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On regular Stein neighborhoods of a union of two totally real planes in $\mathbb{C}^2$

TADEJ STARČIČ (Ljubljana)

Abstract. We find regular Stein neighborhoods of a union of totally real planes $M = (A + iI)\mathbb{R}^2$ and $N = \mathbb{R}^2$ in $\mathbb{C}^2$, provided that the entries of a real 2×2 matrix A are sufficiently small. A key step in our proof is a local construction of a suitable function ρ near the origin. The sublevel sets of ρ are strongly Levi pseudoconvex and admit strong deformation retraction to $M \cup N$.

1. Introduction. The class of Stein manifolds is one of the most important classes of complex manifolds. There are many characterizations of Stein manifolds (see Remmert [16], Grauert [11] and Cartan [3]). Also many classical problems in complex analysis are solvable on Stein manifolds (see the monographs [13] and [14]). Therefore it is a very useful property for a subset of a manifold to have open Stein neighborhoods.

On the other hand, one would also like to understand the topology or the homotopy type of such neighborhoods. Moreover, approximation theorems can be obtained if neighborhoods have further suitable properties (see Chirka [4]). Interesting results in this direction for real surfaces immersed (or embedded) into a complex surface were given by Forstnerič [7, Theorem 2.2] and Slapar [17]. If $\pi : S \to X$ is a smooth immersion of a closed real surface into a complex surface with finitely many special double points and only flat hyperbolic complex points, then $\pi(S)$ has a basis of *regular* Stein neighborhoods; these are open Stein neighborhoods which admit a strong deformation retraction onto $\pi(S)$ (for the precise definition see Section 4). The problem is to find a good plurisubharmonic function locally near every double point [6, 7, 17] or hyperbolic complex point [17]. We add here that elliptic complex points prevent the surface from having a basis of Stein

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neighborhoods due to the existence of Bishop discs [2], while the surface is locally polynomially convex at hyperbolic points by a result of Forstnerič and Stout [9].

In this paper we consider a union of two totally real planes M and N in $\mathbb{C}^2$ with $M \cap N = \{0\}$. Every such union is complex-linearly equivalent to $\mathbb{R}^2 \cup M(A)$, where $M(A)$ is the real span of the columns of the matrix $A + iI$. Moreover, A is a real matrix determined up to real conjugacy and such that $A - iI$ is invertible. By a result of Weinstock [18] each compact subset of $\mathbb{R}^2 \cup M(A)$ is polynomially convex if and only if A has no purely imaginary eigenvalue of modulus greater than one. For matrices A that satisfy this condition it is then reasonable to try to find regular Stein neighborhoods for $\mathbb{R}^2 \cup M(A)$. If $A = 0$ the situation near the origin coincides with the special double point of a real surface immersed in a complex surface, as mentioned above. When A is diagonalizable over $\mathbb{R}$ with $\mathrm{Trace}(A) = 0$, a regular Stein neighborhood basis has been constructed by Slapar [17, Proposition 3].

In Section 4 we prove that regular Stein neighborhoods of $\mathbb{R}^2 \cup M(A)$ in $\mathbb{C}^2$ can be constructed if the entries of A are sufficiently small. An important step in our proof is a local construction of a suitable function ρ near the origin, depending smoothly on the entries of A. Furthermore, ρ is strictly plurisubharmonic in complex tangent directions to its sublevel sets, and such that the sublevel sets shrink down to $M \cup N$. The Levi form of ρ is a homogeneous polynomial of high degree, and it is difficult to control its sign for larger entries of A. It would also be interesting to generalize the construction to the case of a union of two totally real subspaces of maximal dimension in $\mathbb{C}^n$, though the computations of the Levi form would quickly get very lengthy and hard to handle.

Every Stein manifold of dimension n can be realized as a CW-complex of dimension at most n (see Andreotti and Frankel [1]). A natural question related to our problem is whether one can find regular Stein neighborhoods of a handlebody obtained by attaching a totally real handle to a strongly pseudoconvex domain. For results in this direction see the monograph [10] and the papers by Eliashberg [5], Forstnerič and Kozak [8] and others. We shall not consider this matter here.

2. Preliminaries. A real linear subspace in $\mathbb{C}^n$ is called *totally real* if it contains no complex subspace. It is clear that the real dimension of a totally real subspace in $\mathbb{C}^n$ is at most n.

Now let M and N be linear totally real subspaces of real dimension n in $\mathbb{C}^n$, intersecting only at the origin. The next lemma describes the basic properties of the union of such subspaces. It is well known, and not difficult to prove. We refer to [18] for the proof and a short note on linear totally real subspaces in $\mathbb{C}^n$.

LEMMA 2.1. *Let M and N be linear totally real subspaces of real dimension n in $\mathbb{C}^n$ and with $M \cap N = \{0\}$. Then there exists a non-singular complex linear transformation which maps N onto $\mathbb{R}^n \approx (\mathbb{R} \times \{0\})^n \subset \mathbb{C}^n$ and M onto $M(A) = (A + iI)\mathbb{R}^n$, where A is a matrix with real entries and such that i is not an eigenvalue of A. Moreover, any non-singular real matrix S maps $M(A) \cup \mathbb{R}^n$ onto $M(SAS^{-1}) \cup \mathbb{R}^n$.*

Our goal is to construct Stein neighborhoods of a union of totally real planes M and N in $\mathbb{C}^2$, intersecting only at the origin (see Section 4). It is easy to see that non-singular linear transformations map Stein domains onto Stein domains and totally real subspaces onto totally real subspaces. According to Lemma 2.1 the general situation thus reduces to the case $N = \mathbb{R}^2 \approx (\mathbb{R} \times \{0\})^2 \subset \mathbb{C}^2$ and $M = (A + iI)\mathbb{R}^2$, where A satisfies one of three conditions listed below. (In each case we also add an orthogonal complement $M^\perp$ to M, and the squared Euclidean distance function d_M to M in $\mathbb{C}^2 = (\mathbb{R} + i\mathbb{R})^2 \approx \mathbb{R}^4$; they are all given in corresponding real coordinates $(x, y, u, v) \approx (x + iy, u + iv) \in \mathbb{C}^2$.)

