_nB)-P(AH_n\bar G_n)P(B)|+2P(\bar G_n)\\
&\quad\le\alpha(n(1-c))+4P(\bar G_n).
\end{aligned}$$

In the third inequality we have used (2.3). This finishes the proof of (i).

The proof of (ii) is similar, with H,H_n,G_n replaced by H^r,H_n^r, $G_{n,r}$, respectively and $A\in\mathcal{B}^l(0,\infty)$. Namely, we have

$$\alpha_r(n)=\sup_{A,H^r,B}|P(AHB)-P(AH^r)P(B)|,$$

where the supremum is taken over all $A\in\mathcal{B}^l(0,\infty),B\in\mathcal{B}^r(-\infty,-n)=\mathcal{B}^l(-\infty,-n)$ and all H^r of the form (2.2). ∎

Here we can see that $\{\alpha_l(n)\}$ and $\{\alpha_r(n)\}$ tend to zero faster than $\{\alpha^l(n)\}$ and $\{\alpha^r(n)\}$, respectively, defined in Q-S.

THEOREM 2.2.

(i) *If $\{(u_i,t_i)\}$ is β-mixing with mixing function $\beta=\{\beta(n)\}$, then $\{(u_i,t_i,l_i)\}$ is β-mixing with mixing function $\beta_l=\{\beta_l(n)\}$ such that*

$$\beta_l(n)\le\beta(n(1-c))+4P(\bar G_n).$$

(ii) *If $\{(u_i,t_i,l_i)\}$ is β-mixing with mixing function $\beta_l=\{\beta_l(n)\}$, then $\{(u_i,t_i,l_i,r_i)\}$ is β-mixing with mixing function $\beta_r=\{\beta_r(n)\}$ such that*

$$\beta_r(n)\le\beta_l(n(1-c))+4P(\bar G_{n,r}).$$

Proof. By definition of β-mixing we have

$$\beta_l(n) = \sup_{A_i,H,B_j} \sum_{i=1}^{I}\sum_{j=1}^{J} |P(A_iHB_j) - P(A_iH)P(B_j)|$$

where the supremum is taken over all partitions $\{A_i, 1 \le i \le I\}$ and $\{B_j, 1 \le j \le J\}$ of Ω, where $A_i \in \mathcal{B}(-\infty,0)$, $B_j \in \mathcal{B}(n,\infty)$ and H has the form (2.1). Notice that

$$\begin{aligned}
&\sum_{i=1}^{I}\sum_{j=1}^{J} |P(A_iHB_j) - P(A_iH)P(B_j)| \\
&\quad \le \sum_{i=1}^{I}\sum_{j=1}^{J} |P(A_iHG_nB_j) - P(A_iHG_n)P(B_j)| \\
&\qquad + \sum_{i=1}^{I}\sum_{j=1}^{J} |P(A_iH\bar{G}_nB_j) - P(A_iH\bar{G}_n)P(B_j)| \\
&\quad \le \sum_{i=1}^{I}\sum_{j=1}^{J} |P(A_iH_nG_nB_j) - P(A_iH_nG_n)P(B_j)| + 2P(\bar{G}_n) \\
&\quad \le \sum_{i=1}^{I}\sum_{j=1}^{J} |P(A_iH_nB_j) - P(A_iH_n)P(B_j)| \\
&\qquad + \sum_{i=1}^{I}\sum_{j=1}^{J} |P(A_iH_n\bar{G}_nB_j) - P(A_iH_n\bar{G}_n)P(B_j)| + 2P(\bar{G}_n) \\
&\quad \le \beta(n(1-c)) + 4P(\bar{G}_n).
\end{aligned}$$

This finishes the proof of (i).

The proof of (ii) is similar with H, H_n, G_n replaced by $H^r, H_n^r, G_{n,r}$, respectively. Namely, we have

$$\beta_r(n) = \sup_{A_i,H^r,B_j} \sum_{i=1}^{I}\sum_{j=1}^{J} |P(A_iH^rB_j) - P(A_iH^r)P(B_j)|,$$

where the supremum is taken over all partitions $\{A_i, 1 \le i \le I\}$ and $\{B_j, 1 \le j \le J\}$ of Ω, where $A_i \in \mathcal{B}^l(0,\infty)$, $B_j \in \mathcal{B}^r(-\infty,-n) = \mathcal{B}^l(-\infty,-n)$ and all H^r are of the form (2.2). ∎

2.3.3. *Heredity of ρ-mixing from $\{(u_i,t_i)\}$ to $\{(u_i,t_i,l_i,r_i)\}$*

THEOREM 2.3.

(i) *Let $\{(u_i,t_i)\}$ be ρ-mixing with mixing function $\rho = \{\rho(n)\}$. Furthermore suppose that*

$$\sup_n \sup_{A,H_n} \frac{P(AH_n)}{P(AH_nG_n)} \equiv \kappa^2 < \infty, \quad \textit{where} \quad A \in \mathcal{B}(-\infty,0),$$

and

$$\operatorname{ess\,sup} P(C_n \mid \mathcal{B}(n,\infty)) \to 0 \qquad \text{as } n \to \infty,$$

where $C_n = \{X_{nc+1} < \inf_{0<j\le nc} Y_j\}$. *Then* $\{(u_i, t_i, l_i)\}$ *is* ρ*-mixing with mixing function* $\rho_l = \{\rho_l(n)\}$ *where*

$$\rho_l(n) \le \rho(n(1-c))\kappa + (\kappa+1)\operatorname{ess\,sup} P^{1/2}(C_n \mid \mathcal{B}(n,\infty)) + (\kappa+1)P^{1/2}(\bar{G}_n).$$

(ii) *Let* $\{(u_i, t_i, l_i)\}$ *be* ρ*-mixing with mixing function* $\rho_l = \{\rho_l(n)\}$. *Furthermore suppose that*

$$\sup_n \sup_{A, H_n^r} \frac{P(AH_n^r)}{P(AH_n^r G_{n,r})} \equiv \kappa^2 < \infty, \quad \text{where} \quad A \in \mathcal{B}^l(0,\infty),$$

and

$$\operatorname{ess\,sup} P(C_{n,r} \mid \mathcal{B}^l(-\infty,-n)) \to 0 \qquad \text{as } n \to \infty,$$

where $C_{n,r} = \{X_{-nc-1} < \inf_{-nc\le j<0} Y_j^r\}$. *Then* $\{(u_i, t_i, l_i, r_i)\}$ *is* ρ*-mixing with mixing function* $\rho^r = \{\rho^r(n)\}$ *where*

$$\rho^r(n) \le \rho_l(n(1-c))\kappa + (\kappa+1)\operatorname{ess\,sup} P^{1/2}(C_{n,r} \mid \mathcal{B}^l(-\infty,-n)) + (\kappa+1)P^{1/2}(\bar{G}_{n,r}).$$

Proof. Notice that

$$\rho_l(n) = \sup_{A,H,B} \frac{|P(AHB) - P(AH)P(B)|}{P^{1/2}(AH)P^{1/2}(B)},$$

where the supremum is taken over all $A \in \mathcal{B}(-\infty,0)$, $B \in \mathcal{B}(n,\infty)$ and H of the form (2.1). This is because the family of sets AH generates the σ-field $\mathcal{B}^l(-\infty,0)$. But

$$\begin{aligned}
\text{(2.7)} \qquad & \frac{|P(AHB) - P(AH)P(B)|}{P^{1/2}(AH)P^{1/2}(B)} \\
&\le \frac{|P(AHG_nB) - P(AHG_n)P(B)|}{P^{1/2}(AH)P^{1/2}(B)} + \frac{|P(AHB\bar{G}_n) - P(AH\bar{G}_n)P(B)|}{P^{1/2}(AH)P^{1/2}(B)} \\
&\le \frac{|P(AH_nB) - P(AH_n)P(B)|}{P^{1/2}(AH)P^{1/2}(B)} + \frac{|P(AH_n\bar{G}_nB) - P(AH_n\bar{G}_n)P(B)|}{P^{1/2}(AH)P^{1/2}(B)} \\
&\quad + \frac{|P(AH\bar{G}_nB) - P(AH\bar{G}_n)P(B)|}{P^{1/2}(AH)P^{1/2}(B)}.
\end{aligned}$$

