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LEIBNIZ'S RULE ON TWO-STEP NILPOTENT LIE GROUPS

BY

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Abstract. Let $\mathfrak{g}$ be a nilpotent Lie algebra which is also regarded as a homogeneous Lie group with the Campbell–Hausdorff multiplication. This allows us to define a generalized multiplication $f \# g = (f^\vee * g^\vee)^\wedge$ of two functions in the Schwartz class $\mathcal{S}(\mathfrak{g}^*)$, where $^\vee$ and $^\wedge$ are the Abelian Fourier transforms on the Lie algebra $\mathfrak{g}$ and on the dual $\mathfrak{g}^*$ and $*$ is the convolution on the group $\mathfrak{g}$.

In the operator analysis on nilpotent Lie groups an important notion is the one of symbolic calculus which can be viewed as a higher order generalization of the Weyl calculus for pseudodifferential operators of Hörmander. The idea of such a calculus consists in describing the product $f \# g$ for some classes of symbols.

We find a formula for $D^\alpha(f \# g)$ for Schwartz functions f, g in the case of two-step nilpotent Lie groups, which includes the Heisenberg group. We extend this formula to the class of functions f, g such that $f^\vee, g^\vee$ are certain distributions acting by convolution on the Lie group, which includes the usual classes of symbols. In the case of the Abelian group $\mathbb{R}^d$ we have $f \# g = fg$, so $D^\alpha(f \# g)$ is given by the Leibniz rule.

1. Statement of the result. Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension d which is endowed with a family $(\delta_t)_{t>0}$ of dilations. We also regard the vector space $\mathfrak{g}$ as a Lie group with the multiplication law given by the Campbell–Hausdorff formula (see Corwin–Greenleef [2])

$$x \circ y = x + y + r(x, y),$$

where $r(x, y)$ is the (finite) sum of the commutator terms of order at least 2 in the Campbell–Hausdorff series for $\mathfrak{g}$.

This allows us to define a generalized multiplication $f \# g = (f^\vee * g^\vee)^\wedge$ of two functions in the Schwartz class $\mathcal{S}(\mathfrak{g}^*)$, where $^\vee$ and $^\wedge$ are the Abelian Fourier transforms on the Lie algebra $\mathfrak{g}$ and on the dual $\mathfrak{g}^*$. In the case of the Abelian group $\mathbb{R}^d$, one gets $f \# g = fg$.

In the operator analysis on nilpotent Lie groups an important notion is the one of symbolic calculus which can be viewed as a higher order generalization of the Weyl calculus for pseudodifferential operators of Hörmander [7].

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The calculus was created by Melin [10] and developed by Manchon [9] and Głowacki [3], [6], [4]. The idea of the calculus consists in describing the product $f \# g$ for some classes of symbols. One of the obstacles in extending the Weyl calculus to general nilpotent Lie groups is the lack of a formula allowing one to calculate the derivatives of $f \# g$.

In the Abelian case, we have the multidimensional Leibniz rule

$$D^\alpha(fg) = \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} D^\beta f D^\gamma g, \qquad \alpha \in \mathbb{N}^d. \tag{1.1}$$

Let $\mathfrak{h}_n = \mathbb{R}^{2n+1}$ be the Heisenberg Lie algebra with the commutator

$$[x, y] = (0, \ldots, 0, \{x, y\}), \qquad x, y \in \mathfrak{h}_n,$$

where $\{x, y\} = \sum_{i=1}^n (x_i y_{n+i} - x_{n+i} y_i))$, and the Heisenberg group with the Campbell–Hausdorff multiplication. In that case there is a simpler form of $f \# g$ (cf. Głowacki [3, Example 3.3]),

$$\begin{aligned} f \# g(w, \lambda) \\ = c_n \iint f(w + \lambda^{1/2} u, \lambda) g(w + \lambda^{1/2} v, \lambda) e^{i\{u,v\}} \, du \, dv, \qquad w \in \mathbb{R}^{2n},\ \lambda > 0. \end{aligned} \tag{1.2}$$

By the chain rule and integration by parts one gets

$$\begin{aligned} D_{2n+1}(f \# g) = D_{2n+1} f \# g + f \# D_{2n+1} g \\ + \frac{1}{2} \sum_{i=1}^n (D_i f \# D_{n+i} g - D_{n+i} f \# D_i g). \end{aligned} \tag{1.3}$$

A general formula for $D^\alpha(f \# g)$, $\alpha \in \mathbb{N}^{2n+1}$, seems to be more complicated.

The purpose of this note is to find such a "Leibniz formula" in the case of two-step nilpotent Lie groups, which includes the Heisenberg group. By the Fourier transform this is equivalent to finding a formula for $T^\alpha(f * g)$, where $T^\alpha f(x) = x^\alpha f(x)$ and $*$ is the convolution on the group $\mathfrak{g}$.

In the Abelian case, there is a formula for the convolution product corresponding to (1.1),

$$T^\alpha(f *_0 g) = \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} T^\beta f *_0 T^\gamma g, \tag{1.4}$$

where $*_0$ is the standard convolution on $\mathbb{R}^d$.