CASE 1. A is diagonalizable over $\mathbb{R}$, i.e. $A = \left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right]$, $a, d \in \mathbb{R}$,

$$\begin{aligned} M &= \operatorname{Span}\{(a,1,0,0),(0,0,d,1)\}, \\ M^\perp &= \operatorname{Span}\{(1,-a,0,0),(0,0,1,-d)\}, \\ d_M(x,y,u,v) &= \frac{(u-dv)^2}{1+d^2} + \frac{(x-ay)^2}{1+a^2}. \end{aligned} \tag{2.1}$$

CASE 2. A has complex eigenvalues (but i is not an eigenvalue), i.e. $A = \left[\begin{smallmatrix} a & -d \\ d & a \end{smallmatrix}\right]$, $a, d \in \mathbb{R}$, $d \neq 0$, $a^2 + (1-d^2)^2 \neq 0$,

$$\begin{aligned} M &= \operatorname{Span}\{(a,1,d,0),(-d,0,a,1)\}, \\ M^\perp &= \operatorname{Span}\{(0,-d,1,-a),(1,-a,0,d)\}, \\ d_M(x,y,u,v) &= \frac{(u-dy-av)^2}{1+a^2+d^2} + \frac{(x-ay+dv)^2}{1+a^2+d^2}. \end{aligned} \tag{2.2}$$

CASE 3. A is non-diagonalizable, i.e. $A = \left[\begin{smallmatrix} a & d \\ 0 & a \end{smallmatrix}\right]$, $a \in \mathbb{R}$, $d \neq 0$,

$$\begin{aligned} M &= \operatorname{Span}\{(a,1,0,0),(d,0,a,1)\}, \\ M^\perp &= \operatorname{Span}\left\{(0,0,1,-a),\left(1,-a,\frac{-ad}{1+a^2},\frac{-d}{1+a^2}\right)\right\}, \\ d_M(x,y,u,v) &= \frac{(u-av)^2}{1+a^2} + \frac{((1+a^2)(x-ay)-dau-dv)^2}{(1+a^2)((1+a^2)^2+d^2)}. \end{aligned} \tag{2.3}$$

Our construction of Stein domains involves strictly plurisubharmonic functions and strongly pseudoconvex domains. Here we recall the basic definitions and establish the notation.

Given a $\mathcal{C}^2$-function ρ on a complex manifold X, we define the *Levi form* by

$$\mathcal{L}_{(z)}(\rho;\lambda)=\langle\partial\overline{\partial}\rho(z),\lambda\wedge\bar{\lambda}\,\rangle,\qquad z\in X,\,\lambda\in T_z^{1,0}X\approx T_zX,$$

where $T_z^{1,0}X$ is the eigenspace corresponding to the eigenvalue i of the underlying almost complex structure operator J on the complexified tangent bundle $\mathbb{C}\otimes_{\mathbb{R}}TX$. In local holomorphic coordinates $z=(z_1,\dots,z_n)$ we have

$$\mathcal{L}_{(z)}(\rho;\lambda)=\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}(z)\lambda_j\overline{\lambda}_k,\qquad \lambda=\sum_{j=1}^{n}\lambda_j\frac{\partial}{\partial z_j}.$$

A function ρ is *strictly plurisubharmonic* if $\mathcal{L}_{(z)}(\rho;\cdot)$ is a positive definite Hermitian quadratic form for all $z\in X$.

Let $\rho\colon\mathbb{C}^n\to\mathbb{R}$ be a $\mathcal{C}^2$ defining function for a domain $\Omega\subset\mathbb{C}^n$, i.e. $\Omega=\{z\in\mathbb{C}^n\colon\rho(z)<c\}$ and $b\Omega=\{z\in\mathbb{C}^n\colon\rho(z)=c\}$ for some $c\in\mathbb{R}$. If also $d\rho(z)\neq 0$ for every $z\in b\Omega$, we say that Ω has *$\mathcal{C}^2$-boundary.*

A domain $\Omega\subset\mathbb{C}^n$ is *strongly Levi pseudoconvex* if for every $z\in b\Omega$ the Levi form of ρ is positive in all complex tangent directions to $b\Omega$:

$$\mathcal{L}_{(z)}(\rho;\lambda)>0,\qquad z\in b\Omega,\,\lambda\in T_z^{\mathbb{C}}(b\Omega):=T_z(b\Omega)\cap iT_z(b\Omega).$$

If ρ strictly plurisubharmonic in a neighborhood of $b\Omega$, the domain Ω is said to be *strongly pseudoconvex.*

Throughout this paper (z_1,z_2) will be the local holomorphic coordinates and (x,y,u,v) the corresponding real coordinates on $\mathbb{C}^2$ with $z_1=x+iy$ and $z_2=u+iv$. Holomorphic and antiholomorphic derivatives are denoted by $\frac{\partial}{\partial z_1}=\frac{1}{2}\big(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\big)$, $\frac{\partial}{\partial\bar{z}_1}=\frac{1}{2}\big(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\big)$ or briefly by $\frac{\partial\rho}{\partial z_1}=\rho_{z_1}$, $\frac{\partial\rho}{\partial\bar{z}_1}=\rho_{\bar{z}_1}$, and the same for $\frac{\partial}{\partial z_2}$, $\frac{\partial}{\partial\bar{z}_2}$.

