COLLOQUIUM MATHEMATICUM

VOL. 145 2016 NO. 2

AN EASIER PROOF OF THE CANONICAL RAMSEY THEOREM

BY

PIERRE MATET (Caen)

Abstract. We give a new proof of the Canonical Ramsey Theorem of Erdős and Rado.

In Ramsey theory one assigns one of several colors to each element of a structure $\mathcal{A}$, and look for a large, nice substructure $\mathcal{B}$ that is monochromatic (i.e. with one and the same color assigned to each element of $\mathcal{B}$). At the origin of the theory is a result of Ramsey. For a set X and m in the set $\mathbb{N}$ of positive integers, let $[X]^m$ denote the collection of all m-element subsets of X. For n in $\mathbb{N}$, $\mathrm{RT}(n)$ asserts that for any $F : [\mathbb{N}]^n \to \{1, 2\}$, there is an infinite subset A of $\mathbb{N}$ such that F is constant on $[A]^n$. Note that it easily follows from $\mathrm{RT}(n)$ that given k in $\mathbb{N}$, an infinite subset M of $\mathbb{N}$, and $F : [M]^n \to \{1, \dots, k\}$, there is an infinite subset A of M such that F is constant on $[A]^n$. The Infinite Ramsey Theorem [Ram] affirms that $\mathrm{RT}(n)$ holds for every n in $\mathbb{N}$.

Canonical Ramsey theory is a branch of Ramsey theory dealing with situations where we use so many colors that there may not be any large, nice, monochromatic substructure. The archetypal result in this theory is the Canonical Ramsey Theorem of Erdős and Rado. Let $n \in \mathbb{N}$. For any n-element subset x of $\mathbb{N}$, let $\langle x_1, \dots, x_n \rangle$ be the increasing enumeration of x. We let $\mathrm{CRT}(n)$ assert that for any $F : [\mathbb{N}]^n \to \mathbb{N}$, there is an infinite subset A of $\mathbb{N}$ and $L \subseteq \{1, \dots, n\}$ such that for any two elements e, e' of $[A]^n$, $F(e) = F(e')$ if and only if $L \subseteq \{i : e_i = e'_i\}$. Note that $\mathrm{RT}(n)$ easily follows from $\mathrm{CRT}(n)$. The Canonical Ramsey Theorem affirms that $\mathrm{CRT}(n)$ holds for all n in $\mathbb{N}$.

Several proofs appeared in print. The proof of Erdős and Rado [ER] proceeds by induction, as does that of Mileti [M]. A non-inductive proof was published by Rado [Rad]. An important feature of the proofs in [ER] and [Rad] is that $\mathrm{RT}(2n)$ is used in the derivation of $\mathrm{CRT}(n)$. These three

2010 *Mathematics Subject Classification*: Primary 03E02.
Key words and phrases: Canonical Ramsey Theorem.
Received 17 March 2015.
Published online 3 June 2016.

DOI: 10.4064/cm6577-3-2016

proofs are all fairly complicated. Recall that the antilexicographic order $<$ on $[\mathbb{N}]^n$ is defined by $e < e'$ whenever (a) there is i such that $e_i \neq e'_i$, and (b) for the greatest such i, $e_i < e'_i$. Thus for $n = 2$, $e < e'$ if and only if we are in one of the following six cases:

- $e_1 < e_2 < e'_1 < e'_2$,
- $e_1 < e_2 = e'_1 < e'_2$,
- $e_1 < e'_1 < e_2 < e'_2$,
- $e_1 < e'_1 < e_2 = e'_2$,
- $e_1 = e'_1 < e_2 < e'_2$,
- $e'_1 < e_1 < e_2 < e'_2$.

It is due to this relatively small number of possibilities that the proof of CRT(2) is so much easier. Trouble starts with $n = 3$, where we counted 31 cases. This explains why the Canonical Ramsey Theorem is often cited without proof (see e.g. [H]), or with just the (not so representative) proof for $n = 2$ (see e.g. [GRS]). The proof we present is both inductive and straightforward.

