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A NEW GENERALIZED CASSINI DETERMINANT

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Abstract. We extend the notion of Cassini determinant to recently introduced hyperfibonacci sequences. We find the Q-matrix for the rth generation hyperfibonacci numbers and prove an explicit expression of the Cassini determinant for these sequences.

1. Introduction. Given a second order recurrence relation

$$a_{n+2} = \alpha a_{n+1} + \beta a_n, \tag{1.1}$$

where α and β are constants, a sequence $(a_k)_{k\geq 0}$ is called a *solution* of (1.1) if its terms satisfy this recurrence. The set of all solutions of (1.1) forms a *linear space*, meaning that if $(a_k)_{k\geq 0}$ and $(b_k)_{k\geq 0}$ are two solutions then $(a_k + b_k)_{k\geq 0}$ is also a solution, and for any constant c, $(ca_k)_{k\geq 0}$ is a solution. Using these basic properties one can derive the identity

$$a_m b_{m-1} - a_{m-1} b_m = (-\beta)^{m-1}(a_1 b_0 - a_0 b_1), \tag{1.2}$$

where $(a_k)_{k\geq 0}$ and $(b_k)_{k\geq 0}$ are any two solutions of (1.1) [VO]. When $\alpha = \beta = 1$ and the first two initial terms are 0 and 1, relation (1.1) defines the well known *Fibonacci sequence* $(F_k)_{k\geq 0}$. One can find more on this subject in the classical reference [VA]. In the case of the Fibonacci sequence, relation (1.2) with $a_m = F_{n+1}$ and $b_m = F_n$ reduces to

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n \tag{1.3}$$

and it is called *Cassini's identity* [GKP, MI, WZ]. This relation can also be written in matrix form as

$$\det\begin{pmatrix} F_n & F_{n+1} \\ F_{n+1} & F_{n+2} \end{pmatrix} = (-1)^n. \tag{1.4}$$

Stakhov [ST] found a generalization of the Cassini identity for the p-Fibonacci numbers. Krattenthaler and Oller-Marcén [KO] also present a similar result.

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In this paper we study the *hyperfibonacci sequences* defined by

$$F_n^{(r)} = \sum_{k=0}^{n} F_k^{(r-1)}, \quad F_n^{(0)} = F_n, \quad F_0^{(r)} = 0, \quad F_1^{(r)} = 1, \tag{1.5}$$

where $r \in \mathbb{N}$ and F_n is the nth Fibonacci number. The number $F_n^{(r)}$ will be called the nth hyperfibonacci number of the *rth generation*. These sequences were recently introduced by Dill and Mező [DM]. Several interesting number-theoretical and combinatorial properties of these sequences have already been proven, e.g. in [CZ].

Here we define the matrix

$$A_{r,n} = \begin{pmatrix} F_n^{(r)} & F_{n+1}^{(r)} & \cdots & F_{n+r+1}^{(r)} \\ F_{n+1}^{(r)} & F_{n+2}^{(r)} & \cdots & F_{n+r+2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n+r+1}^{(r)} & F_{n+r+2}^{(r)} & \cdots & F_{n+2r+2}^{(r)} \end{pmatrix}$$

and we prove a formula for $\det(A_{r,n})$ extending (1.4).

2. Q-matrix of the hyperfibonacci sequences. According to definition (1.5), we have

$$F_{n+1}^{(r)} = F_n^{(r)} + F_{n+1}^{(r-1)}. \tag{2.1}$$

In the case $r = 1$ the second term $F_{n+1}^{(r-1)}$ is determined by the Fibonacci recurrence relation,

$$F_{n+3}^{(1)} = F_{n+2}^{(1)} + (F_{n+2}^{(1)} - F_{n+1}^{(1)}) + (F_{n+1}^{(1)} - F_n^{(1)}),$$

thus we have

$$F_{n+3}^{(1)} = 2F_{n+2}^{(1)} - F_n^{(1)}. \tag{2.2}$$

Now, iteratively using (2.2) we derive the recurrence relation

$$F_{n+2}^{(1)} = F_{n+1}^{(1)} + F_n^{(1)} + 1 \tag{2.3}$$

by computing

$$\begin{aligned} F_{n+3}^{(1)} &= F_{n+2}^{(1)} + 2F_{n+1}^{(1)} - F_{n-1}^{(1)} - F_n^{(1)} \\ &= F_{n+2}^{(1)} + F_{n+1}^{(1)} + 2F_n^{(1)} - F_{n-2}^{(1)} - F_{n-1}^{(1)} - F_n^{(1)} \\ &= F_{n+2}^{(1)} + F_{n+1}^{(1)} + \cdots + F_3^{(1)} - F_0^{(1)} - F_1^{(1)} - \cdots - F_n^{(1)} \\ &= F_{n+2}^{(1)} + F_{n+1}^{(1)} + 1. \end{aligned}$$