For the first expression on the right hand side of (2.7) we have

$$\begin{aligned}
\frac{|P(AH_nB) - P(AH_n)P(B)|}{P^{1/2}(AH)P^{1/2}(B)} &\le \frac{|P(AH_nB) - P(AH_n)P(B)|}{P^{1/2}(AHG_n)P^{1/2}(B)} \\
&\le \frac{|P(AH_nB) - P(AH_n)P(B)|}{P^{1/2}(AH_n)P^{1/2}(B)} \frac{P^{1/2}(AH_n)}{P^{1/2}(AHG_n)} \le \rho(n(1-c))\kappa.
\end{aligned}$$

For the second expression on the right hand side of (2.7),

$$\begin{aligned}
&\frac{|P(AH_n\bar{G}_nB) - P(AH_n\bar{G}_n)P(B)|}{P^{1/2}(AH)P^{1/2}(B)} \\
&\qquad \le \frac{P(AH_n\bar{G}_nB)}{P^{1/2}(AH)P^{1/2}(B)} + \frac{P(AH_n\bar{G}_n)P(B)}{P^{1/2}(AH)P^{1/2}(B)} \\
&\qquad \le \frac{P^{1/2}(AH_n)P^{1/2}(\bar{G}_nB)}{P^{1/2}(AH)P^{1/2}(B)} + \frac{P^{1/2}(AH_n)P^{1/2}(\bar{G}_n)P(B)}{P^{1/2}(AH)P^{1/2}(B)} \\
&\qquad \le P^{1/2}(\bar{G}_n \,|\, \mathcal{B}(n,\infty))\Big(\frac{P(AH_n)}{P(AH)}\Big)^{1/2} + P^{1/2}(\bar{G}_n)\Big(\frac{P(AH_n)}{P(AH)}\Big)^{1/2} \\
&\qquad \le \big(P^{1/2}(C_n \,|\, \mathcal{B}(n,\infty)) + P^{1/2}(\bar{G}_n)\big)\Big(\frac{P(AH_n)}{P(AH)}\Big)^{1/2} \\
&\qquad \le \big(P^{1/2}(C_n \,|\, \mathcal{B}(n,\infty)) + P^{1/2}(\bar{G}_n)\big)\Big(\frac{P(AH_n)}{P(AH_nG_n)}\Big)^{1/2} \\
&\qquad \le \big(P^{1/2}(C_n \,|\, \mathcal{B}(n,\infty)) + P^{1/2}(\bar{G}_n)\big)\kappa.
\end{aligned}$$

Finally for the third expression we have

$$\begin{aligned}
&\frac{|P(AH\bar{G}_nB) - P(AH\bar{G}_n)P(B)|}{P^{1/2}(AH)P^{1/2}(B)} \\
&\qquad \le \frac{P(AH\bar{G}_nB)}{P^{1/2}(AH)P^{1/2}(B)} + \frac{P(AH\bar{G}_n)P(B)}{P^{1/2}(AH)P^{1/2}(B)} \\
&\qquad \le \frac{P^{1/2}(AH)P^{1/2}(\bar{G}_nB)}{P^{1/2}(AH)P^{1/2}(B)} + \frac{P^{1/2}(AH)P^{1/2}(\bar{G}_n)P(B)}{P^{1/2}(AH)P^{1/2}(B)} \\
&\qquad \le P^{1/2}(\bar{G}_n \,|\, \mathcal{B}(n,\infty)) + P^{1/2}(\bar{G}_n) \le P^{1/2}(C_n \,|\, \mathcal{B}(n,\infty)) + P^{1/2}(\bar{G}_n).
\end{aligned}$$

Combining the above inequalities we get (i); and (ii) can be proved in a similar way. ∎

2.4. Heredity of mixing conditions from $\{(u_i, t_i)\}$ to $\{\tilde{u}_i^*\}$. Let τ_i be the label of the ith seed, $\nu_i = \tau_{i+1} - \tau_i$, X_i^* the position of the ith seed, i.e. $X_i^* = X_{\tau_i}$, and $u_i^* = X_{i+1}^* - X_i^*$, $i \in \mathbb{Z}$, the distance between the ith and $(i+1)$th seeds. Then

$$u_i^* = X_{i+1}^* - X_i^* = X_{\tau_{i+1}} - X_{\tau_i} = \sum_{j=\tau_i}^{\tau_{i+1}-1} u_j.$$

Let $\{(\tilde{u}_i, \tilde{t}_i, \tilde{l}_i, \tilde{r}_i, \tilde{s}_i)\}$ be the Palm version of $\{(u_i, t_i, l_i, r_i, s_i)\}$, i.e. the distribution of $\{(\tilde{u}_i, \tilde{t}_i, \tilde{l}_i, \tilde{r}_i, \tilde{s}_i)\}$ is the conditional distribution of $\{(u_i, t_i, l_i, r_i, s_i)\}$ given $s_0 = 1$. Then the sequence $\{(\tilde{u}_i^*, \tilde{\nu}_i^*)\}$ is stationary, where

$$\tilde{u}_i^* = \tilde{X}_{i+1}^* - \tilde{X}_i^*.$$

Below we formulate a theorem on heredity of mixing properties from $\{(u_i, t_i, l_i, r_i, s_i)\}$ to $\{(\tilde{u}_i, \tilde{t}_i, \tilde{l}_i, \tilde{r}_i)\}$. Case (i) of the theorem is similar to Theorem 5.2 of Q-S, but $\{\tilde{\alpha}_r(n)\}$ here has a better rate of convergence to zero than its analog in Q-S. Furthermore the proof below is much simpler than that in Q-S.

THEOREM 2.4.

(i) *If* $\{(u_i, t_i, l_i, r_i)\}$ *is* α*-mixing with mixing function* $\alpha_r = \{\alpha_r(n)\}$, *then* $\{(\tilde{u}_i, \tilde{t}_i, \tilde{l}_i, \tilde{r}_i)\}$ *is* α*-mixing with mixing function* $\tilde{\alpha}_r = \{\tilde{\alpha}_r(n)\}$ *such that* $\tilde{\alpha}_r(n) = 2\alpha_r(n/2)/P(s_0 = 1)$.

(ii) *If* $\{(u_i, t_i, l_i, r_i)\}$ *is* β*-mixing with mixing function* $\beta_r = \{\beta_r(n)\}$, *then* $\{(\tilde{u}_i, \tilde{t}_i, \tilde{l}_i, \tilde{r}_i)\}$ *is* β*-mixing with mixing function* $\tilde{\beta}_r = \{\tilde{\beta}_r(n)\}$ *such that* $\tilde{\beta}_r(n) = 2\beta_r(n/2)/P(s_0 = 1)$.

(iii) *If* $\{(u_i, t_i, l_i, r_i)\}$ *is* ρ*-mixing with mixing function* $\rho_r = \{\beta_r(n)\}$ *and*
$$\sup_{B \in D(n/2)} \frac{1}{P(s_0 = 1 \mid B)} < \infty, \qquad \sup_{A \in D_{-n/2}} \frac{1}{P(s_0 = 1 \mid A)} < \infty,$$
where D_k *and* $D(k)$ *are the* σ*-fields defined in the proof, then* $\{(\tilde{u}_i, \tilde{t}_i, \tilde{l}_i, \tilde{r}_i)\}$ *is* ρ*-mixing with mixing function* $\tilde{\rho} = \{\tilde{\rho}(n)\}$, *where*
$$\tilde{\rho}(n) = 2 \max\left(\sup_{B \in D(n/2)} \frac{1}{P^{1/2}(s_0 = 1 \mid B)}, \sup_{A \in D_{-n/2}} \frac{1}{P^{1/2}(s_0 = 1 \mid A)} \right).$$

Proof. The proof of (i) is similar to the proof of Theorem 5.2 of Q-S, and the proof of (ii) is similar to that of (i).