In the general case of nilpotent Lie groups Głowacki [5] showed that

$$T^\alpha(f * g) = T^\alpha f * g + f * T^\alpha g + \sum_{\substack{l(\beta)+l(\gamma)=l(\alpha) \\ 0<l(\beta)<l(\alpha)}} c_{\beta,\gamma} T^\beta f * T^\gamma g \tag{1.5}$$

for $\alpha \neq 0$ and for any Schwartz functions f, g on $\mathfrak{g}$. Here, $c_{\beta,\gamma}$ are real constants and $l(\alpha)$ is the homogeneous length of the multiindex α (see Section 2). Notice that this formula does not give exact values of $c_{\beta,\gamma}$, and the condition $l(\beta)+l(\gamma)=l(\alpha)$ does not characterize precisely the pairs (β,γ) which appear in (1.5) with a nonzero constant coefficient $c_{\beta,\gamma}$.

In order to formulate the main result we introduce some notation. Let $X_1,\ldots,X_d$ be a base of the vector space $\mathfrak{g}$. Suppose that $A=(a_{i,j,k})_{i,j,k}$ is the matrix of the structure constants of $\mathfrak{g}$ which are given by

$$[X_i,X_j]=\sum_{k=1}^{d} a_{i,j,k}X_k,\qquad 1\le i,j\le d.$$

Let $D=\{(i,j,k): a_{i,j,k}\neq 0\}$ and $\sigma\in\mathbb{N}^D$. We denote by $\sigma_{[0]},\sigma_{[1]},\sigma_{[2]},\in\mathbb{N}^d$ the multiindices

$$\sigma_{[0],k}=\sum_{i,j}\sigma_{(i,j,k)},\qquad \sigma_{[1],i}=\sum_{j,k}\sigma_{(i,j,k)},\qquad \sigma_{[2],j}=\sum_{i,k}\sigma_{(i,j,k)}.$$

For $\alpha,\beta\in\mathbb{N}^d$, $\sigma\in\mathbb{N}^D$ and $\beta+\sigma_{[0]}\le\alpha$ we define the generalized multinomial coefficient

$$\binom{\alpha}{\beta}_{\sigma}=\frac{\alpha!}{\beta!\sigma!(\alpha-\beta-\sigma_{[0]})!}. \tag{1.6}$$

Note that for an Abelian group we have $\sigma_{[0]}=\sigma_{[1]}=\sigma_{[2]}=\mathbf{0}$ and $\binom{\alpha}{\beta}_{\sigma}=\binom{\alpha}{\beta}$.

Our main result is the following.

THEOREM 1.1. *Suppose that $\mathfrak{g}$ is a two-step nilpotent Lie group with the Campbell–Hausdorff multiplication. For any Schwartz functions f, g on $\mathfrak{g}$ and every multiindex $\alpha\in\mathbb{N}^d$,*

$$T^{\alpha}(f*g)=\sum_{\beta+\gamma+\sigma_{[0]}=\alpha}\binom{\alpha}{\beta}_{\sigma}c_{\sigma}T^{\beta+\sigma_{[1]}}f*T^{\gamma+\sigma_{[2]}}g, \tag{1.7}$$

where the (nonzero) constants c_σ are given by

$$c_{\sigma}=2^{-\sum_{i,j,k}\sigma_{(i,j,k)}}\prod_{i,j,k}a_{i,j,k}^{\sigma_{(i,j,k)}},\qquad \sigma\in\mathbb{N}^D.$$

An analogous formula for more than two functions is given in Proposition 3.4 below. Moreover, in Corollary 3.10 we show that the above formula is still valid for tempered distributions whose convolution with Schwartz class functions is in the Schwartz class.

Applying the Fourier transform to (1.7) we get an equivalent formula for $D^{\alpha}(f\mathbin{\#}g)$ for Schwartz functions f,g on $\mathfrak{g}^*$. We extend this formula to a certain class of functions that includes the classes of symbols $S^{\mathbf{m}}(\mathfrak{g}^*,\mathbf{g})$ which are admissible in Głowacki's calculus [6] (see Subsection 3.4).

In Subsection 3.5 we illustrate our results in the case of the Heisenberg group.

2. Two-step nilpotent Lie groups. Let $\mathfrak{g}$ be a Lie algebra of dimension d endowed with a one-parameter family $(\delta_t)_{t>0}$ of group automorphisms, called *dilations*. Let $p_1 = 1$ and $p_2 = 2$ be the exponents of homogeneity of the dilations. Let

$$\mathfrak{g}_1 = \{x \in \mathfrak{g} : \delta_t(x) = t^{p_1}x\}, \qquad \mathfrak{g}_2 = \{x \in \mathfrak{g} : \delta_t(x) = t^{p_2}x\}.$$

Then $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $\mathfrak{g}$ is a two-step nilpotent Lie algebra. Let $d_1 = \dim \mathfrak{g}_1$.