If ρ defines a domain $\Omega\subset\mathbb{C}^2$, we have

$$T_z^{\mathbb{C}}(b\Omega)=\Big\{(w_1,w_2)\colon\frac{\partial\rho}{\partial z_1}(z)\,w_1+\frac{\partial\rho}{\partial z_2}(z)\,w_2=0\Big\},$$

and we consider the following vector in the complex tangent direction to $b\Omega$:

$$\lambda_\rho=\left(\frac{\partial\rho}{\partial z_2},-\frac{\partial\rho}{\partial z_1}\right)\in T^{\mathbb{C}}(b\Omega).\tag{2.4}$$

A straightforward calculation then gives

$$\begin{aligned}\mathcal{L}(\rho;\lambda_\rho)&=\rho_{z_1\bar{z}_1}\rho_{z_2}\overline{\rho}_{z_2}+\rho_{z_2\bar{z}_2}\rho_{z_1}\overline{\rho}_{z_1}-\rho_{z_2\bar{z}_1}\rho_{z_1}\overline{\rho}_{z_2}-\rho_{z_1\bar{z}_2}\rho_{z_2}\overline{\rho}_{z_1}\\&=\rho_{z_1\bar{z}_1}|\rho_{z_2}|^2+\rho_{z_2\bar{z}_2}|\rho_{z_1}|^2-2\,\mathrm{Re}(\rho_{z_2\bar{z}_1}\rho_{z_1}\overline{\rho}_{z_2}).\end{aligned}\tag{2.5}$$

In terms of real partial derivatives, we have

$$\begin{aligned}\mathcal{L}(\rho;\lambda_\rho) &= \frac{1}{16}\left(\frac{\partial^2\rho}{\partial x^2}+\frac{\partial^2\rho}{\partial y^2}\right)\left(\left(\frac{\partial\rho}{\partial u}\right)^2+\left(\frac{\partial\rho}{\partial v}\right)^2\right)\\ &+\frac{1}{16}\left(\frac{\partial^2\rho}{\partial u^2}+\frac{\partial^2\rho}{\partial v^2}\right)\left(\left(\frac{\partial\rho}{\partial x}\right)^2+\left(\frac{\partial\rho}{\partial y}\right)^2\right)\\ &-\frac{1}{8}\left(\frac{\partial^2\rho}{\partial x\partial u}+\frac{\partial^2\rho}{\partial y\partial v}\right)\left(\frac{\partial\rho}{\partial x}\frac{\partial\rho}{\partial u}+\frac{\partial\rho}{\partial y}\frac{\partial\rho}{\partial v}\right)\\ &+\frac{1}{8}\left(-\frac{\partial^2\rho}{\partial x\partial v}+\frac{\partial^2\rho}{\partial y\partial u}\right)\left(\frac{\partial\rho}{\partial v}\frac{\partial\rho}{\partial x}-\frac{\partial\rho}{\partial y}\frac{\partial\rho}{\partial u}\right).\end{aligned}\tag{2.6}$$

3. Local construction at the intersection. In this section we give a local construction of regular Stein neighborhoods near $M\cap N=\{0\}$ of the union of two totally real planes $M,N\subset\mathbb{C}^2$. Our goal is to find a function $\rho\colon\mathbb{C}^n\to\mathbb{R}$ satisfying the following properties:

(1) $M\cup N=\{\rho=0\}=\{\nabla\rho=0\}$,
(2) $\Omega_\epsilon=\{\rho<\epsilon\}$ is strongly Levi pseudoconvex for any sufficiently small $\epsilon>0$.

Observe that in this case the flow of the negative gradient vector field $-\nabla\rho$ gives a strong deformation retraction of Ω_ϵ onto $M\cup N$.

To fulfil conditions (1) and (2) one might take linear combinations of products of squared Euclidean distance functions to M and N in $\mathbb{C}^2$ respectively. However, the Levi form of such a function would be a polynomial of high degree, and therefore difficult to control. To simplify the situation we prefer to work with homogeneous polynomials. The following lemma is a preparation for our key result, Lemma 3.3.

Lemma 3.1. *Let A, M and d_M be as in* (2.1), (2.2) *or* (2.3), *and let $N=\mathbb{R}^2$ with $d_N(x,y,u,v)=y^2+v^2$. Then the function*

$$\rho=d_M^{\alpha+1}d_N^{\beta}+d_M^{\alpha}d_N^{\beta+1},\qquad \alpha,\beta\ge 1,$$

has the following properties:

(1) *$M\cup N=\{\rho=0\}=\{\nabla\rho=0\}$.*
(2) *There exist constants $r,\epsilon_0>0$ such that ρ is strictly plurisubharmonic on $(\{d_M<\epsilon_0\}\cup\{d_N<\epsilon_0\})\setminus(M\cup N\cup\overline{B}_r)$, where B_r is the ball centered at 0 and with radius r. In addition, for $\alpha=\beta=1$ the Levi form of ρ is positive on a neighborhood of $(M\cup N)\setminus\{0\}$, and for $\alpha,\beta\ge 2$ it vanishes on $M\cup N$.*
(3) *For any $\epsilon>0$ and $\Omega_\epsilon=\{\rho<\epsilon\}$ the Levi form of ρ in the complex tangent direction to the boundary $b\Omega_\epsilon$ is*

$$\mathcal{L}(\rho;\lambda_\rho)=\frac{1}{k}\,d_M^{3\alpha-2}d_N^{3\beta-2}P,\qquad \lambda_\rho\in T^{\mathbb{C}}(b\Omega_\epsilon),$$

where k is a positive polynomial in the entries of A, and P is a homogeneous polynomial of degree 10 in the variables x,y,u,v and with coefficients depending polynomially on the entries of A.

Proof. Property (1) is an immediate consequence of the definition of ρ.