(iii) Let D_k denote the σ-field generated by $\{(\tilde{u}_i, \tilde{t}_i, \tilde{l}_i, \tilde{r}_i),\ i \in \mathbb{Z},\ i \le k\}$, and $D(k)$ the σ-field generated by $\{(\tilde{u}_i, \tilde{t}_i, \tilde{l}_i, \tilde{r}_i),\ i \in \mathbb{Z},\ i > k\}$. Notice that
$$\begin{aligned}
\tilde{\rho}(n) &= \sup_k \sup_{A,B} \frac{|\tilde{P}(AB) - \tilde{P}(A)\tilde{P}(B)|}{\tilde{P}^{1/2}(A)\tilde{P}^{1/2}(B)} \\
&= \max\left(\sup_{k < -n/2} \sup_{A,B} \frac{|\tilde{P}(AB) - \tilde{P}(A)\tilde{P}(B)|}{\tilde{P}^{1/2}(A)\tilde{P}^{1/2}(B)}, \right. \\
&\qquad \left. \sup_{k \ge -n/2} \sup_{A,B} \frac{|\tilde{P}(AB) - \tilde{P}(A)\tilde{P}(B)|}{\tilde{P}^{1/2}(A)\tilde{P}^{1/2}(B)} \right),
\end{aligned}$$
where the suprema are taken over all $A \in D_k$ and $B \in D(n + k)$.

In the case $k \ge -n/2$ we have
$$\begin{aligned}
\frac{|\tilde{P}(AB) - \tilde{P}(A)\tilde{P}(B)|}{\tilde{P}^{1/2}(A)\tilde{P}^{1/2}(B)} &= \frac{|P(AB \mid s_0 = 1) - P(A \mid s_0 = 1)P(B \mid s_0 = 1)|}{P^{1/2}(A \mid s_0 = 1)P^{1/2}(B \mid s_0 = 1)} \\
&= P(s_0 = 1)\frac{|P(AB \mid s_0 = 1) - P(A \mid s_0 = 1)P(B \mid s_0 = 1)|}{P^{1/2}(A, s_0 = 1)P^{1/2}(B, s_0 = 1)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{|P(AB, s_0 = 1) - P(A, s_0 = 1)P(B)|}{P^{1/2}(A, s_0 = 1)P^{1/2}(B, s_0 = 1)} \\
&\quad + \frac{P(A \mid s_0 = 1)|P(B, s_0 = 1) - P(B)P(s_0 = 1)|}{P^{1/2}(A, s_0 = 1)P^{1/2}(B, s_0 = 1)} \\
&= \frac{|P(AB, s_0 = 1) - P(A, s_0 = 1)P(B)|}{P^{1/2}(A, s_0 = 1)P^{1/2}(B)} \frac{P^{1/2}(B)}{P^{1/2}(B, s_0 = 1)} \\
&\quad + \frac{P(A \mid s_0 = 1)|P(B, s_0 = 1) - P(B)P(s_0 = 1)|}{P^{1/2}(B)P^{1/2}(s_0 = 1)} \\
&\qquad\qquad \times \frac{P^{1/2}(B)P^{1/2}(s_0 = 1)}{P^{1/2}(B, s_0 = 1)P^{1/2}(A, s_0 = 1)} \\
&\le \rho(n/2) \frac{1}{P^{1/2}(s_0 = 1 \mid B)} + \rho(n/2)P(A \mid s_0 = 1) \\
&\qquad\qquad \times \frac{1}{P^{1/2}(s_0 = 1 \mid B)} \frac{1}{P^{1/2}(A \mid s_0 = 1)} \\
&\le \rho(n/2) \frac{1}{P^{1/2}(s_0 = 1 \mid B)} + \rho(n/2)P^{1/2}(A \mid s_0 = 1) \frac{1}{P^{1/2}(s_0 = 1 \mid B)} \\
&\le 2\rho(n/2) \frac{1}{P^{1/2}(s_0 = 1 \mid B)} \\
&\le 2\rho(n/2) \sup_{B \in D(n+k)} \frac{1}{P^{1/2}(s_0 = 1 \mid B)}.
\end{aligned}
$$

In a similar way, in the case $k < -n/2$ we get

$$
\frac{|\tilde{P}(AB) - \tilde{P}(A)\tilde{P}(B)|}{\tilde{P}^{1/2}(A)\tilde{P}^{1/2}(B)} \le 2\rho(n/2) \sup_{A \in D_k} \frac{1}{P^{1/2}(s_0 = 1 \mid A)}.
$$

Combining the two inequalities we get the assertion of the theorem. ∎

Now we formulate a theorem on heredity of mixing properties from $\{(\tilde{u}_i, \tilde{t}_i, \tilde{l}_i, \tilde{r}_i)\}$ to $\{\tilde{u}_i^*\}$.

THEOREM 2.5.

(i) *If* $\{(\tilde{u}_i, \tilde{t}_i, \tilde{l}_i, \tilde{r}_i, \tilde{s}_i)\}$ *is* α*-mixing with mixing function* $\tilde{\alpha} = \{\tilde{\alpha}(n)\}$, *then* $\{(\tilde{u}_i^*, \tilde{t}_i^*)\}$ *is* α*-mixing with the same mixing function.*

(ii) *If* $\{(\tilde{u}_i, \tilde{t}_i, \tilde{l}_i, \tilde{r}_i, \tilde{s}_i)\}$ *is* β*-mixing with mixing function* $\tilde{\beta} = \{\tilde{\beta}(n)\}$, *then* $\{(\tilde{u}_i^*, \tilde{t}_i^*)\}$ *is* β*-mixing with the same mixing function.*

(iii) *If* $\{(\tilde{u}_i, \tilde{t}_i, \tilde{l}_i, \tilde{r}_i, \tilde{s}_i)\}$ *is* ρ*-mixing with mixing function* $\tilde{\rho} = \{\tilde{\rho}(n)\}$, *then* $\{(\tilde{u}_i^*, \tilde{t}_i^*)\}$ *is* ρ*-mixing with the same mixing function.*

Proof. The proof in all cases is similar to the proof of Theorem 5.3 in Q-S. ∎

Immediately from the above theorem we get the following corollary.

COROLLARY 2.1.

(i) *Let* $\{(u_i, t_i)\}$ *be stationary and* α*-mixing with mixing function* $\alpha = \{\alpha(n)\}$. *Then* $\{(\tilde{u}_i^*, \tilde{t}_i^*)\}$ *is stationary and* α*-mixing with mixing function* $\alpha^* = \{\alpha^*(n)\}$, *where*

$$\alpha^*(n) = 2\alpha_r(n/2)/P(s_0 = 1).$$

(ii) *Let* $\{(u_i, t_i)\}$ *be stationary and* β*-mixing with mixing function* $\beta = \{\beta(n)\}$. *Then* $\{(\tilde{u}_i^*, \tilde{t}_i^*)\}$ *is stationary and* β*-mixing with mixing function* $\beta^* = \{\beta^*(n)\}$, *where*

$$\beta^*(n) = 2\beta_r(n/2)/P(s_0 = 1).$$

(iii) *Let* $\{(u_i, t_i)\}$ *be stationary and* ρ*-mixing with mixing function* $\rho = \{\rho(n)\}$. *Then* $\{(\tilde{u}_i^*, \tilde{t}_i^*)\}$ *is stationary and* ρ*-mixing with mixing function* $\rho^* = \{\rho^*(n)\}$, *where*

$$\rho^*(n) = 2\max\left(\sup_{B\in D(n/2)} \frac{1}{P^{1/2}(s_0 = 1 \mid B)}, \sup_{A\in D_{-n/2}} \frac{1}{P^{1/2}(s_0 = 1 \mid A)}\right).$$

3. FCLT for the number of seeds

3.1. Notation and relations. Let us define the following processes:

$$N(t) = \#\Big\{i \ge 1 : \sum_{j=0}^{i-1} u_j \le t\Big\}, \qquad N^*(t) = \#\Big\{i \ge 1 : \sum_{j=0}^{i-1} u_j^* + X_0^* \le t\Big\},$$

$$\tilde{N}^*(t) = \#\Big\{i \ge 1 : \sum_{j=0}^{i-1} \tilde{u}_j^* \le t\Big\}, \qquad M(t) = \sum_{j=0}^{[t]} s_j, \quad t \ge 0.$$

Here $N(t)$ is the number of points in $(0, t]$ of the marked point process $\{(\hat{X}_i, \hat{t}_i)\}$, while $N^*(t)$ and $\tilde{N}^*(t)$ are the numbers of seeds in $(0, t]$ of the processes $\{(X_i^*, t_i^*)\}$ and $\{(\hat{X}_i^*, \hat{t}_i^*)\}$, respectively. $M(n)$ is the number of seeds among the points $(X_1, t_1), \ldots, (X_n, t_n)$.