The vector space $\mathfrak{g}$ is also regarded as a Lie group with the multiplication

$$x \circ y = x + y + \tfrac{1}{2}[x, y].$$

The exponential map is then the identity map. From antisymmetry and the Jacobi identity,

$$a_{i,j,k} + a_{j,i,k} = 0, \qquad \sum_k (a_{i,j,k}a_{k,l,m} + a_{j,l,k}a_{l,i,m} + a_{l,i,k}a_{k,j,m}) = 0.$$

Moreover, the homogeneous structure of $\mathfrak{g}$ gives $a_{i,j,k} = 0$ if any of the conditions $i = j$, $\max(i, j) \geq k$, $\max(i, j) > d_1$, $k \leq d_1$ is satisfied. For every $k > d_1$ we have $(x \circ y)_k = x_k + y_k + r_k(x, y)$, where

$$r_k(x, y) = \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} a_{i,j,k}x_iy_j.$$

Let $T_jf(x) = x_jf(x)$, $D_jf(x) = i\partial_jf(x)$ and

$$T^\alpha f(x) = x_1^{\alpha_1}\cdots x_d^{\alpha_d}f(x), \qquad D^\alpha f(x) = D_1^{\alpha_1}\cdots D_d^{\alpha_d}f(x).$$

Let $|\alpha| = \sum_{i=1}^d \alpha_i$ be the *length* of $\alpha \in \mathbb{N}^d$. Denote by $l(\alpha)$ the *homogeneous length* of α, i.e.

$$l(\alpha) = p_1(\alpha_1 + \cdots + \alpha_{d_1}) + p_2(\alpha_{d_1+1} + \cdots + \alpha_d).$$

The Schwartz space is denoted by $\mathcal{S}(\mathfrak{g})$. Let Lebesgue measures $dx, d\xi$ on $\mathfrak{g}$ and $\mathfrak{g}^*$ be normalized so that the relationship between a function $f \in \mathcal{S}(\mathfrak{g})$ and its Abelian Fourier transform $\widehat{f} \in \mathcal{S}(\mathfrak{g}^*)$ is given by

$$\widehat{f}(\xi) = \int_{\mathfrak{g}} e^{-ix\xi}f(x)\,dx, \qquad f(x) = \int_{\mathfrak{g}^*} e^{ix\xi}\widehat{f}(\xi)\,d\xi.$$

The Fourier transform extends by duality to the space of tempered distributions.

The normalized Lebesgue measure on the vector space $\mathfrak{g}$ is a Haar measure on the group $\mathfrak{g}$. The convolution $*$ on $\mathfrak{g}$ is given by

$$f * g(x) = \int_{\mathfrak{g}} f(x \circ y^{-1})g(y)\,dy. \tag{2.1}$$

Recall some notation that we have already introduced in Section 1. For the group $\mathfrak{g}$ and $\sigma \in \mathbb{N}^D$ we defined the d-dimensional multiindices $\sigma_{[0]}, \sigma_{[1]}, \sigma_{[2]} \in \mathbb{N}^d$. We also defined the generalized multinomial coefficient

$\binom{\alpha}{\beta}_\sigma$ for $\alpha, \beta \in \mathbb{N}^d$ and $\sigma \in \mathbb{N}^D$. Let us also denote by c_σ the constants which appeared in (1.7), i.e.

$$c_\sigma = 2^{-\sum_{i,j,k} \sigma(i,j,k)} \prod_{i,j,k} a_{i,j,k}^{\sigma(i,j,k)}, \qquad \sigma \in \mathbb{N}^D. \tag{2.2}$$

3. Leibniz's rule

3.1. Multinomial theorem. The following proposition is a generalization of the multinomial theorem on $\mathbb{R}^d$. This will be crucial in the proof of Theorem 1.1.

PROPOSITION 3.1. *For any $x, y \in \mathfrak{g}$ and every multiindex $\alpha \in \mathbb{N}^d$,*

$$(x \circ y)^\alpha = \sum_{\beta+\gamma+\sigma_{[0]}=\alpha} \binom{\alpha}{\beta}_\sigma c_\sigma x^{\beta+\sigma_{[1]}} y^{\gamma+\sigma_{[2]}} g, \tag{3.1}$$

where the (nonzero) constants c_σ are given by (2.2).

Proof. Let $\alpha \in \mathbb{N}^d$. Then

$$\begin{aligned}(x \circ y)^\alpha &= \prod_{k=1}^{d} (x \circ y)_k^{\alpha_k} = \prod_{l=1}^{d_1} (x_l + y_l)^{\alpha_l} \prod_{k=d_1+1}^{d} (x_k + y_k + r_k(x,y))^{\alpha_k} \\
&= \prod_{l=1}^{d_1} \sum_{\beta_l+\gamma_l=\alpha_l} \binom{\alpha_l}{\beta_l} x_l^{\beta_l} y_l^{\gamma_l} \prod_{k=d_1+1}^{d} \sum_{\beta_k+\gamma_k+\tau_k=\alpha_k} \binom{\alpha_k}{\beta_k \gamma_k \tau_k} x_k^{\beta_k} y_k^{\gamma_k} r_k(x,y)^{\tau_k} \\
&= \sum_{\substack{\beta_l+\gamma_l=\alpha_l \\ 1 \le l \le d_1}} \sum_{\substack{\beta_k+\gamma_k+\tau_k=\alpha_k \\ d_1+1 \le k \le d}} \prod_{l=1}^{d_1} \binom{\alpha_l}{\beta_l} \prod_{k=d_1+1}^{d} \binom{\alpha_k}{\beta_k \gamma_k \tau_k} \\
&\quad \times \prod_{l=1}^{d_1} x_l^{\beta_l} \prod_{k=d_1+1}^{d} x_k^{\beta_k} \prod_{l=1}^{d_1} y_l^{\gamma_l} \prod_{k=d_1+1}^{d} y_k^{\gamma_k} \prod_{k=d_1+1}^{d} r_k(x,y)^{\tau_k}.\end{aligned} \tag{3.2}$$