Next, we fix $m,n\geq 1$, and for any $\lambda=\sum_{j=1}^{2}\lambda_j\frac{\partial}{\partial z_j}\in T(\mathbb{C}^2)$ we obtain

$$\begin{aligned}\mathcal{L}(d_M^m d_N^n;\lambda)&=md_M^{m-1}d_N^n\mathcal{L}(d_M;\lambda)+(m-1)md_M^{m-2}d_N^n\Big|\sum_{j=1}^{2}\frac{\partial d_M}{\partial z_j}\lambda_j\Big|^2\\&\quad+2mnd_N^{n-1}d_M^{m-1}\operatorname{Re}\Big(\Big(\sum_{j=1}^{2}\frac{\partial d_M}{\partial z_j}\lambda_j\Big)\Big(\sum_{j=1}^{2}\frac{\partial d_N}{\partial \overline{z}_j}\overline{\lambda}_j\Big)\Big)\\&\quad+nd_N^{n-1}d_M^m\mathcal{L}(d_N;\lambda)+(n-1)nd_N^{n-2}d_M^m\Big|\sum_{j=1}^{2}\frac{\partial d_N}{\partial z_j}\lambda_j\Big|^2.\end{aligned}\tag{3.1}$$

It is well known and easy to check that the functions d_M and d_N are strictly plurisubharmonic. Moreover, there exists a constant $c>0$ such that

$$\mathcal{L}(d_M;\lambda)\geq c|\lambda|^2,\quad \mathcal{L}(d_N;\lambda)\geq c|\lambda|^2,\qquad \lambda\in T(\mathbb{C}^2).$$

For some constant $b>0$ we also have

$$\Big|\Big(\sum_{j=1}^{2}\frac{\partial d_M}{\partial z_j}\lambda_j\Big)\Big(\sum_{j=1}^{2}\frac{\partial d_N}{\partial \overline{z}_j}\overline{\lambda}_j\Big)\Big|\leq b\sqrt{d_N d_M}\,|\lambda|^2,\qquad \lambda\in T(\mathbb{C}^2).$$

Therefore, if we are sufficiently far away from N and close enough to M, but not on M, the term $md_M^{m-1}d_N^n\mathcal{L}(d_M;\lambda)$ in (3.1) will dominate the third term in (3.1), and will thus make $\mathcal{L}(d_M^m d_N^n;\lambda)$ positive there, for all λ. Similarly, the term $nd_N^{n-1}d_M^m\mathcal{L}(d_N;\lambda)$ makes $\mathcal{L}(d_M^m d_N^n;\lambda)$ positive, provided that we are far away from M and close to N, but not on N. Hence $\rho=d_M^{\alpha+1}d_N^{\beta}+d_M^{\alpha}d_N^{\beta+1}$ satisfies the first part of (2). Clearly, since ∇d_M vanishes on M and ∇d_N vanishes on N, the Levi form of ρ is positive on $(M\cup N)\setminus\{0\}$ for $\alpha=\beta=1$, and vanishes on $M\cup N$ for $\alpha,\beta\geq 2$. This concludes the proof of (2).

To prove (3) we need to factor $\mathcal{L}(\rho;\lambda_\rho)$ (see (2.5)) into a product of $d_M^{3\alpha-2}d_N^{3\beta-2}$ and a polynomial in x,y,u,v with coefficients depending on the entries of A. Here we have

$$\begin{aligned}\lambda_\rho&=\big((\alpha+1)d_N^{\beta}d_M^{\alpha}+\alpha d_N^{\beta+1}d_M^{\alpha-1}\big)\lambda_{d_M}\\&\quad+\big((\beta+1)d_N^{\beta}d_M^{\alpha}+\beta d_N^{\beta-1}d_M^{\alpha+1}\big)\lambda_{d_N}.\end{aligned}\tag{3.2}$$

Firstly, since $\sum_{j=1}^{2}\frac{\partial d_M}{\partial z_j}\lambda_{d_M j}=0$ and $\sum_{j=1}^{2}\frac{\partial d_N}{\partial z_j}\lambda_{d_N j}=0$, we can clearly factor $\sum_{j=1}^{2}\frac{\partial d_M}{\partial z_j}\lambda_{\rho j}$ and $\sum_{j=1}^{2}\frac{\partial d_N}{\partial z_j}\lambda_{\rho j}$ respectively into a product of d_M^{α} or d_N^{β} and a polynomial in x,y,u,v.

Next, we observe that $d_M^{2\alpha-2}$ (respectively $d_N^{2\beta-2}$) factors out of $\mathcal{L}(d_M;\lambda_\rho)$ (respectively $\mathcal{L}(d_N;\lambda_\rho)$), trivially. An easy and straightforward computation by using (2.6) shows further that $\mathcal{L}(d_N;\lambda_{d_N}) = \frac{1}{2}d_N$, while if d_M is as in case (2.1), (2.2) or (2.3), we obtain, respectively,

$$\mathcal{L}(d_M;\lambda_{d_M}) = \begin{cases} \frac{1}{2}d_M, \\ \dfrac{((1+a^2+d^2)^2-4d^2)}{2(1+a^2+d^2)^2}d_M, \\ \dfrac{(1+a^2)^2}{2((1+a^2)^2+d^2)}d_M. \end{cases}$$

Hence $d_M^{2\alpha-1}$ and $d_N^{2\beta-1}$ factor out of $\mathcal{L}(d_M;\lambda_\rho)$ and $\mathcal{L}(d_N;\lambda_\rho)$, respectively.

From (3.1) applied for $m=\alpha+1$, $n=\beta$ and $m=\alpha$, $n=\beta+1$ it now follows immediately that $\mathcal{L}(\rho;\lambda_\rho)$ factors into a product of $d_M^{3\alpha-2}d_N^{3\beta-2}$ and a polynomial in x,y,u,v. However, there are terms of $\mathcal{L}(\rho;\lambda_\rho)$ which include $d_M^{3\alpha+3}$ as a factor. For d_M as in case (2.1) we then see that $\mathcal{L}(\rho;\lambda_\rho)$ is of the form

$$\mathcal{L}(\rho;\lambda_\rho) = \frac{1}{(1+a^2)^5(1+d^2)^5}d_M^{3\alpha-2}d_N^{3\beta-2}P, \tag{3.3}$$

where P is a homogeneous polynomial of degree 10 in x,y,u,v and the coefficients of P are polynomials in a, d. If d_M is of the form (2.2) or (2.3), we have

$$\mathcal{L}(\rho;\lambda_\rho) = \frac{1}{(1+a^2+d^2)^5}d_M^{3\alpha-2}d_N^{3\beta-2}P \tag{3.4}$$

and

$$\mathcal{L}(\rho;\lambda_\rho) = \frac{1}{(1+a^2)^5((1+a^2)^2+d^2)^5}d_M^{3\alpha-2}d_N^{3\beta-2}P, \tag{3.5}$$

respectively, where P again has all the properties required. ∎

We note here that by choosing suitable substitutions, it is possible to compute the polynomial P in Lemma 3.1(3) explicitly, but on the other hand this might involve very long expansions of polynomials. (See also the proof of Lemma 3.3 for this approach in the special case $A=0$.)