Furthermore let us define the processes

$$U_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} (u_j - a), \qquad U_n^*(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} (u_j^* - a_1),$$

$$\tilde{U}_n^*(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} (\tilde{u}_j^* - a_1), \quad \hat{M}_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} (s_j - Es_0), \quad t \ge 0,$$

and

$$N_n(t) = (N(nt) - nt/a)/\sqrt{n/a^3}, \qquad N_n^*(t) = (N^*(nt) - nt/a_1)/\sqrt{n/a_1^3},$$
$$\tilde{N}_n^*(t) = (\tilde{N}^*(nt) - nt/a_1)/\sqrt{n/a_1^3}, \; M_n(t) = (M(nt) - ntEs_0)/\sqrt{n}, \quad t \ge 0,$$

where $a = Eu_1$ and $a_1 = E\tilde{u}_1^* = a/Es_0$.

The aim of this section is to give FCLT for the number of seeds, i.e. to give conditions for the following convergences:

$$\frac{1}{\sigma}\tilde{U}_n^* \xrightarrow{\mathcal{D}} \mathcal{W}, \quad \frac{1}{\sigma}U_n^* \xrightarrow{\mathcal{D}} \mathcal{W}, \quad \frac{1}{\sigma}\tilde{N}_n^* \xrightarrow{\mathcal{D}} \mathcal{W}, \quad \frac{1}{\sigma}N_n^* \xrightarrow{\mathcal{D}} \mathcal{W}, \tag{3.8}$$

$$\frac{1}{\sigma}M_n \xrightarrow{\mathcal{D}} \mathcal{W}, \quad \frac{1}{\sigma}\tilde{M}_n \xrightarrow{\mathcal{D}} \mathcal{W}, \tag{3.9}$$

where $\mathcal{W}$ is a standard Wiener process, σ^2 is some finite positive number and $\xrightarrow{\mathcal{D}}$ means weak convergence in the function space $D[0,\infty)$ with the Skorokhod J_1 topology. We will show FCLT for $\tilde{U}_n^*$, and to get the other convergences in (3.8) we will use relations between convergences of $\tilde{U}_n^*$ and $\tilde{N}_n^*$ and of $\tilde{U}_n^*$ and U_n^*. These relations are formulated in the following lemma.

LEMMA 3.1.

(i) *The following equivalence holds:*

$$\left[\frac{1}{\sigma}\tilde{U}_n^* \xrightarrow{\mathcal{D}} \mathcal{W}\right] \equiv \left[\frac{1}{\sigma}\tilde{N}_n^* \xrightarrow{\mathcal{D}} \mathcal{W}\right]. \tag{3.10}$$

(ii) *If* $\{u_i^*, i \in \mathbb{Z}\}$ *is* α*-mixing then*

$$\left[\frac{1}{\sigma}\tilde{U}_n^* \xrightarrow{\mathcal{D}} \mathcal{W}\right] \equiv \left[\frac{1}{\sigma}U_n^* \xrightarrow{\mathcal{D}} \mathcal{W}\right] \equiv \left[\frac{1}{\sigma}N_n^* \xrightarrow{\mathcal{D}} \mathcal{W}\right]. \tag{3.11}$$

(iii) *We have*

$$\left[\frac{1}{\sigma}\hat{M}_n \xrightarrow{\mathcal{D}} \mathcal{W}\right] \equiv \left[\frac{1}{\sigma}M_n \xrightarrow{\mathcal{D}} \mathcal{W}\right]. \tag{3.12}$$

The equivalence (3.10) was given in Szczotka (1986, Proposition 2.11) and also in (6.1) of Q-S. The implication

$$\left[\frac{1}{\sigma}\tilde{U}_n^* \xrightarrow{\mathcal{D}} \mathcal{W}\right] \Rightarrow \left[\frac{1}{\sigma}U_n^* \xrightarrow{\mathcal{D}} \mathcal{W}\right],$$

under α-mixing of $\{u_i^*\}$, was given in Theorem 6.2 of Q-S, and the proof of the reverse implication is similar to the proof of Theorem 6.2 of Q-S, so it is omitted here. The second equivalence in (3.11) is an obvious consequence of the first and of (3.10). The equivalence (3.12) is obvious.

3.2. Main results on FCLT. In this subsection we give five versions of FCLT for the procesess of the number of seeds. Namely, we give five different sets of conditions under which the convergences in (3.8) hold. The first set of conditions appears in Theorem 3.1 below. It is based on the thinning of the process $\{(X_i, t_i)\}$ by a process $\{s_i\}$, where a stationary sequence $\{(u_i, s_i)\}$ satisfies two-dimensional FCLT. Here, the thinning process $\{s_i\}$ is general, the s_i assume values 0 or 1, and they are not necessarily related to the s_i defined in Section 2. The second set of conditions appears in Theorem 3.2, which gives conditions for (3.14) to hold in case (ii) of Theorem 3.1. The third and fourth sets of conditions occur in Theorems 3.3 and 3.4, which use some rates of convergences to zero of the α-mixing function of the sequence $\{(u_i, t_i)\}$. The fifth set of conditions concerns the case when $\{u_i\}$ is ρ-mixing (Theorem 3.6).

A pair $(\mathcal{W}_1, \mathcal{W}_2)$ of processes is said to be a *two-dimensional Wiener process* if $\mathcal{W}_1$ and $\mathcal{W}_2$ are standard Wiener processes with covariance

$$\operatorname{cov}(\mathcal{W}_1(t_1), \mathcal{W}_2(t_2)) = \sigma_{1,2} \min(t_1, t_2).$$

Convergence to that process is called here *two-dimensional FCLT* in the product space $D[0,\infty) \times D[0,\infty)$ with the product Skorokhod J_1 topology.

THEOREM 3.1 (FCLT under thinning). *Let $\{(u_i, s_i)\}$ be stationary.*

(i) *If $(U_n, U_n^*) \xrightarrow{\mathcal{D}} (\sigma_1 \mathcal{W}_1, \sigma_2 \mathcal{W}_2)$, then*

$$M_n \xrightarrow{\mathcal{D}} a_1^{-1}(\sigma_1 \mathcal{W}_1 - \sigma_2 \mathcal{W}_2 \circ \gamma_1), \tag{3.13}$$

where $\gamma_1(t) = tEs_0$, $0 < \sigma_1$, $\sigma_2 < \infty$ and $(\mathcal{W}_1, \mathcal{W}_2)$ is a two-dimensional Wiener process.

(ii) *If $(U_n, M_n) \xrightarrow{\mathcal{D}} (\sigma_1 \mathcal{W}_3, \sigma_4 \mathcal{W}_4)$, then*

$$U_n^* \xrightarrow{\mathcal{D}} \sigma_1 \mathcal{W}_3 \circ \gamma_2 - a_1 \sigma_4 \mathcal{W}_4 \circ \gamma_2, \tag{3.14}$$

where $\gamma_2(t) = t/Es_0$, $t \geq 0$, $0 < \sigma_1, \sigma_3 < \infty$ and $(\mathcal{W}_3, \mathcal{W}_4)$ is a two-dimensional Wiener process.