Let $D_k = \{(i,j) : (i,j,k) \in D\}$. Clearly, $(i,j) \in D_k$ if $a_{i,j,k} \neq 0$. Thus,

$$\begin{aligned}r_k(x,y)^{\tau_k} &= \Big(\frac{1}{2} \sum_{(i,j) \in D_k} a_{i,j,k} x_i y_j \Big)^{\tau_k} \\
&= 2^{-\tau_k} \sum_{\sum_{(i,j)\in D_k} \tau_{k,i,j} = \tau_k} \binom{\tau_k}{\dots \tau_{k,i,j} \dots} \prod_{(i,j)\in D_k} (a_{i,j,k} x_i y_j)^{\tau_{k,i,j}} \\
&= 2^{-\tau_k} \sum_{\sum_{(i,j)\in D_k} \tau_{k,i,j} = \tau_k} \binom{\tau_k}{\dots \tau_{k,i,j} \dots} \prod_{(i,j)\in D_k} a_{i,j,k}^{\tau_{k,i,j}} \prod_{(i,j)\in D_k} x_i^{\tau_{k,i,j}} y_j^{\tau_{k,i,j}}.\end{aligned} \tag{3.3}$$

Here, $\binom{\tau_k}{\dots\tau_{k,i,j}\dots}$ denotes the multinomial coefficient

$$\binom{\tau_k}{\dots\tau_{k,i,j}\dots} = \frac{\tau_k!}{\prod_{(i,j)\in D_k}\tau_{k,i,j}!}.$$

By using (3.3), the expression (3.2) is equal to

$$\sum_{\substack{\beta_l+\gamma_l=\alpha_l\\ 1\le l\le d_1}}\ \sum_{\substack{\beta_k+\gamma_k+\tau_k=\alpha_k\\ d_1+1\le k\le d}}\ \sum_{\sum_{(i,j)\in D_k}\tau_{k,i,j}=\tau_k} \prod_{l=1}^{d_1}\binom{\alpha_l}{\beta_l}\prod_{k=d_1+1}^{d}\left(\binom{\alpha_k}{\beta_k\gamma_k\tau_k}\binom{\tau_k}{\dots\tau_{k,i,j}\dots}2^{-\tau_k}\prod_{(i,j)\in D_k}a_{i,j,k}^{\tau_{k,i,j}}\right)$$
$$\times\prod_{l=1}^{d_1}x_l^{\beta_l}\prod_{k=d_1+1}^{d}\Big(x_k^{\beta_k}\prod_{(i,j)\in D_k}x_i^{\tau_{k,i,j}}\Big)\prod_{l=1}^{d_1}y_l^{\gamma_l}\prod_{k=d_1+1}^{d}\Big(y_k^{\gamma_k}\prod_{(i,j)\in D_k}y_j^{\tau_{k,i,j}}\Big). \tag{3.4}$$

If we denote $\sigma_{(i,j,k)} = \tau_{k,i,j}$, then $\sigma\in\mathbb{N}^D$. Moreover,

$$\prod_{l=1}^{d_1}\binom{\alpha_l}{\beta_l}\prod_{k=d_1+1}^{d}\binom{\alpha_k}{\beta_k\gamma_k\sigma_k}\binom{\sigma_k}{\dots\sigma_{(i,j,k)}\dots} = \frac{\alpha!}{\beta!\gamma!\sigma!} = \binom{\alpha}{\beta}_\sigma.$$

The conditions $\beta_l+\gamma_l=\alpha_l$, $l=1,\dots,d_1$, and $\sum_{(i,j)\in D_k}\tau_{k,i,j}=\tau_k$, $\beta_k+\gamma_k+\tau_k=\alpha_k$, $k=d_1+1,\dots,d$, can be simply written as $\beta+\gamma+\sigma_{[0]}=\alpha$. Recall that the numbers c_σ are given by (2.2). Thus, (3.4) is equal to