Before stating our key lemma we prove the following result on homogeneous polynomials.

Lemma 3.2. *Let $Q,R\in\mathbb{R}[x_1,\dots,x_m]$ be real homogeneous polynomials in m variables and of even degree s. Assume further that Q vanishes at the origin and is positive elsewhere. Then $Q\ge\epsilon_0|R|$ for any sufficiently small constant $\epsilon_0>0$ with equality precisely at the origin.*

Proof. We denote by $\|x\| = \sqrt{x_1^2+\cdots+x_m^2}$ the standard Euclidean norm of $x=(x_1,\dots,x_m)\in\mathbb{R}^m$.

Since Q vanishes at the origin and is positive elsewhere, there exists a constant $c > 0$ such that $Q(x) \geq c$ for all x on the unit sphere, i.e. $\|x\| = 1$. Also, there exists a constant $C > 0$ such that $|R(x)| \leq C$ for any x on the unit sphere.

However, homogeneous polynomials are uniquely determined by their values on the unit sphere. Thus $Q(x) \geq c\|x\|^s$ and $|R(x)| \leq C\|x\|^s$ for any x, and with equalities precisely at the origin. The conclusion follows. ∎

The following lemma is essential in the proof of Theorem 4.1, where we construct Stein neighborhoods.

LEMMA 3.3. *Let A, M, d_M, N and d_N be as in Lemma* 3.1, *and let*

$$\rho = d_M^2 d_N + d_M d_N^2.$$

If the entries of A are sufficiently close to zero, then for any $\epsilon > 0$ the sublevel set $\Omega_\epsilon = \{\rho < \epsilon\}$ is strongly Levi pseudoconvex.

Proof. By Lemma 3.1 the Levi form of $\rho = d_M^2 d_N + d_M d_N^2$ in the complex tangent direction λ_ρ (see (2.4)) to the boundary of $\Omega_\epsilon = \{\rho < \epsilon\}$ is

$$\mathcal{L}(\rho;\lambda_\rho) = \frac{1}{k}\, d_M d_N P, \qquad \lambda_\rho \in T^{\mathbb{C}}(b\Omega_\epsilon), \tag{3.6}$$

where k is a positive polynomial in a, d (see (3.3), (3.4) or (3.5)), and P is a homogeneous polynomial of degree 10 in x, y, u, v. Furthermore, the coefficients of P are polynomials in a, d; these are the entries of A (see (2.1), (2.2) or (2.3)).

We now write P as a sum of two polynomials in x, y, u, v:

$$P = Q + R, \tag{3.7}$$

where the coefficients of Q do not depend on a or d, and the coefficients of R are polynomials in a, d without constant term.

Observe further that for $a = d = 0$ the Levi form in (3.6) is equal to the product $(x^2+u^2)(y^2+v^2)\,Q$. On the other hand, it is precisely the Levi form of the function

$$\rho_0(x,y,u,v) = (x^2+u^2)^2(v^2+y^2) + (x^2+u^2)(v^2+y^2)^2$$

in the complex tangent direction λ_{ρ_0} to the boundary of its sublevel set, which means that

$$\mathcal{L}(\rho_0;\lambda_{\rho_0}) = (x^2+u^2)(y^2+v^2)Q. \tag{3.8}$$

To simplify the computation of the Levi form of ρ_0 by using (2.5) and (3.1), we now replace certain expressions by new variables. We introduce the notation

$$V = v^2 + y^2, \qquad Z = u^2 + x^2, \qquad \omega = V + Z, \tag{3.9}$$

and we apply (3.1) for $d_M = Z$, $d_N = V$ in the cases $m = 2$, $n = 1$ and $m = 1$, $n = 2$. After adding the expressions obtained and slightly regrouping like terms, we get

$$\begin{aligned} \mathcal{L}(\rho_0;\lambda) &= (2ZV + V^2)\mathcal{L}(Z;\lambda) + (Z^2 + 2VZ)\mathcal{L}(V;\lambda) \\ &\quad + (4Z + 4V)\,\mathrm{Re}\bigg(\Big(\sum_{j=1}^{2} \frac{\partial Z}{\partial z_j}\lambda_j\Big)\Big(\sum_{j=1}^{2} \frac{\partial V}{\partial \overline{z}_j}\overline{\lambda}_j\Big)\bigg) \\ &\quad + 2V\bigg|\sum_{j=1}^{2} \frac{\partial Z}{\partial z_j}\lambda_j\bigg|^2 + 2Z\bigg|\sum_{j=1}^{2} \frac{\partial V}{\partial z_j}\lambda_j\bigg|^2. \end{aligned} \tag{3.10}$$

Next, observe that

$$\frac{\partial Z}{\partial z_1} = x, \qquad \frac{\partial Z}{\partial z_2} = u, \qquad \frac{\partial V}{\partial z_1} = -iy, \qquad \frac{\partial V}{\partial z_2} = -iv,$$

and by (2.4),

$$\lambda_Z = (u, -x), \qquad \lambda_V = (-iv, iy). \tag{3.11}$$

By taking $\alpha = \beta = 1$ and $d_M = Z$, $d_N = V$ in (3.2), we further deduce that

$$\lambda_{\rho_0} = (Z + \omega)V\lambda_Z + (V + \omega)Z\lambda_V. \tag{3.12}$$