When $r = 2$ we use the same approach to get the recurrence for the second generation of hyperfibonacci numbers,

$$F_{n+2}^{(2)} = F_{n+1}^{(2)} + F_n^{(2)} + n + 2. \tag{2.4}$$

Namely, in this case the second term in (2.1) is determined by the recurrence relation (2.3). This means that again we can perform the $(n{+}1)$-step iterative procedure, this time using

$$F^{(2)}_{n+3} = 2F^{(2)}_{n+2} - F^{(2)}_{n} + 1. \tag{2.5}$$

The fact that terms indexed by 3 through n cancel each other and that $n+1$ ones remain, completes the proof of (2.4).

Recall that *polytopic numbers* are a generalization of square and triangular numbers. These numbers can be represented by a regular geometrical arrangement of equally spaced points. The nth regular r-topic number $P^{(r)}_n$ is equal to

$$P^{(r)}_n = \binom{n+r-1}{r}. \tag{2.6}$$

When $r = 3$, the ith step of the iterative procedure described above results in an extra i, which sum to a triangular number $\binom{n+3}{2}$ after the final $(n+1)$st iteration. Furthermore, in the next case we add the ith triangular number in the ith step of the iteration. According to the properties of polytopic numbers, these numbers sum to the tetrahedral number $\binom{n+4}{3}$. In general, in the ith step of the iteration we add the ith regular $(r-1)$-topic number, and the sum of these numbers after the final step of the procedure is the regular polytopic number $\binom{n+r}{r-1}$. Now we collect all this reasoning into

Lemma 2.1. *The difference between the nth r-generation hyperfibonacci number and the sum of its two predecessors is the nth regular $(r-1)$-topic number,*

$$F^{(r)}_{n+2} = F^{(r)}_{n+1} + F^{(r)}_{n} + \binom{n+r}{r-1}. \tag{2.7}$$

We can also write (2.7) as

$$F^{(r)}_{n+2} = F^{(r)}_{n+1} + F^{(r)}_{n} + P^{(r-1)}_{n+2}.$$

Hyperfibonacci sequences can be defined by the vector recurrence relation

$$\begin{pmatrix} F^{(r)}_{n+1} \\ F^{(r)}_{n+2} \\ \vdots \\ F^{(r)}_{n+r+2} \end{pmatrix} = Q_{r+2} \begin{pmatrix} F^{(r)}_{n} \\ F^{(r)}_{n+1} \\ \vdots \\ F^{(r)}_{n+r+1} \end{pmatrix} \tag{2.8}$$

where Q_{r+2} is a square matrix

$$Q_{r+2} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ q_1 & q_2 & q_3 & \cdots & q_{r+1} & q_{r+2} \end{pmatrix}. \tag{2.9}$$

In order to determine $q_1, \ldots, q_{r+2}$ we use the fact that terms from $-r$ through 0 of the rth generation hyperfibonacci numbers take values

$$0, \ldots, \pm 1, 0, 0, \ldots, 0, 1, r+1, \ldots.$$

This follows from Lemma 2.1 since

$$\binom{(n-2)+r}{r-1} = \frac{n(n+1)(n+2)\cdots(n+r-2)}{(r-1)!}. \tag{2.10}$$

These expressions are obviously 0 for $n = 0, -1, \ldots, -r$.

In particular, when $n = -r+2$ we get

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ F_2^{(r)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ q_1 & q_2 & q_3 & \cdots & q_{r+1} & q_{r+2} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

meaning that $q_{r+2} = F_2^{(r)}$. In the same way we obtain relations for all elements of Q_{r+2},

$$\begin{aligned} q_{r+2} &= F_2^{(r)}, \\ q_{r+1} &= F_3^{(r)} - F_2^{(r)} q_{r+2}, \\ q_r &= F_4^{(r)} - F_3^{(r)} q_{r+2} - F_2^{(r)} q_{r+1}, \\ &\cdots \\ q_1 &= F_{r+3}^{(r)} - F_{r+2}^{(r)} q_{r+2} - \cdots - F_2^{(r)} q_2. \end{aligned}$$

This reasoning gives Theorem 2.2.