$$\sum_{\beta+\gamma+\sigma_{[0]}=\alpha}\binom{\alpha}{\beta}_\sigma c_\sigma\prod_{k=1}^{d}\Big(x_k^{\beta_k}\prod_{(i,j)\in D_k}x_i^{\sigma(i,j,k)}\Big)\prod_{k=1}^{d}\Big(y_k^{\gamma_k}\prod_{(i,j)\in D_k}y_j^{\sigma(i,j,k)}\Big)$$
$$=\sum_{\beta+\gamma+\sigma_{[0]}=\alpha}\binom{\alpha}{\beta}_\sigma c_\sigma\prod_{i=1}^{d}x_i^{\beta_i+\sum_{j,k:(i,j)\in D_k}\sigma(i,j,k)}\prod_{j=1}^{d}y_j^{\gamma_j+\sum_{i,k:(i,j)\in D_k}\sigma(i,j,k)}$$
$$=\sum_{\beta+\gamma+\sigma_{[0]}=\alpha}\binom{\alpha}{\beta}_\sigma c_\sigma x^{\beta+\sigma_{[1]}}y^{\gamma+\sigma_{[2]}}.\ ∎$$

3.2. Convolution rule

Proof of Theorem 1.1. By (2.1) we have

$$T^\alpha(f*g)(x) = x^\alpha(f*g)(x) = \int_{\mathfrak{g}} x^\alpha f(x\circ y^{-1})g(y)\,dy. \tag{3.5}$$

Applying (3.1) we get

$$x^\alpha = ((x\circ y^{-1})\circ y)^\alpha = \sum_{\beta+\gamma+\sigma_{[0]}=\alpha}\binom{\alpha}{\beta}_\sigma c_\sigma(x\circ y^{-1})^{\beta+\sigma_{[1]}}y^{\gamma+\sigma_{[2]}}. \tag{3.6}$$

The conclusion follows from combining (3.6) and (3.5). ∎

As a consequence, we get a relationship between the exponents $\beta + \sigma_{[1]}$ and $\gamma + \sigma_{[2]}$ on the right hand side in (1.7) in terms of homogeneous length, as in (1.5).

COROLLARY 3.2. *Formula* (1.5) *holds.*

Proof. Let $\beta + \gamma + \sigma_{[0]} = \alpha$, where $\alpha, \beta, \gamma \in \mathbb{N}^d$ and $\sigma \in \mathbb{N}^D$. By a direct calculation,

$$\begin{aligned} l(\beta + \sigma_{[1]}) + l(\gamma + \sigma_{[2]}) &= \sum_{i=1}^{d_1} \Big(\beta_i + \sum_{j,k} \sigma_{(i,j,k)}\Big) + 2 \sum_{k=d_1+1}^{d} \beta_k \\ &+ \sum_{j=1}^{d_1} \Big(\gamma_j + \sum_{i,k} \sigma_{(i,j,k)}\Big) + 2 \sum_{k=d_1+1}^{d} \gamma_k = \sum_{i=k}^{d_1} \alpha_k + 2 \sum_{i=d_1+1}^{d} \alpha_k = l(\alpha). \ \blacksquare \end{aligned}$$

If we compare the coefficients on both sides of $T^{\alpha^1+\alpha^2}(f * g) = T^{\alpha_1}(T^{\alpha_2}(f * g))$, obtained from Theorem 1.1, we get the following identity.

COROLLARY 3.3. *For any* $\alpha^1, \alpha^2, \beta \in \mathbb{N}^d$ *and* $\sigma \in \mathbb{N}^D$,

$$\binom{\alpha_1 + \alpha_2}{\beta}_{\sigma} = \sum_{b(\alpha_1,\alpha_2,\beta,\sigma)} \binom{\alpha^1}{\beta^1}_{\sigma^1} \binom{\alpha^2}{\beta^2}_{\sigma^2}, \tag{3.7}$$

where $b(\alpha_1, \alpha_2, \beta, \sigma)$ *is the set*
$\{(\beta^1, \beta^2, \sigma^1, \sigma^2) : \beta^1 + \beta^2 = \beta,\ \sigma^1 + \sigma^2 = \sigma,\ \beta^1 + \sigma^1_{[0]} \le \alpha^1,\ \beta^2 + \sigma^2_{[0]} \le \alpha^2\}$.

Notice that this is the analogue of the combinatorial identity

$$\binom{n_1 + n_2}{k} = \sum_{\substack{k_1+k_2=k \\ k_1 \le n_1,\, k_2 \le n_2}} \binom{n_1}{k_1}\binom{n_2}{k_2}, \qquad n_1, n_2, k \in \mathbb{N}.$$

As in Theorem 1.1, we can find a convolution rule for more than two functions. First, we extend our notation a little. For $n \in \mathbb{N}$ let

$$D^{(n)} = \{(i, j, k, r, s) : a_{i,j,k} \neq 0,\ 1 \le r < s \le n\}.$$

Notice that if $n = 2$, then $D^{(2)}$ is essentially the same as D. For $\tau \in \mathbb{N}^{D^{(n)}}$ we consider the following multiindices in $\mathbb{N}^d$:

$$\tau_{[0],k} = \sum_{i,j,r,s} \tau_{(i,j,k,r,s)}, \qquad k = 1, \dots, d,$$

$$\tau_{[m],l} = \sum_{j,k,s} \tau_{(l,j,k,m,s)} + \sum_{i,k,r} \tau_{(i,l,k,r,m)}, \qquad m = 1, \dots, n,\ l = 1, \dots, d.$$