An easy computation gives

$$\mathcal{L}(V;\lambda) = \mathcal{L}(Z;\lambda) = \tfrac{1}{2}|\lambda|^2, \qquad \lambda \in T(\mathbb{C}^2). \tag{3.13}$$

By combining (3.9), (3.11), (3.12), (3.13), and by regrouping terms, we now get

$$\begin{aligned} \mathcal{L}(V;\lambda_{\rho_0}) &= \tfrac{1}{2}\big((Z+\omega)^2V^2(u^2+x^2) + (V+\omega)^2Z^2(v^2+y^2)\big) \\ &= \tfrac{1}{2}\big((Z+\omega)^2V^2Z + (V+\omega)^2Z^2V\big) \\ &= \tfrac{1}{2}VZ\big(VZ(Z+V) + 4\omega VZ + \omega^2(V+Z)\big) \\ &= \tfrac{1}{2}VZ\omega(5VZ + \omega^2). \end{aligned} \tag{3.14}$$

It is also easy to calculate

$$\sum_{j=1}^{2} \frac{\partial Z}{\partial z_j}\lambda_{\rho_0 j} = -i(V+\omega)Z\Delta, \qquad \sum_{j=1}^{2} \frac{\partial V}{\partial z_j}\lambda_{\rho_0 j} = i(Z+\omega)V\Delta, \tag{3.15}$$

where $\Delta = xv - uy$. By using (3.9) and (3.15) we can regroup and simplify the sum of the last three terms in (3.10). We obtain

$$\begin{aligned} &-4\omega(V+\omega)(Z+\omega)VZ\Delta^2 + 2V((V+\omega)Z\Delta)^2 + 2Z((Z+\omega)V\Delta)^2 \\ &\quad = -2VZ\Delta^2\big(2\omega(V+\omega)(Z+\omega) - (Z(V+\omega)^2 + V(Z+\omega)^2)\big) \\ &\quad = -2VZ\Delta^2\big(2\omega(VZ + \omega(V+Z) + \omega^2) - ((ZV+\omega^2)(V+Z) + 4\omega VZ)\big) \\ &\quad = -2VZ\Delta^2\big(2\omega(VZ + 2\omega^2) - (5\omega VZ + \omega^3)\big) = -6VZ\omega\Delta^2(\omega^2 - VZ). \end{aligned}$$

Finally, we have

$$\begin{aligned}\mathcal{L}(\rho_0;\lambda_{\rho_0}) &= \tfrac{1}{2}VZ\omega(5VZ+\omega^2)(4ZV+V^2+Z^2)-6VZ\omega\Delta^2\left(\omega^2-VZ\right)\\ &= \tfrac{1}{2}VZ\omega\big((5VZ+\omega^2)(2ZV+\omega^2)-12\Delta^2(\omega^2-VZ)\big).\end{aligned}$$

After replacing ω, V, Z, Δ in the above expression back by the variables x, y, u, v, and comparing them to (3.8), we obtain the factorization

$$Q(x,y,u,v)=\tfrac{1}{2}(x^2+y^2+u^2+v^2)P_0(x,y,u,v), \tag{3.16}$$

where P_0 is a homogeneous polynomial of degree 8 in x, y, u, v.

Next, we examine the sign of the polynomial P_0. We use the Cauchy–Schwarz inequality

$$\Delta^2=(xv-yu)^2\le(x^2+u^2)(y^2+v^2)=VZ$$

to see that

$$\begin{aligned}P_0 &= (5VZ+\omega^2)(2ZV+\omega^2)-12\Delta^2\left(\omega^2-VZ\right)\\ &\ge 22(VZ)^2-5(VZ)\omega^2+\omega^4\ge 22\big(VZ-\tfrac{5}{44}\omega^2\big)^2+\tfrac{63}{88}\omega^4.\end{aligned}$$

This proves that P_0 and hence also Q (see (3.16)) vanish at the origin and are positive everywhere else. Moreover,

$$P_0(x,y,u,v)\ge\tfrac{63}{88}(x^2+y^2+u^2+v^2)^4. \tag{3.17}$$

We now show that the polynomial P in (3.7) vanishes at the origin and is positive elsewhere, provided that the entries a, d of A are chosen sufficiently small. Recall that the polynomial R (see (3.7)) is of the form

$$R(x,y,u,v)=\sum_{|\alpha|=10}S_\alpha(a,d)x^{\alpha_1}y^{\alpha_2}u^{\alpha_3}v^{\alpha_4}, \tag{3.18}$$

where $\alpha=(\alpha_1,\ldots,\alpha_4)$ is a multiindex, and S_α is a polynomial in a, d. Remember also that all S_α are without constant terms and hence $S_\alpha(0,0) = 0$. We denote by N_0 the number of terms of R. Since Q is a homogeneous polynomial of degree 10 (see (3.16)), which is positive everywhere except at the origin, we can use Lemma 3.2 to get a constant $\epsilon_0>0$ such that

$$\frac{1}{N_0}Q\ge\epsilon_0|x^{\alpha_1}y^{\alpha_2}u^{\alpha_3}v^{\alpha_4}|,\qquad \alpha=(\alpha_1,\ldots,\alpha_4),\ |\alpha|=10, \tag{3.19}$$

where equality holds precisely at the origin. By a continuity argument, we also have $|S_\alpha(a,d)|<\epsilon_0$ for all a, d small enough, and uniformly for all coefficients S_α of R. It then follows from (3.19) that for all sufficiently small a and d, we have $Q\ge|R|$, with equality precisely at the origin. This implies that P vanishes at the origin and is positive elsewhere. Finally, the Levi form of ρ (see (3.6)) is then positive in the complex tangent direction to $b\Omega_\epsilon$ for any ϵ. ∎