THEOREM 2.2. *For the hypefibonacci sequences we have*

$$A_{r,n} = Q_{r+2}^n A_{r,0}, \tag{2.11}$$

where $A_{r,n}$ is defined in the introduction.

Proof. Relation (2.8) in expanded form can be written as $A_{r,n} = Q_{r+2}A_{r,n-1}$. Now the statement of the theorem follows immediately:

$$A_{r,n} = Q_{r+2}A_{r,n-1} = Q_{r+2}^2 A_{r,n-2} = Q_{r+2}^n A_{r,0}. \ \blacksquare$$

The elements $q_1, \dots, q_{r+2}$ can be given explicitly. In particular, expressions for q_r, q_{r+1}, q_{r+2} are

$$q_{r+2} = 1 + r, \qquad q_{r+1} = 1 - \binom{r+1}{2}, \qquad q_r = \frac{r^3 - 7r}{6}.$$

As an example, we calculate the hyperfibonacci numbers $F_3^{(2)}, F_4^{(2)}, \dots, F_9^{(2)}$ of the second generation, collected in the matrix $A_{2,3}$.

For the second generation of the hyperfibonacci sequences we have

$$A_{2,0} = \begin{pmatrix} 0 & 1 & 3 & 7 \\ 1 & 3 & 7 & 14 \\ 3 & 7 & 14 & 26 \\ 7 & 14 & 26 & 46 \end{pmatrix}, \qquad Q_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & -2 & 3 \end{pmatrix},$$

according to (1.5) and (2.9). Now we determine the matrix $A_{2,3}$ from Theorem 2.2:

$$A_{2,3} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & -2 & 3 \end{pmatrix}^3 \begin{pmatrix} 0 & 1 & 3 & 7 \\ 1 & 3 & 7 & 14 \\ 3 & 7 & 14 & 26 \\ 7 & 14 & 26 & 46 \end{pmatrix} = \begin{pmatrix} 7 & 14 & 26 & 46 \\ 14 & 26 & 46 & 79 \\ 26 & 46 & 79 & 133 \\ 46 & 79 & 133 & 221 \end{pmatrix}.$$

Note that the eigenvalues of Q_4 are $\phi, 1, 1, \bar{\phi}$, where

$$\phi = \frac{1+\sqrt{5}}{2}, \qquad \bar{\phi} = \frac{1-\sqrt{5}}{2}.$$

The class of matrices (2.9) has some further interesting properties. Here we point out that all these matrices have determinant -1:

LEMMA 2.3. *For all* $r \in \mathbb{N}$,

$$\det(Q_{r+2}) = -1.$$

Proof. We prove this by comparing the determinants of $A_{r,-r}$ and $A_{r,-r-1}$,

$$A_{r,-r} = Q_{r+2}A_{r,-r-1}.$$

We have

$$\det(A_{r,-r-1}) = \det\begin{pmatrix} (-1)^r & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \iddots & \vdots \\ 0 & 0 & 1 & \cdots & F^{(r)}_{r-2} \\ 0 & 1 & r+1 & \cdots & F^{(r)}_{r-1} \end{pmatrix}_{r\times r}$$

$$= (-1)^r \det\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & r+1 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & F^{(r)}_{r-3} & F^{(r)}_{r-2} \\ 1 & r+1 & \cdots & F^{(r)}_{r-2} & F^{(r)}_{r-1} \end{pmatrix}_{(r-1)\times(r-1)}$$

$$= (-1)^{r+1}(-1)^{\lfloor (r-1)/2\rfloor+1} = (-1)^{\lfloor r/2\rfloor}.$$

On the other hand, $\det(A_{r,-r}) = (-1)^{\lfloor r/2\rfloor}$, which proves that

$$\det(A_{r,-r}) = -\det(A_{r,-r-1}). \tag{2.12}$$

Now, the statement follows from the Binet–Cauchy theorem. ■

It is worth noting that in [LLS], the authors give some properties of the k-generalized Fibonacci Q-matrix.