For $\alpha, \beta^1, \dots, \beta^n \in \mathbb{N}^d$, $\tau \in \mathbb{N}^{D^{(n)}}$ and $\sum_{m=1}^n \beta^m + \tau_{[0]} = \alpha$ we also denote

$$\binom{\alpha}{\beta^1 \dots \beta^n}_{\tau} = \frac{\alpha!}{\beta^1! \cdots \beta^n! \tau!}, \qquad \tilde{c}_\tau = 2^{-\sum_{i,j,k,r,s} \tau_{(i,j,k,r,s)}} \prod_{i,j,k,r,s} a_{i,j,k}^{\tau_{(i,j,k,r,s)}}.$$

PROPOSITION 3.4. *Let $f_1, \dots, f_n$ be Schwartz functions on $\mathfrak{g}$. For every $\alpha \in \mathbb{N}^d$,*

$$(3.8)\qquad T^\alpha(f_1 * \dots * f_n) = \sum_{\beta^1+\dots+\beta^n+\tau_{[0]}=\alpha} \binom{\alpha}{\beta^1 \dots \beta^n}_\tau \tilde{c}_\tau T^{\beta^1+\tau_{[1]}} f_1 * \dots * T^{\beta^n+\tau_{[n]}} f_n.$$

Proof. As in the proof of Proposition 3.1, we find a formula for $(y^1 \circ \dots \circ y^n)^\alpha$, where $y^1, \dots, y^n \in \mathfrak{g}$. We get

$$(3.9)\qquad (y^1 \circ \dots \circ y^n)^\alpha = \prod_{k=1}^d \Big(y_k^1 + \dots + y_k^n + \frac{1}{2} \sum_{a_{i,j,k} \neq 0} a_{i,j,k} \sum_{r<s} y_i^r y_j^s \Big)^{\alpha_k}$$
$$= \sum_{\beta^1+\dots+\beta^n+\tau_{[0]}=\alpha} \binom{\alpha}{\beta^1 \dots \beta^n}_\tau \tilde{c}_\tau (y^1)^{\beta^1+\tau_{[1]}} \dots (y^n)^{\beta^n+\tau_{[n]}}.$$

If we apply (3.9) to $y^1 = x^1 \circ (x^2)^{-1}, \dots, y^{n-1} = x^{n-1} \circ (x^n)^{-1}$, $y^n = x^n$, where $x^1, \dots, x^n$ are integral variables in the convolution, we get the conclusion. ∎

3.3. $\mathcal{S}$-convolvers. Let A be a tempered distribution on $\mathfrak{g}$, i.e. a continuous linear functional on $\mathcal{S}(\mathfrak{g})$. The convolution of a Schwartz function f on $\mathfrak{g}$ on the right with a tempered distribution A is defined by

$$f * A(x) = \langle A, \widetilde{f_x} \rangle,$$

where $\widetilde{f_x}(y) = f(xy^{-1})$. Let $\widetilde{A}$ denote the distribution given by $\langle \widetilde{A}, f \rangle = \langle A, \widetilde{f} \rangle$. We say that a distribution $A \in \mathcal{S}'(\mathfrak{g})$ is a *right $\mathcal{S}$-convolver* on a nilpotent Lie group $\mathfrak{g}$ if $f * A \in \mathcal{S}(\mathfrak{g})$ whenever $f \in \mathcal{S}(\mathfrak{g})$. We define left $\mathcal{S}$-convolvers in a similar way. A is called an *$\mathcal{S}$-convolver* if it is both a left and right $\mathcal{S}$-convolver. By Proposition 2.5 in Corwin [1], the space of $\mathcal{S}$-convolvers is closed under convolution and multiplication by polynomials. We have

$$f * (A * B) = (f * A) * B, \quad \langle A * B, f \rangle = \langle B, \widetilde{A} * f \rangle.$$

Formula (1.7) is also valid for $\mathcal{S}$-convolvers in place of Schwartz functions on a two-step nilpotent Lie group.

COROLLARY 3.5. *If A, B are $\mathcal{S}$-convolvers on $\mathfrak{g}$, then*

$$(3.10)\qquad T^\alpha(A * B) = \sum_{\beta+\gamma+\sigma_{[0]}=\alpha} \binom{\alpha}{\beta}_\sigma c_\sigma T^{\beta+\sigma_{[1]}} A * T^{\gamma+\sigma_{[2]}} B.$$

Proof. We prove (3.10) by induction on the length of α. Let $T^{e_k} = T_k$, $k = 1, \dots, d_1$. Suppose first that A is a Schwartz function. Then

$$\langle T_k(A * B), f \rangle = \langle A * B, T_k f \rangle = \langle B, \widetilde{A} * T_k f \rangle.$$

By (1.7), this is equal to

$$\langle B, T_k(\widetilde{A} * f) - T_k\widetilde{A} * f\rangle = \langle T_k B, \widetilde{A} * f\rangle - \langle B, T_k\widetilde{A} * f\rangle.$$

As $\widetilde{T_k\widetilde{A}} = -T_k A$, the base step is done when A is a Schwartz function. If A is an $\mathcal{S}$-convolver, then we can repeat the same reasoning using the just proven formula

$$T_k(f * A) = T_k f * A + f * T_k A, \qquad f \in \mathcal{S}(\mathfrak{h}),$$

instead of the case $\alpha = e_k$ in (1.7).