Remark 3.4. By analyzing the part of the proof of Lemma 3.3 where Lemma 3.2 was applied, we can tell how small the entries of the matrix A in the assumption of Lemma 3.3 can be. By combining (3.16) and (3.17) we see that

$$Q(x,y,u,v) \geq \tfrac{63}{176}(x^2+y^2+u^2+v^2)^5. \tag{3.20}$$

As we expect the entries a, d of A to be smaller than one, we can roughly estimate the coefficients S_α of R (see (3.7) and (3.18)) by $|S_\alpha(a,d)| \leq N_\alpha \max\{|a|,|d|\}$, where N_α denotes the sum of the moduli of the coefficients of S_α. Thus

$$N_1\, N_0 \max\{|a|,|d|\}(x^{10}+y^{10}+u^{10}+v^{10}) \geq |R(x,y,u,v)|, \tag{3.21}$$

where $N_1 = \max_{|\alpha|=10} N_\alpha$ and N_0 is the number of terms of R. It follows from (3.20) and (3.21) that for any $|a|,|d| < \frac{63}{176 N_0 N_1}$ we have $Q \geq |R|$, with equality precisely at the origin.

Remark 3.5. The conclusion of Lemma 3.3 holds, for instance, also for the function $d_M^2 d_N^2 + d_M d_N^3$. One might expect to prove even more. But on the other hand, it is not clear at the moment how that would improve the conclusion of the lemma for larger entries of A.

4. Regular Stein neighborhoods of the union of totally real planes. A system $\{\Omega_\epsilon\}_{\epsilon\in(0,1)}$ of open Stein neighborhoods of a set S in a complex manifold X is called *regular* if for every $\epsilon \in (0,1)$ we have

(1) $\Omega_\epsilon = \bigcup_{t<\epsilon} \Omega_t$, $\overline{\Omega}_\epsilon = \bigcap_{t>\epsilon} \Omega_t$,
(2) $S = \bigcap_{\epsilon\in(0,1)} \Omega_\epsilon$ is a strong deformation retract of Ω_ϵ.

Theorem 4.1. *Let A be a real 2×2 matrix such that $A - iI$ is invertible. Further, let $M = (A+iI)\mathbb{R}^2$ and $N = \mathbb{R}^2$. If the entries of A are sufficiently small, then $M \cup N$ has a regular system of strongly pseudoconvex Stein neighborhoods in $\mathbb{C}^2$. Moreover, away from the origin the neighborhoods coincide with sublevel sets of the squared Euclidean distance functions to M and N, respectively.*

As noted in Section 2, the general case of the union of two totally real planes intersecting at the origin reduces to the situation described in Theorem 4.1. Furthermore, we may assume that M is as in one of the three cases (2.1), (2.2) or (2.3).

Proof of Theorem 4.1. Lemma 3.3 furnishes a function $\rho = d_M^2 d_N + d_M d_N^2$, where d_M and d_N respectively are squared Euclidean distance functions to M and N in $\mathbb{C}^2$. For any $\epsilon > 0$, the sublevel set $\Omega_\epsilon = \{\rho < \epsilon\}$ is strongly Levi pseudoconvex. Also, the Levi form of ρ is positive on $(M \cup N) \setminus \{0\}$ and we have $\{\rho = 0\} = \{\nabla\rho = 0\} = M \cup N$ (see Lemma 3.1).

We proceed by patching ρ away from the origin with the squared distance functions. First, we choose open balls B_r and B_{2r} centered at 0 and with radii r and $2r$. Next, for every $\epsilon > 0$ we set

$$T_{\epsilon,M} = \{z \in \mathbb{C}^2 \setminus \overline{B}_r \colon d_M(z) < \epsilon\}, \quad T_{\epsilon,N} = \{z \in \mathbb{C}^2 \setminus \overline{B}_r \colon d_N(z) < \epsilon\}$$

and observe that for ϵ small enough the union $T_\epsilon = T_{\epsilon,M} \cup T_{\epsilon,N}$ is disjoint. We now glue the function ρ on B_{2r} with the restrictions $\rho_M = d_M|_{T_{\epsilon,M}}$ and $\rho_N = d_N|_{T_{\epsilon,N}}$:

$$\rho_0(z) = \theta(z)\rho(z) + (1-\theta(z))\rho_M(z) + (1-\theta(z))\rho_N(z), \quad z \in B_{2r} \cup T_\epsilon.$$

Here θ is a smooth cut-off function which is supported on B_{2r} and equals one on B_r. More precisely, $\theta = \chi(|z_1|^2 + |z_2|^2)$, where χ is another cut-off function with $\chi(t) = 1$ for $t \leq r$, and $\chi(t) = 0$ for $t \geq 2r$. Observe that ρ_0 coincides with ρ on $\overline{B}_r$, and with d_M or d_N on $T_{\epsilon,M} \setminus B_{2r}$ and $T_{\epsilon,N} \setminus B_{2r}$ respectively.

It is immediate that $\{\rho_0 = 0\} = M \cup N$ and that $\nabla\rho_0$ vanishes on $M \cup N$. On $(B_{2r} \setminus \overline{B}_r) \setminus (M \cup N)$, but close to $M \cup N$, we have $\nabla\theta$ close to tangent directions to $M \cup N$, and $\nabla\rho_M$ and $\nabla\rho_N$ are close to normal directions to M and N respectively. After possibly choosing ϵ smaller and shrinking T_ϵ, we get $\{\nabla\rho_0 = 0\} = M \cup N$. Finally, if ϵ is sufficiently small, the flow of the negative gradient vector field $-\nabla\rho_0$ gives us a deformation retraction of $\Omega_\epsilon = \{\rho_0 < \epsilon\}$ onto $M \cup N$.