We found it convenient to replace CRT(n) with an equivalent statement denoted by $\Phi(n)$. Given $G : [[\mathbb{N}]^n]^2 \to \{1, 2\}$, we abbreviate $G(\{e, e'\})$ as $G(e, e')$. We say that G is 1-*transitive* if $G(e, e'') = 1$ whenever e, e', e'' are three distinct elements of $[\mathbb{N}]^n$ such that $G(e, e') = G(e', e'') = 1$. We let $\Phi(n)$ assert that for any 1-transitive $G : [[\mathbb{N}]^n]^2 \to \{1, 2\}$, there is an infinite subset A of $\mathbb{N}$, and $L \subseteq \{1, \ldots, n\}$, such that for any two distinct elements e, e' of $[A]^n$, $G(e, e') = 1$ if and only if $L \subseteq \{i : e_i = e'_i\}$.

Lemma. *$\Phi(n)$ and* CRT(n) *are equivalent.*

Proof. We start by showing that $\Phi(n)$ implies CRT(n). For $F : [\mathbb{N}]^n \to \mathbb{N}$, define $G : [[\mathbb{N}]^n]^2 \to \{1, 2\}$ by $G(e, e') = 1$ if and only if $F(e) = F(e')$. Let A and L be as in the statement of $\Phi(n)$. Then for any two distinct elements e, e' of $[A]^n$, $F(e) = F(e')$ if and only if $L \subseteq \{i : e_i = e'_i\}$.

Let us now deal with the converse. Given a 1-transitive $G : [[\mathbb{N}]^n]^2 \to \{1, 2\}$, consider the equivalence relation $\equiv$ on $[\mathbb{N}]^n$ defined by $e \equiv e'$ if and only if either $e = e'$, or $e \neq e'$ and $G(e, e') = 1$. Now define $F : [\mathbb{N}]^n \to \mathbb{N}$ so that for any e, e' in $[\mathbb{N}]^n$, $F(e) = F(e')$ if and only if $e \equiv e'$. We may find an infinite subset A of $\mathbb{N}$ and $L \subseteq \{1, \ldots, n\}$ such that for e, e' in $[A]^n$, $F(e) = F(e')$ if and only if $L \subseteq \{i : e_i = e'_i\}$. Now fix $e < e'$ in $[A]^n$. Then $G(e, e') = 1$ if and only if $F(e) = F(e')$ if and only if $L \subseteq \{i : e_i = e'_i\}$. ∎

Theorem. *$\Phi(n)$ holds for every n in $\mathbb{N}$.*

Proof. We proceed by induction. To prove $\Phi(1)$, fix a 1-transitive $f : [[\mathbb{N}]^1]^2 \to \{1, 2\}$. Define an equivalence relation $\equiv$ on $\mathbb{N}$ by $r \equiv s$ if and only if either $r = s$, or $r \neq s$ and $f(\{r\}, \{s\}) = 1$. First suppose that there is an infinite equivalence class V. Then f takes the constant value 1 on

$[[V]^1]^2$, which is equivalent to saying that for any two distinct elements e, e' of $[V]^1$, $f(e,e') = 1$ if and only if $\emptyset \subseteq \{i : e_i = e'_i\}$. Now suppose that every equivalence class is finite. Pick a subset W of $\mathbb{N}$ such that $W \cap Y$ has size 1 for each equivalence class Y. Then f is identically 2 on $[[W]^1]^2$, which is the same as saying that for any two distinct elements e, e' of $[W]^1$, $f(e,e') = 1$ if and only if $\{1\} \subseteq \{i : e_i = e'_i\}$.

Now suppose that $\Phi(m)$ holds for every m in N with $m \leq n$. To show $\Phi(n+1)$, let $F : [[\mathbb{N}]^{n+1}]^2 \to \{1,2\}$ be 1-transitive. Using induction, define $g : [\mathbb{N}]^n \to \{1,2\}$, three functions G, H, J from $[[\mathbb{N}]^n]^2$ to $\{1,2\}$, and a_i and B_i for i in N so that:

(1) B_i is an infinite subset of $\mathbb{N}$.
(2) $a_i = \min(B_i)$.
(3) $B_{i+1} \subseteq \{m \in B_i : m > a_i\}$.
(4) For $i \leq n$, $B_i = \{m \in \mathbb{N} : m \geq i\}$.
(5) For any $e \in [\{1,\dots,a_i\}]^n$ and any two distinct elements a, a' of B_{i+1}, we have $F(e \cup \{a\}, e \cup \{a'\}) = g(e)$.
(6) Let $e < e'$ in $[\{1,\dots,a_i\}]^n$. Then
   - $F(e \cup \{a\}, e' \cup \{a\}) = G(e,e')$ for all a in B_{i+1};
   - $F(e \cup \{a\}, e' \cup \{a'\}) = H(e,e')$ whenever $a, a' \in B_{i+1}$ and $a < a'$;
   - $F(e \cup \{a\}, e' \cup \{a'\}) = J(e,e')$ whenever $a, a' \in B_{i+1}$ and $a > a'$.

Note that (5) is obtained by applying $\Phi(1)$ repeatedly (once for each e), whereas for (6) we appeal to RT(1) (once for each (e,e') with $e < e'$) and RT(2) (twice for each (e,e') with $e < e'$). Set $A = \{a_1, a_2, \dots\}$. By RT(n), we may find an infinite subset B of A and $j \in \{1,2\}$ such that g is identically j on $[B]^n$.

Case 1: $j = 1$. Define $K : [[B]^n]^2 \to \{1,2\}$ by $K(e,e') = F(e \cup \{a\}, e' \cup \{a\})$, where a is the least q in B such that $\max(e) < q$ and $\max(e') < q$.

Claim 1. *Let e, e', e'' be three distinct elements of $[B]^n$ such that $K(e,e') = K(e',e'') = 1$. Then $K(e,e'') = 1$.*

Proof. Let a (respectively, b, c) be the least q in B such that $\max(e) < q$ and $\max(e') < q$ (respectively, $\max(e') < q$ and $\max(e'') < q$, $\max(e) < q$ and $\max(e'') < q$). Select d in B such that $d > \max(\{a,b,c\})$. Now since $1 = g(e) = K(e,e') = g(e') = K(e,e'') = g(e'')$, we have $1 = F(e \cup \{c\}, e \cup \{d\}) = F(e \cup \{d\}, e \cup \{a\}) = F(e \cup \{a\}, e' \cup \{a\}) = F(e' \cup \{a\}, e' \cup \{d\}) = F(e' \cup \{d\}, e' \cup \{b\}) = F(e' \cup \{b\}, e'' \cup \{b\}) = F(e'' \cup \{b\}, e'' \cup \{d\}) = F(e'' \cup \{d\}, e'' \cup \{c\})$. By 1-transitivity of F, it follows that $1 = F(e \cup \{c\}, e'' \cup \{c\}) = K(e,e'')$, which completes the proof of Claim 1.

By Claim 1 and $\Phi(n)$, there must be an infinite subset C of B and $L \subseteq \{1,\dots,n\}$ such that for any two distinct elements e, e' of $[C]^n$, $K(e,e') = 1$

if and only if $L \subseteq \{i : e_i = e'_i\}$. Now given two distinct elements x, x' of $[C]^{n+1}$, let $e = \{x_1, \dots, x_n\}$, $e' = \{x'_1, \dots, x'_n\}$ and a be the least q in B such that both x_n and x'_n are less than q. Then $L \subseteq \{i : x_i = x'_i\}$ if and only if $L \subseteq \{i : e_i = e'_i\}$ if and only if $K(e, e') = 1$ if and only if $F(e \cup \{a\}, e' \cup \{a\}) = 1$ if and only if $F(e \cup \{a\}, x') = 1$ if and only if $F(x, x') = 1$, since $F(e' \cup \{a\}, x') = 1 = F(e \cup \{a\}, x)$.

Case 2: $j = 2$. We may find an infinite subset D of B, and three subsets M, S, P of $\{1, \dots, n\}$ such that for any two distinct elements e, e' of $[D]^n$:

- $G(e, e') = 1$ if and only if $M \subseteq \{i : e_i = e'_i\}$;
- $H(e, e') = 1$ if and only if $S \subseteq \{i : e_i = e'_i\}$;
- $J(e, e') = 1$ if and only if $P \subseteq \{i : e_i = e'_i\}$.