3. Cassini's identity in matrix form

LEMMA 3.1.

$$\det\begin{pmatrix} F_n-1 & F_{n+1}-1 & F_{n+2}-1 \\ F_{n+1}-1 & F_{n+2}-1 & F_{n+3}-1 \\ F_{n+2}-1 & F_{n+3}-1 & F_{n+4}-1 \end{pmatrix} = (-1)^n, \qquad n \geq 0. \tag{3.1}$$

Proof. Using the definition of Fibonacci numbers and elementary transformations on rows and columns of determinants we get

$$\det\begin{pmatrix} F_n-1 & F_{n+1}-1 & F_{n+2}-1 \\ F_{n+1}-1 & F_{n+2}-1 & F_{n+3}-1 \\ F_{n+2}-1 & F_{n+3}-1 & F_{n+4}-1 \end{pmatrix}$$

$$= \det\begin{pmatrix} F_n-1 & F_{n+1}-1 & F_{n+2}-1 \\ F_{n+1}-1 & F_{n+2}-1 & F_{n+3}-1 \\ F_n+F_{n+1}-1 & F_{n+1}+F_{n+2}-1 & F_{n+2}+F_{n+3}-1 \end{pmatrix}$$

$$= \det \begin{pmatrix} F_n - 1 & F_{n+1} - 1 & F_{n+2} - 1 \\ F_{n+1} - 1 & F_{n+2} - 1 & F_{n+3} - 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \det \begin{pmatrix} F_n - 1 & F_{n+1} - 1 & F_n + F_{n+1} - 1 \\ F_{n+1} - 1 & F_{n+2} - 1 & F_{n+1} + F_{n+2} - 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \det \begin{pmatrix} F_n - 1 & F_{n+1} - 1 & 1 \\ F_{n+1} - 1 & F_{n+2} - 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} = \det \begin{pmatrix} F_n & F_{n+1} & 0 \\ F_{n+1} & F_{n+2} & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= -(F_n F_{n+2} - F_{n+1}^2) = (-1)^n. \blacksquare$$

Lemma 3.2. *For the first generation of hyperfibonacci sequences,* $(F_n^{(1)})_{n\geq 0}$,

$$\det \begin{pmatrix} F_n^{(1)} & F_{n+1}^{(1)} & F_{n+2}^{(1)} \\ F_{n+1}^{(1)} & F_{n+2}^{(1)} & F_{n+3}^{(1)} \\ F_{n+2}^{(1)} & F_{n+3}^{(1)} & F_{n+4}^{(1)} \end{pmatrix} = (-1)^n.$$

Proof. By using the relation

$$F_n^{(1)} = F_{n+2} - 1 \tag{3.2}$$

(which follows immediately from the elementary Fibonacci identity $\sum_{k=0}^{n} F_k = F_{n+2} - 1$, $n \geq 0$) and Lemma 3.1, we have

$$\det \begin{pmatrix} F_n^{(1)} & F_{n+1}^{(1)} & F_{n+2}^{(1)} \\ F_{n+1}^{(1)} & F_{n+2}^{(1)} & F_{n+3}^{(1)} \\ F_{n+2}^{(1)} & F_{n+3}^{(1)} & F_{n+4}^{(1)} \end{pmatrix} = \det \begin{pmatrix} F_{n+2} - 1 & F_{n+3} - 1 & F_{n+4} - 1 \\ F_{n+3} - 1 & F_{n+4} - 1 & F_{n+5} - 1 \\ F_{n+4} - 1 & F_{n+5} - 1 & F_{n+6} - 1 \end{pmatrix}$$

$$= (-1)^n. \blacksquare$$

Theorem 3.3. *For all* $r \in \mathbb{N}$ *and* $n \in \mathbb{Z}$,

$$\det(A_{r,n}) = (-1)^{n+\lfloor (r+3)/2 \rfloor}. \tag{3.3}$$

Proof. Using elementary transformations on matrices and Lemma 2.3 we get

$$\det(A_{r,0}) = \det \begin{pmatrix} F_0^{(r)} & F_1^{(r)} & \cdots & F_{r-2}^{(r)} & F_{r-1}^{(r)} \\ F_1^{(r)} & F_2^{(r)} & \cdots & F_{r-1}^{(r)} & F_r^{(r)} \\ \vdots & \vdots & & \vdots & \vdots \\ F_{r-2}^{(r)} & F_{r-1}^{(r)} & \cdots & F_{2r-4}^{(r)} & F_{2r-3}^{(r)} \\ F_{r-1}^{(r)} & F_r^{(r)} & \cdots & F_{2r-3}^{(r)} & F_{2r-2}^{(r)} \end{pmatrix}$$