Now, let $T^{e_k} = T_k$, $k = d_1 + 1, \ldots, d$. If A is a Schwartz function, then

$$\begin{aligned}\langle T_k(A * B), f\rangle &= \langle A * B, T_k f\rangle = \langle B, \widetilde{A} * T_k f\rangle \\ &= \langle B, T_k(\widetilde{A} * f) - T_k\widetilde{A} * f - \frac{1}{2}\sum_{(i,j)\in D_k} a_{i,j,k} T_i\widetilde{A} * T_j f\rangle \\ &= \langle T_k B, \widetilde{A} * f\rangle + \langle T_k A * B\rangle + \frac{1}{2}\sum_{(i,j)\in D_k} a_{i,j,k}\langle T_i A * B, T_j f\rangle.\end{aligned}$$

We get $\sum_{(i,j)\in D_k} a_{i,j,k} T_j T_i A = 0$ from the antisymmetry of the structure constants on $\mathfrak{g}$, and so

$$T_k(A * B) = T_k A * B + A * T_k B + \frac{1}{2}\sum_{(i,j)\in D_k} a_{i,j,k} T_i A * T_j B \tag{3.11}$$

whenever A is a Schwartz function. Similarly to the case of T^{e_k} for $k = 1, \ldots, d_1$, we find that (3.11) also holds when A is an $\mathcal{S}$-convolver.

Now, assume that (3.10) holds for a multiindex α. The inductive step follows from (3.7). ∎

3.4. Leibniz's rule for $f \# g$. Applying the Fourier transform to (1.7) we get an equivalent formula for the derivatives of $f \# g$.

COROLLARY 3.6. *If $\alpha \in \mathbb{N}^d$ and $f, g \in \mathcal{S}(\mathfrak{g}^*)$, then*

$$D^\alpha(f \# g) = \sum_{\beta+\gamma+\sigma_{[0]}=\alpha} \binom{\alpha}{\beta}_\sigma c_\sigma D^{\beta+\sigma_{[1]}} f \# D^{\gamma+\sigma_{[2]}} g, \tag{3.12}$$

where the constants c_σ are given by (2.2).

The above formula is valid under some weaker smoothness conditions for functions, which is essential for applying these results and for a better understanding of symbolic calculus on two-step nilpotent Lie groups.

Let $\mathbf{m}_1$, $\mathbf{m}_2$ be $\mathbf{g}$-weights on $\mathfrak{g}^*$ (for more details see Głowacki [6]) and

$$S^{\mathbf{m}}(\mathfrak{g}^*, \mathbf{g}) = \{a \in C^\infty(\mathfrak{g}^*) : |D^\alpha a(x)| \le \mathbf{m}(x)\rho(x)^{-l(\alpha)}\},$$

where $\rho(x) = 1 + \|x\|$, $\|\cdot\|$ being the homogeneous norm on $\mathfrak{g}^*$. A typical example of weight is $\mathbf{m}(x) = \rho(x)^N$, $N \in \mathbb{R}$. Notice that if a distribution A satisfies $\widehat{A} \in S^{\mathbf{m}}(\mathfrak{g}^*, \mathbf{g})$ for some weight $\mathbf{m}$, then one can write A as a sum of a tempered distribution with compact support and a Schwartz function. Thus A is an $\mathcal{S}$-convolver on $\mathfrak{g}$. If $a \in S^{\mathbf{m}_1}(\mathfrak{g}^*, \mathbf{g})$ and $b \in S^{\mathbf{m}_2}(\mathfrak{g}^*, \mathbf{g})$, then, by the calculus of Głowacki [6], we have $a \# b \in S^{\mathbf{m}_1\mathbf{m}_2}(\mathfrak{g}^*, \mathbf{g})$ and a certain continuity of the product $\#$, which is sufficient to deduce from Corollary 3.5 the following.

COROLLARY 3.7. *Formula* (3.12) *holds for functions a and b such that $a^\vee$ and $b^\vee$ are $\mathcal{S}$-convolvers on $\mathfrak{g}$. In particular, if $a \in S^{\mathbf{m}_1}(\mathfrak{g}^*, \mathbf{g})$ and $b \in S^{\mathbf{m}_2}(\mathfrak{g}^*, \mathbf{g})$, then $D^\alpha(a \# b)$ is given by* (3.12), *which can also be understood pointwise.*

3.5. Heisenberg group. The Heisenberg group/algebra $\mathfrak{h}_n$ was introduced in Section 1. Let us recall that multiplication on $\mathfrak{h}_n$ is given by

$$x \circ y = \big(x_1 + y_1, \ldots, x_{2n} + y_{2n}, x_{2n+1} + y_{2n+1} + \tfrac{1}{2}\{x, y\}\big). \tag{3.13}$$