Notice that the process $\xi(t) := \sigma_1 \mathcal{W}_3(t/Es_0) - a_1 \sigma_4 \mathcal{W}_4(t/Es_0)$, $t \geq 0$, has independent increments and $E\xi^2(t) = t\sigma^2$, where

$$\sigma^2 = (\sigma_1^2 - 2\sigma_1\sigma_4 a_1 \sigma_{12} + a_1^2 \sigma_4^2)/Es_0.$$

So, if $\sigma^2 > 0$ and if $\{u_i^*\}$ is strongly mixing, then by (3.11) and (3.10) we get the convergences (3.8) with this σ.

The following theorem gives conditions for the assumption $(U_n, M_n) \xrightarrow{\mathcal{D}} (\sigma_1 \mathcal{W}_3, \sigma_4 \mathcal{W}_4)$ of Theorem 3.1(ii) to hold with $s_i = l_i r_i$.

THEOREM 3.2 (FCLT for (U_n, M_n)). *Let $\{(u_i, t_i), i \in \mathbb{Z}\}$ be strongly mixing with mixing function $\alpha = \{\alpha(n)\}$ and suppose that for some $0 < \delta < \infty$ the following inequalities and convergences hold as $n \to \infty$:*

$$Eu_1^{2+\delta} < \infty, \tag{3.15}$$

$$n\alpha(n)^{\delta/(2+\delta)} \to 0, \tag{3.16}$$

$$n\Big(P\Big(\sum_{j=1}^n u_j \leq t_1\Big)\Big)^{\delta/(2+\delta)} \to 0, \qquad n\Big(P\Big(\sum_{j=1}^n u_{-j} \leq t_0\Big)\Big)^{\delta/(2+\delta)} \to 0, \tag{3.17}$$

$$\frac{1}{n} E\Big(\sum_{j=1}^n \bar{u}_j\Big)^2 \to \sigma_1^2, \qquad \frac{1}{n} E\Big(\sum_{j=1}^n \bar{s}_j\Big)^2 \to \sigma_2^2, \tag{3.18}$$

$$\frac{1}{n}E\Big(\sum_{j=1}^{n}\bar{u}_j\Big)\Big(\sum_{j=1}^{n}\bar{s}_j\Big) \to \sigma_1\sigma_2\sigma_{12}, \tag{3.19}$$

where $0 < \sigma_1^2, \sigma_2^2 < \infty$ *and the matrix* $\left(\begin{smallmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{smallmatrix}\right)$ *is positive definite. Then*

$$(U_n, \hat{M}_n) \xrightarrow{\mathcal{D}} (\sigma_1\mathcal{W}_1, \sigma_2\mathcal{W}_2) \quad \text{and} \quad (U_n, M_n) \xrightarrow{\mathcal{D}} (\sigma_1\mathcal{W}_1, \sigma_2\mathcal{W}_2). \tag{3.20}$$

The above theorem also gives conditions for $\hat{M}_n, M_n \xrightarrow{\mathcal{D}} \sigma\mathcal{W}$ to hold, but these conditions are stronger than those in the following theorem.

THEOREM 3.3 (FCLT for the number of seeds). *Let* $\{(u_i, t_i)\}$ *be strongly mixing with mixing function* $\alpha = \{\alpha(n)\}$ *and suppose that*

$$n\alpha(n) \to 0, \tag{3.21}$$

$$nP\Big(\sum_{j=1}^{n} u_j \le t_1\Big) \to 0, \qquad nP\Big(\sum_{j=1}^{n} u_{-j} \le t_0\Big) \to 0, \tag{3.22}$$

$$\frac{1}{n}E\Big(\sum_{j=1}^{n}(s_j - Es_0)\Big)^2 \to \sigma^2, \qquad 0 < \sigma^2 < \infty. \tag{3.23}$$

Then

$$\hat{M}_n, M_n \xrightarrow{\mathcal{D}} \sigma\mathcal{W}. \tag{3.24}$$

THEOREM 3.4 (Third set of conditions for FCLT). *Let* $\{(u_i, t_i)\}$ *be strongly mixing with mixing function* $\alpha = \{\alpha(n)\}$ *and suppose that for some* $\varepsilon > 2$ *and* $\delta > 0$ *such that* $\varepsilon > 2(1 + 2/\delta)$ *the following conditions hold:*

$$Eu_1^{\varepsilon}, Et_1^{\varepsilon} < \infty, \tag{3.25}$$

$$\sum_{n=1}^{\infty}(\alpha(n))^{\delta/(2+\delta)} < \infty. \tag{3.26}$$

Then $\sigma^2 \equiv \operatorname{var}(\tilde{u}_1^*) + 2\sum_{k=2}^{\infty}\operatorname{cov}(\tilde{u}_1^*, \tilde{u}_k^*) < \infty$. *If* $\sigma^2 > 0$ *then the convergences in* (3.8) *hold with this* σ.

The difference between this theorem and Theorem 6.1 of Q-S is that here $\varepsilon > 2 + 4/\delta$, while the latter theorem assumes that $\varepsilon > 4 + 4/\delta$.

The following theorem gives a weaker condition on the rates of convergence to zero of the mixing function than Theorem 3.4, but it additionally assumes a condition on the process $\{\bar{N}(t), t \ge 0\}$, where $\bar{N}(t) = \#\{k \ge 1 : \sum_{j=1}^{k} u_j \le t\}$.

THEOREM 3.5 (Fourth set of conditions for FCLT). *Let* $\{(u_i, t_i)\}$ *be strongly mixing with mixing function* $\alpha = \{\alpha(n)\}$ *and suppose that for some*

$1/2 < \lambda < 1$,

$$\limsup_k E\left(\frac{1}{k}\bar{N}(k)\right)^{2/\lambda} < \infty. \tag{3.27}$$

Furthermore suppose that for δ *such that* $\lambda = \delta/(2+\delta)$ *the following conditions hold:*

$$\sum_{n=1}^{\infty}(\alpha(n))^{\delta/(2+\delta)} < \infty, \tag{3.28}$$

$$Eu_1^{2+\delta}, Et_1^{2+\delta} < \infty. \tag{3.29}$$

Then the convergences in (3.8) *hold with* $\sigma^2 \equiv \mathrm{var}(\tilde{u}_1^*) + 2\sum_{k=2}^{\infty}\mathrm{cov}(\tilde{u}_1^*, \tilde{u}_k^*)$, *if it is positive.*

THEOREM 3.6 (FCLT for ρ-mixing). *Let the assumptions of Theorem* 2.3 *be satisfied. Furthermore, assume that condition* (3.29) *holds and* $\sigma_n^2 \equiv \mathrm{Var}(\tilde{U}_n^*(1)) \to \sigma^2$, $0 < \sigma^2 < \infty$. *Then* (3.8) *holds.*

3.3. Auxiliary lemmas. To prove the main results we will use the following lemmas.

LEMMA 3.2. *If* $\{b_n, n \geq 0\}$ *is a nonincreasing sequence of nonnegative numbers with* $b_0 = 1$, *then for any* $0 < c < 1$,

$$\sum_{n=1}^{\infty} b_{[nc]} \leq \left(1 + \frac{1}{c}\right)\sum_{n=1}^{\infty} b_n. \tag{3.30}$$

LEMMA 3.3. *If for some* $0 < \lambda < 1$ *the following conditions hold:*

$$\sum_{n=1}^{\infty}(\alpha(n))^{\lambda} < \infty, \tag{3.31}$$

$$\sum_{n=1}^{\infty}\Big(P\Big(\sum_{j=1}^{n} u_j \leq t_1\Big)\Big)^{\lambda} < \infty, \quad \sum_{n=1}^{\infty}\Big(P\Big(\sum_{j=1}^{n} u_{-j} \leq t_0\Big)\Big)^{\lambda} < \infty, \tag{3.32}$$

then

$$\sum_{n=1}^{\infty}(\alpha_r(n))^{\lambda} < \infty. \tag{3.33}$$

LEMMA 3.4. *If for some* $\kappa > 2$,

$$Eu_1^{\kappa}, Et_1^{\kappa} < \infty, \tag{3.34}$$

then for any $0 < \lambda < 1$,

$$\Big(P\Big(\sum_{j=1}^{n} u_j < t_1\Big)\Big)^{\lambda} \leq K_1 n^{-\lambda\kappa/2}, \quad \Big(P\Big(\sum_{j=1}^{n} u_{-j} < t_0\Big)\Big)^{\lambda} \leq K_2 n^{-\lambda\kappa/2}, \tag{3.35}$$

where K_1 *and* K_2 *are some constants not depending on* n.