$$= -\det \begin{pmatrix} 0 & F_0^{(r)} & \cdots & F_{r-3}^{(r)} & F_{r-2}^{(r)} \\ F_0^{(r)} & F_1^{(r)} & \cdots & F_{r-2}^{(r)} & F_{r-1}^{(r)} \\ \vdots & \vdots & & \vdots & \vdots \\ F_{r-3}^{(r)} & F_{r-2}^{(r)} & \cdots & F_{2r-5}^{(r)} & F_{2r-4}^{(r)} \\ F_{r-2}^{(r)} & F_{r-1}^{(r)} & \cdots & F_{2r-4}^{(r)} & F_{2r-3}^{(r)} \end{pmatrix}$$

$$= (-1)^r \det \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & F_2^{(r)} \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \cdots & F_{r-2}^{(r)} & F_{r-1}^{(r)} \\ 1 & F_2^{(r)} & \cdots & F_{r-1}^{(r)} & F_r^{(r)} \end{pmatrix}$$

$$= (-1)^r(-1)^{\lfloor (r+2)/2 \rfloor} \det \begin{pmatrix} 1 & F_2^{(r)} & \cdots & F_{r-1}^{(r)} & F_r^{(r)} \\ 0 & 1 & \cdots & F_{r-2}^{(r)} & F_{r-1}^{(r)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & F_2^{(r)} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = (-1)^{\lfloor (r+3)/2 \rfloor}.$$

According to Theorem 2.2 we obtain

$$\begin{aligned} \det(A_{r,n}) &= \det(Q_{r+2})^n \det(A_{r,0}) = (-1)^n \det(A_{r,0}) \\ &= (-1)^n(-1)^{\lfloor (r+3)/2 \rfloor} = (-1)^{n+\lfloor (r+3)/2 \rfloor}. \ \blacksquare \end{aligned}$$

Let $M = M^{(m,n,r)}$ be a matrix with $M_{i,j} = F_{n+i+j-2}^{(r)}$, $1 \le i, j \le m$. Theorem 3.3 can be restated as

$$\det(M^{(n,r,r+2)}) = (-1)^{n+\lfloor (r+3)/2 \rfloor}.$$

Finally, let us show that for $m > r+2$,

$$\det(M^{(m,n,r)}) = 0. \tag{3.4}$$

The proof is by performing elementary transformations on $M^{(m,n,r)}$ leading to a matrix having one column consisting of zeros.

Take a look at the ith row of $M^{(m,n,r)}$:

$$[F_{n+i-1}^{(r)} \ F_{n+i}^{(r)} \ F_{n+i+1}^{(r)} \ \cdots \ F_{n+i+j-2}^{(r)} \ \cdots \ F_{n+i+m-3}^{(r)} \ F_{n+i+m-2}^{(r)}].$$

Using (2.1) and subtracting the jth element from the $(j+1)$st for $j = m-1, m-2, \ldots, 2, 1$ (thus simulating subtracting column j from column

$j+1$ in $M^{(m,n,r)}$) we get

$$[F^{(r)}_{n+i-1}\ F^{(r-1)}_{n+i}\ F^{(r-1)}_{n+i+1}\ \cdots\ F^{(r-1)}_{n+i+j-2}\ \cdots\ F^{(r-1)}_{n+i+m-3}\ F^{(r-1)}_{n+i+m-2}].$$

We can repeat the process for $j = m-1, m-2, \ldots, 2$ to get

$$[F^{(r)}_{n+i-1}\ F^{(r-1)}_{n+i}\ F^{(r-2)}_{n+i+1}\ \cdots\ F^{(r-2)}_{n+i+j-2}\ \cdots\ F^{(r-2)}_{n+i+m-3}\ F^{(r-2)}_{n+i+m-2}].$$

After repeating the process r times (for $j = m-1, m-2, \ldots, r$), we get

$$[F^{(r)}_{n+i-1}\ F^{(r-1)}_{n+i}\ F^{(r-2)}_{n+i+1}\ \cdots\ F^{(1)}_{n+i+r-2}\ F_{n+i+r-1}\ \cdots\ F_{n+i+m-2}].$$

Since $m > r+2$ we have $n+i+r-1 \le n+i+m-4$, so the above row contains

$$[\cdots F_{n+i+r-1}\ F_{n+i+r}\ F_{n+i+r+1} \cdots]$$

at positions $r-1$, r and $r+1$. Subtracting the first two elements from the third, we get

$$[\cdots F_{n+i+r-1}\ F_{n+i+r}\ 0 \cdots].$$

That way we arrive at a matrix with a column consisting of zeros, whose determinant is therefore zero.

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