There is a remarkable relationship between the convolution structure of the Heisenberg group and the Weyl calculus for pseudodifferential operators, which was explained e.g. in Howe [8]. For $\lambda = 1$ in (1.2) one obtains the Weyl formula for the symbol of the composition of two pseudodifferential operators (cf. Głowacki [3, Example 3.3])

$$a \#_W b(\xi) = \iint a(\xi + u) b(\xi + v) e^{i\{u,v\}}\, du\, dv.$$

It is easy to see that $D^\alpha(a \#_W b)$ is given by the (noncommutative) Leibniz rule

$$D^\alpha(a \#_W b) = \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} D^\alpha a \#_W D^\gamma b. \tag{3.14}$$

Let $f, g \in \mathcal{S}(\mathfrak{h}_n)$. It is directly checked that for $i = 1, \ldots, 2n$,

$$T_i(f * g) = T_i f * g + f * T_i g.$$

If $\alpha \in \mathbb{N}^{2n+1}$ and $\alpha_{2n+1} = 0$, then

$$T^\alpha(f * g) = \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} T^\alpha f * T^\gamma g,$$

which corresponds to (3.14). On the other hand, by the relation

$$x_{2n+1} = (x \circ y^{-1})_{2n+1} + y_{2n+1} + \frac{1}{2} \sum_{i=1}^{n} \big((x \circ y^{-1})_i y_{n+i} - (x \circ y^{-1})_{n+i} y_i\big)$$

we also get (cf. (1.3))

$$T_{2n+1}(f*g) = T_{2n+1}f*g + f*T_{2n+1}g + \frac{1}{2}\sum_{i=1}^{n}(T_i f * T_{n+i}g - T_{n+i}f * T_i g).$$

Higher order formulas are more complicated, for instance

$$\begin{aligned} T_{2n+1}^2(f*g) = T_{2n+1}^2 f*g &+ f*T_{2n+1}^2 g + 2T_{2n+1}f*T_{2n+1}g \\ &+ \sum_{i=1}^{n}(T_{2n+1}T_i f * T_{n+i}g + T_i f * T_{2n+1}T_{n+i}g \\ &\qquad - T_{2n+1}T_{n+i}f * T_i g - T_{n+i}f * T_{2n+1}T_i g) \\ &+ \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n}(T_jT_i f * T_{n+j}T_{n+i}g - T_{n+j}T_i f * T_jT_{n+i}g \\ &\qquad - T_jT_{n+i}f * T_{n+j}T_i g + T_{n+j}T_{n+i}f * T_jT_i g). \end{aligned}$$

We find a general formula for $T_{2n+1}^k(f*g)$, $k\in\mathbb{N}$, as a corollary of Theorem 1.1. Let us first illustrate the notation by using it in the case of the Heisenberg group. The matrix A is given by

$$a_{i,n+i,2n+1} = 1, \quad a_{n+i,i,2n+1} = -1, \quad i = 1,\dots,n,$$

and $a_{i,j,k} = 0$ otherwise. We have

$$D = \{(1,n{+}1,2n{+}1),\dots,(n,2n,2n{+}1),(n{+}1,1,2n{+}1),\dots,(2n,n,2n{+}1)\}.$$

Let $\sigma\in\mathbb{N}^D$. Then $\sigma_{[1]}$, $\sigma_{[2]}$, $\sigma_{[0]}$ are given by

$$\begin{aligned} \sigma_{[1]} &= (\sigma_{(1,n+1,2n+1)},\dots,\sigma_{(2n,n,2n+1)},0), \\ \sigma_{[2]} &= (\sigma_{(n+1,1,2n+1)},\dots,\sigma_{(n,2n,2n+1)},0), \\ \sigma_{[0]} &= \Big(0,\dots,0,\sum_{i=1}^{n}(\sigma_{(i,n+i,2n+1)} + \sigma_{(n+i,i,2n+1)})\Big). \end{aligned}$$

If $\sigma_c = \sigma_{[0],2n+1}$, then $T_{2n+1}^k(f*g)$, $k\in\mathbb{N}$, is given by

$$\begin{aligned} T_{2n+1}^k(f*g) = &\sum_{\substack{\{l,m\in\mathbb{N},\sigma\in\mathbb{N}^D:\\ l+m+\sigma_c=k\}}} \frac{k!}{l!m!\sigma!}2^{-\sigma_c}(-1)^{\sum_{i=1}^{n}\sigma_{(n+i,i,2n+1)}} \\ &\cdot T_1^{\sigma(1,n+1,2n+1)}\cdots T_{2n}^{\sigma(2n,n,2n+1)}T_{2n+1}^{l}f * T_1^{\sigma(n+1,1,2n+1)}\cdots T_{2n}^{\sigma(n,2n,2n+1)}T_{2n+1}^{m}g. \end{aligned}$$

As in the procedure described in Subsection 3.4, one gets an extension of the rule for $\mathcal{S}$-convolvers on $\mathfrak{h}_n$ and a formula for the derivatives of the product $a\,\#\,b$.

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