LEMMA 3.5. *If $\{(u_i, t_i), i \in \mathbb{Z}\}$ is strongly mixing with mixing function $\alpha = \{\alpha(n)\}$ and for some $\delta > 2$ the following conditions hold:*

$$Eu_1^{2+\delta}, Et_1^{2+\delta} < \infty, \tag{3.36}$$

$$\sum_{n=1}^{\infty} (\alpha(n))^{\delta/(2+\delta)} < \infty, \tag{3.37}$$

then

$$\sum_{n=1}^{\infty} (\alpha_r(n))^{\delta/(2+\delta)} < \infty. \tag{3.38}$$

LEMMA 3.6. *If for some $1/2 < \lambda < 1$, we have $Et_1^{2/(1-\lambda)} < \infty$ and*

$$\sup_n E\left(\frac{1}{n} E\bar{N}(n)\right)^{2/\lambda} < \infty, \tag{3.39}$$

then for that λ the conditions in (3.32) hold.

Some sufficient conditions for the convergence

$$\lim_{n\to\infty} \frac{1}{n} E\Big(\sum_{j=1}^{n} y_j\Big)^2 = \sigma^2, \quad 0 < \sigma^2 < \infty,$$

where $\{y_j\}$ is a centered stationary strongly mixing sequence of random variables, are given by Doukhan–Massart–Rio's Theorem (see for example the first assertion of Theorem 1.2 in Merlevède & Peligrad (2000)). We do not recall it here.

3.4. Proofs of the main results on FCLT

Proof of Theorem 3.1. Define the processes

$$\theta_{n,1}(t) = \frac{1}{\sqrt{n}}\Big(\sum_{j=1}^{[nt]} u_j - \sum_{j=1}^{M(nt)} u_j^*\Big), \quad t \ge 0,$$

$$\theta_{n,2}(t) = \frac{nt - [nt]}{\sqrt{n}} a_1 E s_0, \qquad t \ge 0.$$

Then

$$U_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{M(nt)} (u_j^* - a_1) + \frac{1}{\sqrt{n}}(M(nt) - ntEs_0)a_1 + \theta_{n,1}(t) + \theta_{n,2}(t). \tag{3.40}$$

Denote $\gamma_{n,3}(t) = M(nt)/n$ and $\gamma_{n,4} = \gamma_{n,3}(t) + 1/n$, $t \ge 0$. Then

$$M_n(t) = a_1^{-1}(U_n(t) - U_n^*(\gamma_{n,3}(t))) - a_1^{-1}(\theta_{n,1}(t) + \theta_{n,2}(t)). \tag{3.41}$$

Furthermore for any $b \ge 0$ we have

$$\sup_{0\le t\le b}\theta_{n,1}(t)\le \sup_{0\le t\le b}\frac{1}{\sqrt{n}}u^*_{M(nt)+1}$$
$$\le \sup_{0\le t\le b}(U^*_n(\gamma_{n,4}(t))-U^*_n(\gamma_{n,4}(t-)))+\frac{1}{\sqrt{n}}a_1,$$
$$\sup_{0\le t\le b}\theta_{n,2}(t)\le \frac{1}{\sqrt{n}}a_1Es_0.$$

Hence from $U^*_n\xrightarrow{\mathcal{D}}\sigma_2\mathcal{W}_2$ and the continuous mapping theorem (see Theorem 5.1 of Billingsley (1968)), we get $\theta_{n,1}\xrightarrow{p}0e$ and $\theta_{n,2}\xrightarrow{p}0e$, where $e(t)=t$, $t\ge 0$.

In view of $\frac{1}{n}M(nt)=\frac{1}{n}\sum_{j=1}^{[nt]}s_j$ and the law of large numbers we have $\gamma_{n,3}\xrightarrow{p}eEs_0$ and $\gamma_{n,4}\xrightarrow{p}eEs_0$, as $n\to\infty$. This together with the convergences of $\theta_{n,1}$ and $\theta_{n,2}$, the representation (3.40) and the continuous mapping theorem gives (3.13).

To prove (3.14) notice that

$$\sum_{j=1}^{[nt]}(u^*_j-a_1)=\sum_{j=1}^{\tau_{[nt]}}(u_j-a)-\frac{a}{Es_0}\sum_{j=1}^{\tau_{[nt]}}(s_j-Es_0),\quad t\ge 0,$$

where τ_n is the label of the nth seed and $\sum_{j=1}^{\tau_n}s_j=n$.

Denoting $\gamma_{n,5}(t)=\tau_{[nt]}/n$ and $\gamma_5(t)=\frac{1}{Es_0}e$, $t\ge 0$, we get

$$U^*_n(t)=U_n(\gamma_{n,5}(t))-a_1M_n(\gamma_{n,5}(t)),\quad t\ge 0.$$

This together with $\sup_t|M_n(t)-\hat{M}_n(t)|\le 1/\sqrt{n}$ and $\gamma_{n,5}\xrightarrow{\mathcal{D}}\gamma_5$ in $D[0,\infty)$ and the assumed conditions gives (3.14), finishing the proof of Theorem 3.1. ∎

Proof of Theorem 3.2. To prove $(U_n,M_n)\xrightarrow{\mathcal{D}}(\sigma_1\mathcal{W}_1,\sigma_2\mathcal{W}_2)$ we first show $(U_n,\hat{M}_n)\xrightarrow{\mathcal{D}}(\sigma_1\mathcal{W}_1,\sigma_2\mathcal{W}_2)$, which by Lemma 3.1(iii) gives $(U_n,M_n)\xrightarrow{\mathcal{D}}(\sigma_1\mathcal{W}_1,\sigma_2\mathcal{W}_2)$, where $(\mathcal{W}_1,\mathcal{W}_2)$ is a two-dimensional Wiener process with covariance $E(\mathcal{W}_1(t)\mathcal{W}_2(s))=\min(t,s)\sigma_{1,2}$ for any $t,s\ge 0$. But to prove $(U_n,\hat{M}_n)\xrightarrow{\mathcal{D}}(\sigma_1\mathcal{W}_1,\sigma_2\mathcal{W}_2)$ it is enough to show $b_1U_n+b_2\hat{M}_n\xrightarrow{\mathcal{D}}b_1\sigma_1\mathcal{W}_1+b_2\sigma_2\mathcal{W}_2$ for any constants b_1 and b_2. To do so we use three times Corollary 1.1, Remark 1.1 and Theorem 1.4 of Merlevède & Peligrad (2000): first we apply them to the sequence $\{\bar{u}_i\}$, then to $\{\bar{s}_i\}$ and finally to $\{y_i=b_1\bar{u}_i+b_2\bar{s}_i\}$.

Since $\{\bar{u}_i\}$ is strongly mixing with mixing function $\{\alpha(n)\}$ satisfying (3.16), and since conditions (3.15) and $\frac{1}{n}E(\sum_{j=1}^n\bar{u}_j)^2\to\sigma_1^2$, $0<\sigma_1^2<\infty$, are satisfied, by case (i) of Corollary 1.1, Remark 1.1 and Theorem 1.4 of Merlevède & Peligrad (2000) we get $U_n\xrightarrow{\mathcal{D}}\sigma_1\mathcal{W}_1$, which also implies that $\{U_n\}$ is tight in $D[0,\infty)$ with the Skorokhod J_1 topology.