Claim 2. $S = P = \{1, \dots, n\}$.

Proof. Suppose otherwise, and pick $e < e'$ in $[D]^n$ such that $1 \in \{H(e, e'), J(e, e')\}$. Select a, a', a'' in D such that $\max(e') < a < a' < a''$. Now $H(e, e') \neq 1$, since otherwise $F(e \cup \{a\}, e' \cup \{a'\}) = 1 = F(e \cup \{a\}, e' \cup \{a''\})$, and therefore $F(e' \cup \{a'\}, e' \cup \{a''\}) = 1$. Hence, $J(e, e') = 1$. But then $F(e \cup \{a''\}, e' \cup \{a'\}) = 1 = F(e \cup \{a''\}, e' \cup \{a\})$. It follows that $F(e' \cup \{a\}, e' \cup \{a'\}) = 1$. This contradiction completes the proof of Claim 2.

We set $Q = M \cup \{n+1\}$. Now let $x < x'$ in $[D]^{n+1}$. Set $e = \{x_1, \dots, x_n\}$ and $e' = \{x'_1, \dots, x'_n\}$. Observe that if $F(x, x') = 1$, then $e \neq e'$ (since $j = 2$), and moreover, by Claim 2, $x_{n+1} = x'_{n+1}$. Hence, $Q \subseteq \{i : x_i = x'_i\}$ if and only if $x_{n+1} = x'_{n+1}$ and $M \subseteq \{i : e_i = e'_i\}$ if and only if $x_{n+1} = x'_{n+1}$ and $G(e, e') = 1$ if and only if $x_{n+1} = x'_{n+1}$ and $F(e \cup \{x_{n+1}\}, e' \cup \{x'_{n+1}\}) = 1$ if and only if $x_{n+1} = x'_{n+1}$ and $F(x, x') = 1$ if and only if $F(x, x') = 1$. ∎

Corollary. $\mathrm{CRT}(n)$ *holds for every* n *in* $\mathbb{N}$.

Let us observe that $\mathrm{RT}(n)$ can be reformulated in the same spirit. Say that $G : [[\mathbb{N}]^n]^2 \to \{1, 2\}$ is 2-*antitransitive* if $G(e, e'') = 1$ whenever e, e', e'' are three distinct elements of $[\mathbb{N}]^n$ such that $G(e, e') = G(e', e'') = 2$. We let $\Psi(n)$ assert that for any 1-transitive, 2-antitransitive $G : [[\mathbb{N}]^n]^2 \to \{1, 2\}$, there is an infinite subset A of N such that G takes the constant value 1 on $[[A]^n]^2$.

The proof of the following is similar to that of the above lemma.

Observation. *The following are equivalent:*

(i) $\mathrm{RT}(n)$.
(ii) $\Psi(n)$.
(iii) *For any* 1-*transitive,* 2-*antitransitive* $G : [[\mathbb{N}]^n]^2 \to \{1, 2\}$, *there is an infinite subset* A *of* $\mathbb{N}$, *and* $L \subseteq \{1, \dots, n\}$ *such that for any distinct* $e, e' \in [A]^n$, $G(e, e') = 1$ *if and only if* $L \subseteq \{i : e_i = e'_i\}$.

REFERENCES

[ER] P. Erdős and R. Rado, *A combinatorial theorem*, J. London Math. Soc. 25 (1950), 249–255.

[GRS] R. L. Graham, B. L. Rothschild and J. H. Spencer, *Ramsey Theory*, Wiley-Interscience Ser. Discrete Math., Wiley, New York, 1980.

[H] L. J. Halbeisen, *Combinatorial Set Theory*, Springer Monogr. Math., Springer, London, 2012.

[M] J. R. Mileti, *The Canonical Ramsey Theorem and computability theory*, Trans. Amer. Math. Soc. 360 (2008), 1309–1341.

[Rad] R. Rado, *Note on canonical partitions*, Bull. London Math. Soc. 18 (1986), 123–126.

[Ram] F. P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. 30 (1930), 264–286.

Pierre Matet
Université de Caen – CNRS
Laboratoire de Mathématiques
BP 5186
14032 Caen Cedex, France
E-mail: pierre.matet@unicaen.fr