Now notice that by Theorem 2.1 the sequence $\{\bar{s}_i\}$ is strongly mixing with mixing function $\bar{\alpha}_r=\{\bar{\alpha}_r(n)\}$, where $\bar{\alpha}_r(n)=\alpha_r(n)/Es_i=\alpha_r(n)/p$.

But by Lemma 2.2 we have

$$\alpha_r(n) \le \alpha(n(1-c)^2) + 4P\Big(\sum_{j=1}^{[nc(1-c)]} u_j \le t_1\Big) + 4P\Big(\sum_{j=1}^{[nc]} u_{-j} \le t_0\Big) \tag{3.42}$$

with $0 < c < 1$. Hence by (3.16)–(3.19) we get

$$n\alpha_r(n) \to 0 \quad \text{and} \quad \frac{1}{n}E\Big(\sum_{j=1}^{n} \bar{s}_j\Big)^2 \to \sigma_2^2, \qquad 0 < \sigma_2 < \infty.$$

Hence by case (ii) of Corollary 1.1, Remark 1.1 and Theorem 1.4 in Merlevède & Peligrad (2000) we get $\hat{M}_n \xrightarrow{\mathcal{D}} \sigma_2\mathcal{W}_2$. This implies that the sequences $\{\hat{M}_n\}$ and $\{M_n\}$ are tight in $D[0,\infty)$ with the Skorokhod J_1 topology.

By tightness of $\{U_n\}$ and $\{M_n\}$ we get tightness of $\{(U_n, M_n)\}$. Therefore it is enough to prove the weak convergence of the finite-dimensional distributions of (U_n, M_n) to $(\sigma_1\mathcal{W}_1, \sigma_2\mathcal{W}_2)$. Since $\{(u_i, s_i)\}$ is stationary and strongly mixing with mixing function $\{\alpha_r(n)\}$, the sequence of processes (U_n, M_n) has asymptotically independent increments (see Billingsley (1968, p. 157)). Hence it is enough to show that for any numbers b_1, b_2 we have $b_1U_n(t) + b_2M_n(t) \xrightarrow{\mathcal{D}} \sigma_1b_1\mathcal{W}_1(t) + \sigma_2b_2\mathcal{W}_2(t)$. But

$$b_1U_n(t) + b_2M_n(t) = \frac{1}{\sqrt{n}}\sum_{j=1}^{[nt]}(b_1\bar{u}_j + b_2\bar{s}_j),$$

so the sequence $\{y_i \equiv b_1\bar{u}_j + b_2\bar{s}_j\}$ is α-mixing with mixing function $\bar{\alpha}_r = \{\bar{\alpha}_r(n)\}$. Now (3.42) for $\lambda = \delta/(2+\delta)$ implies

$$\begin{aligned} n(\alpha_r(n))^\lambda \le n3^\lambda(\alpha(n(1-c)^2))^\lambda &+ n3^\lambda4^\lambda\Big(P\Big(\sum_{j=1}^{[nc(1-c)]} u_j \le t_1\Big)\Big)^\lambda \\ &+ n3^\lambda4^\lambda\Big(P\Big(\sum_{j=1}^{[nc]} u_{-j} \le t_0\Big)\Big)^\lambda. \end{aligned} \tag{3.43}$$

By (3.16) and (3.17) we get

$$n(\alpha_r(n))^{\delta/(2+\delta)} \to 0 \qquad \text{as } n \to \infty. \tag{3.44}$$

Since

$$\begin{aligned} \frac{1}{n}E\Big(\sum_{j=1}^{n} y_j\Big)^2 &= \frac{1}{n}E\Big(b_1\sum_{j=1}^{n}\bar{u}_j + b_2\sum_{j=1}^{n}\bar{s}_j\Big)^2 \\ &= b_1^2\frac{1}{n}E\Big(\sum_{j=1}^{n}\bar{u}_j\Big)^2 + 2b_1b_2\frac{1}{n}E\Big(\sum_{j=1}^{n}\bar{u}_j\Big)\Big(\sum_{j=1}^{n}\bar{s}_j\Big) \\ &\quad + b_2^2\frac{1}{n}E\Big(\sum_{j=1}^{n}\bar{s}_j\Big)^2, \end{aligned}$$

by (3.18) and (3.19) we have

$$\frac{1}{n}E\Big(\sum_{j=1}^{n} y_j\Big)^2 \to b_1^2\sigma_1^2 + 2b_1b_2\sigma_1\sigma_2\sigma_{12} + b_2^2\sigma_2^2 \equiv \sigma^2$$

and $0 < \sigma^2 < \infty$. Hence for the processes $Y_n(t) = \frac{1}{\sigma\sqrt{n}}\sum_{j=1}^{[nt]} y_j$, $n \ge 1$, we have $Y_n \xrightarrow{\mathcal{D}} \mathcal{W}$, which implies

$$\frac{1}{\sqrt{n}}\sum_{j=1}^{[nt]} y_j \xrightarrow{\mathcal{D}} b_1\sigma_1\mathcal{W}_1(t) + b_2\sigma_2\mathcal{W}_2(t) \quad \text{for all } t \ge 0.$$

Since the sequence of processes $\frac{1}{\sqrt{n}}\sum_{j=1}^{[nt]} y_j$ has asymptotically independent increments we get $(U_n, \hat{M}_n) \xrightarrow{\mathcal{D}} (\sigma_1\mathcal{W}, \sigma_2\mathcal{W}_2)$, which by Lemma 3.1 gives $(U_n, M_n) \xrightarrow{\mathcal{D}} (\sigma_1\mathcal{W}, \sigma_2\mathcal{W}_2)$. This finishes the proof. ∎

Proof of Theorem 3.3. Since $s_i = \ell_i s_i$, Theorem 2.1 implies that $\{s_i - p, i \in \mathbb{Z}\}$ is α-mixing with mixing function $\alpha_r = \{\alpha_r(n)\}$. By Theorem 2.1 we get

$$\alpha_r(n) \le \alpha(n(1-c)^2) + 4P\Big(\sum_{j=1}^{[nc(1-c)]} u_j \le t_1\Big) + 4P\Big(\sum_{j=1}^{[nc]} u_{-j} \le t_0\Big). \tag{3.45}$$

Hence by the assumption $n\alpha(n) \to 0$ and (3.22) we get $n\alpha_r(n) \to 0$. Now using Theorem 1.4 of Merlevède & Peligrad (2000), together with Remark 1.1 there, and next applying (3.23) and (3.30), we get the assertion of the theorem. ∎

Proof of Theorem 3.4. By the assumption and Corollary 2.1 the sequence $\{\tilde{u}_i^* - a_1\}$ is stationary and strongly mixing with mixing function $\{\tilde{\alpha}(n)\}$ where $\tilde{\alpha}(n) = \alpha_r(n)/P(s_0 = 1)$. To prove the assertion of the theorem it is enough to show that the assumptions of Theorem 1.7 of Peligrad (1986, p. 202) are fulfilled, i.e. $E(\tilde{u}_1^*)^{2+\delta} < \infty$ and $\sum_{n=1}^{\infty}(\alpha_r(n))^{\delta/(2+\delta)} < \infty$. Then $\sigma^2 \equiv \mathrm{var}(\tilde{u}_1^*) + 2\sum_{k=2}^{\infty}\mathrm{cov}(\tilde{u}_1^*, \tilde{u}_k^*)$ and if $\sigma^2 > 0$, then $\frac{1}{\sigma}\tilde{U}_n^* \xrightarrow{\mathcal{D}} \mathcal{W}$. This and Lemma 3.1 give the other convergences in (3.8).

Now, by (3.25) and Lemma 5.4 of Q-S we get $E(\tilde{u}_1^*)^{2+\delta} < \infty$. Finiteness of $\sum_{n=1}^{\infty}(\alpha_r(n))^{\delta/(2+\delta)}$ follows from the assumption and Lemma 3.5. ∎

Proof of Theorem 3.5. The proof runs much as the proof of Theorem 3.4 with the difference that to get (3.32) we use Lemma 3.6. ∎

Proof of Theorem 3.6. Using Theorem 2.3, then Lemma 5.4 of Q-S, Theorem A of Herrndorf (1984) and the equivalences in Lemma 3.1 we get the assertion of the theorem. ∎

3.5. Proofs of auxiliary lemmas

Proof of Lemma 3.2. The lemma follows immediately from

$$\sum_{n=1}^{\infty} b_{[nc]} \le \int_0^{\infty} b_{[cx]}\,dx \le \frac{1}{c}\int_0^{\infty} b_{[x]}\,dx \le \Big(1+\frac{1}{c}\Big)\sum_{n=1}^{\infty} b_n. \ \blacksquare$$

Proof of Lemma 3.3. By Lemma 2.1 in Section 2.3.2,

$$\alpha_r(n) \le \alpha(n(1-c)^2) + 4P\Big(\sum_{j=1}^{[nc(1-c)]} u_j \le t_1\Big) + 4P\Big(\sum_{j=1}^{[nc]} u_{-j} \le t_0\Big) \tag{3.46}$$

for some $0 < c < 1$. But for any $0 < \lambda < 1$ and any $c_1, c_2, c_3 > 0$ we have

$$(c_1 + c_2 + c_3)^\lambda \le 3^\lambda \max\{c_1^\lambda, c_2^\lambda, c_3^\lambda\} \le 3^\lambda (c_1^\lambda + c_2^\lambda + c_3^\lambda). \tag{3.47}$$

This and (3.46) give

$$\begin{aligned}(\alpha_r(n))^\lambda \le 3^\lambda(\alpha(n(1-c)^2))^\lambda &+ 3^\lambda 4^\lambda \Big(P\Big(\sum_{j=1}^{[nc(1-c)]} u_j \le t_1\Big)\Big)^\lambda \\ &+ 3^\lambda 4^\lambda \Big(P\Big(\sum_{j=1}^{[nc]} u_{-j} \le t_0\Big)\Big)^\lambda.\end{aligned}$$

Now using Lemma 3.2 and the other assumptions of Lemma 3.3 we get the assertion. $\blacksquare$

Below, for brevity we write $P^\lambda(A) = (P(A))^\lambda$ for any event A and $0 < \lambda < 1$.

Proof of Lemma 3.4. Notice that

$$P\Big(\sum_{j=1}^{n} u_j \le t_1\Big) \le P(t_1 > na/2) + P\Big(\sum_{j=1}^{n} u_j \le t_1,\ t_1 \le na/2\Big).$$

Denoting $b_n := P^\lambda(\sum_{j=1}^n u_j \le t_1)$ and using inequality (3.47) with two components we get

$$\begin{aligned}b_n = P^\lambda\Big(\sum_{j=1}^{n} u_j \le t_1\Big) &\le 2^\lambda P^\lambda(t_1 > na/2) + 2^\lambda P^\lambda\Big(\sum_{j=1}^{n} u_j \le t_1,\ t_1 \le na/2\Big) \\ &\le 2^\lambda P^\lambda(t_1 > na/2) + 2^\lambda P^\lambda\Big(\sum_{j=1}^{n}(u_j - a) \le -na/2\Big).\end{aligned}$$

Now Chebyshev's inequality yields

$$b_n \le 2^\lambda (2/a)^{\kappa\lambda} n^{-\kappa\lambda}\Big((Et_1^\kappa)^\lambda + \Big(E\Big|\sum_{j=1}^{n}(u_j - a)\Big|^\kappa\Big)^\lambda\Big).$$

Applying Theorem 1 of Yokoyama (1980) to the second term in the brackets we get

$$b_n \le (2/a)^{\kappa\lambda} n^{-\kappa\lambda}\big((Et_1^\kappa)^\lambda + K_0 n^{\kappa\lambda/2}(E|u_1 - a|^\kappa)^\lambda\big) \le K_1 n^{-\lambda\kappa/2},$$

where K_1 is a constant independent of n.

The second inequality in (3.35) follows from

$$\begin{aligned} P^\lambda\Big(\sum_{j=1}^n u_{-j} \le t_0\Big) &\le 2^\lambda P^\lambda(t_0 > na/2) + 2^\lambda P^\lambda\Big(\sum_{j=1}^n (u_{-j} - a) \le -na/2\Big) \\ &= 2^\lambda P^\lambda(t_1 > na/2) + 2^\lambda P^\lambda\Big(\sum_{j=1}^n (u_j - a) \le -na/2\Big) \end{aligned}$$

and from the proof of the first inequality in (3.35). ∎

Proof of Lemma 3.5. Let $\lambda = \frac{\delta}{2+\delta}$. Then from the assumption of the lemma we have $\lambda\kappa/2 = \frac{\delta}{2+\delta}(1 + \delta/2) = \delta/2 > 1$. Hence by Lemma 3.4 we get convergence of the series in (3.32), and by (3.37) we get convergence in (3.31), which implies convergence (3.33), which is (3.38) for the chosen λ. ∎

Proof of Lemma 3.6. Recall that $\bar N(t) = \#\{i \ge 1 : \sum_{j=1}^i u_j \le t\}$, so $P(\sum_{j=1}^n u_j \le t_1) = P(\bar N(t_1) \ge n)$. Hence writing

$$L := \sum_{n=1}^\infty P^\lambda\Big(\sum_{j=1}^n u_j \le t_1\Big) = \sum_{n=1}^\infty \big(nP(\bar N(t_1) \ge n)\big)^\lambda n^{-\lambda}$$

and using Hölder's inequality with $p = 1/\lambda$, $q = 1/(1-\lambda)$ we get

$$\begin{aligned} L &\le \Big(\sum_{n=1}^\infty \big(nP(\bar N(t_1) \ge n)\big)^{\lambda p}\Big)^{1/p} \cdot \Big(\sum_{n=1}^\infty n^{-\lambda q}\Big)^{1/q} \\ &= \Big(\sum_{n=1}^\infty nP(\bar N(t_1) \ge n)\Big)^\lambda \cdot \Big(\sum_{n=1}^\infty n^{-\lambda/(1-\lambda)}\Big)^{1-\lambda}. \end{aligned}$$

The last series is finite because $1/2 < \lambda < 1$. Now notice that

$$\begin{aligned} E(\bar N(t_1))^2 &= \sum_{n=1}^\infty E\bar N^2(t_1)\mathbb{I}(n-1 < t_1 \le n) \\ &\le \sum_{n=1}^\infty E\bar N^2(n)\mathbb{I}(n-1 < t_1 \le n) \\ &= \sum_{n=1}^\infty E\Big(\frac{1}{n}\bar N(n)\Big)^2 (n^2\mathbb{I}(n-1 < t_1 \le n)). \end{aligned}$$

Using Hölder's inequality with $p = 1/\lambda$, $q = 1/(1-\lambda)$ we get

$$\begin{aligned} E(\bar{N}(t_1))^2 &\le \sum_{n=1}^{\infty}\Big(E\Big(\frac{1}{n}\bar{N}(n)\Big)^{2p}\Big)^{1/p}\Big(\sum_{n=1}^{\infty} n^{2q}P(n-1<t_1\le n)\Big)^{1/q} \\ &\le \sup_n\Big(E\Big(\frac{1}{n}\bar{N}(n)\Big)^{2/\lambda}\Big)^{\lambda}\Big(\sum_{n=1}^{\infty} n^{2/(1-\lambda)}P(n-1<t_1\le n)\Big)^{1-\lambda} \\ &\le \sup_n\Big(E\Big(\frac{1}{n}\bar{N}(n)\Big)^{2/\lambda}\Big)^{\lambda}(Et_1^{2/(1-\lambda)})^{1-\lambda}. \end{aligned}$$

Hence L is finite, which finishes the proof. ■

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A. Dziwisz, W. Szczotka
Institute of Mathematics
University of Wrocław
50-384 Wrocław, Poland
E-mail: atel@plusnet.pl
wladyslaw.szczotka@math.uni.wroc.pl