It remains to verify that Ω_ϵ is indeed Stein, provided that ϵ is chosen small enough. Since ρ, d_M, d_N and their gradients all vanish on $M \cup N$, for $z \in M \cup N$ and any $\lambda \in T_z(\mathbb{C}^2)$ we have

$$\mathcal{L}_{(z)}(\rho_0;\lambda) = \theta(z)\mathcal{L}_{(z)}(\rho;\lambda) + (1-\theta(z))\mathcal{L}_{(z)}(\rho_M;\lambda) + (1-\theta(z))\mathcal{L}_{(z)}(\rho_N;\lambda).$$

The Levi form of ρ_0 is thus positive on $(M \cup N) \setminus \{0\}$. By choosing ϵ small enough, it is then positive on $\Omega_\epsilon \setminus B_r$. Furthermore, as ρ_0 coincides with ρ on B_r, the Levi form of ρ_0 is positive in the complex tangent direction to $b\Omega_\epsilon$ (by Lemma 3.3).

We now use a standard argument to get a strictly plurisubharmonic function in all directions also on $b\Omega_\epsilon \cap B_r$. We set a new defining function for Ω_ϵ:

$$\tilde{\rho}(z) = (\rho_0(z) - \epsilon)e^{C(\rho_0(z)-\epsilon)}, \tag{4.1}$$

where C is a large constant (to be chosen). By computation we get

$$\mathcal{L}_{(z)}(\tilde{\rho};\lambda) = \mathcal{L}_{(z)}(\rho_0;\lambda) + 2C\left|\sum_{j=1}^{2} \frac{\partial\rho_0}{\partial z_j}(z)\lambda_j\right|^2, \quad z \in b\Omega_\epsilon,\ \lambda \in T_z(\mathbb{C}^2).$$

After taking C large enough the Levi form of $\tilde{\rho}$ becomes positive in all directions on $b\Omega_\epsilon$. This proves strong pseudoconvexity of Ω_ϵ. Since the restrictions of plurisubharmonic functions to analytic sets are plurisubharmonic and satisfy the maximum principle (see [12]), we cannot have any compact analytic subset of positive dimension in $\mathbb{C}^2$. As $\Omega_\epsilon \subset \mathbb{C}^2$ is strongly pseudoconvex, it is Stein by a result of Grauert (see [11, Proposition 5]). ■

Remark 4.2. The assumption of taking the entries of A in Theorem 4.1 sufficiently small is essential, and enables the application of Lemma 3.3; see Remark 3.4 for an estimate of how small the entries of A can be.

Lemma 3.3 can also be applied to give an extension of a result on certain closed real surfaces immersed into a complex surface ([7, Theorem 2.2] and [17, Theorem 2]).

Proposition 4.3. *Let $\pi\colon S \to X$ be a smooth immersion of a closed real surface into a Stein surface satisfying the following properties:*

(1) *π has only transverse double points (no multiple points) $p_1, \ldots, p_k$, and in a neighborhood of each point p_j ($j \in \{1, \ldots, k\}$) there exist holomorphic coordinates $\psi_j\colon U_j \to V_j \subset \mathbb{C}^2$ such that $\psi_j(p_j) = 0$, $\psi_j(\tilde{S} \cap U_j) = (\mathbb{R}^2 \cup M_j) \cap V_j$, where $\tilde{S} = \pi(S)$ and $M_j = (A_j + iI)\mathbb{R}^2$ with $A_j - iI$ invertible.*
(2) *π has finitely many complex points $p_{k+1}, \ldots, p_m$, which are flat hyperbolic.*

If the entries of A_j for every $j \in \{1, \ldots, k\}$ are sufficiently close to zero, then $\tilde{S}$ has a regular strongly pseudoconvex Stein neighborhood basis in X.

The proofs given in [7, Theorem 2.2] and [17, Theorem 2] apply mutatis mutandis to our situation. For completeness we include a sketch.

Proof of Proposition 4.3. By Lemma 3.3 for every $j \in \{1, \ldots, k\}$ there exists a smooth non-negative function $\rho_j\colon V_j \to \mathbb{R}$ which is strictly plurisubharmonic away from the origin and whose sublevel sets $\{\rho_j < \epsilon\}$ are strongly Levi pseudoconvex. Furthermore, $\{\rho_j = 0\} = \{\nabla\rho_j = 0\} = (\mathbb{R}^2 \cup M_j) \cap V_j$ (see also Lemma 3.1). Next we set $\varphi_j = \rho_j \circ \psi_j\colon U_j \to \mathbb{R}$ and observe that φ_j inherits the above properties from ρ_j.

By [17, Lemma 8] for every $j \in \{k+1, \ldots, m\}$ there exists a small neighborhood U_j of p_j and a smooth non-negative function $\varphi\colon U_j \to \mathbb{R}$ which is strictly plurisubharmonic on $U_j \setminus \{p_j\}$ and such that $\{\varphi_j = 0\} = \{\nabla\varphi_j = 0\} = \tilde{S} \cap U_j$.

Further, let $\varphi_0 = d_{\tilde{S}}$ and d_p be the squared distance functions to $\tilde{S}$ and to $p \in \tilde{S}$ respectively in X, relative to some Riemannian metric on X. It is well known that the squared distance function to a smooth totally real submanifold is strictly plurisubharmonic in a neighborhood of the submanifold (see e.g. [17, Proposition 2] or [15, Proposition 4.1]). Therefore φ_0 is strictly plurisubharmonic in some open neighborhood U_0 of $\tilde{S} \setminus \{p_1, \ldots, p_m\}$.

We now patch the functions φ_j for all $j \in \{0, 1, \dots, m\}$. First, we denote $U = \bigcup_{j=0}^{m} U_j$ and let $r\colon U \to \tilde{S}$ be defined as $r(z) = p$ if $d_{\tilde{S}}(z) = d_p(z)$. The map r is well defined and smooth, provided that the sets U_j are chosen small enough. Next, we choose a partition of unity $\{\theta_j\}_{0 \le j \le m}$ subordinated to $\{U_j \cap \tilde{S}\}_{0 \le j \le m}$, and such that for every $j \in \{1, \dots, m\}$ the function θ_j equals one near p_j. Finally, we define

$$\rho(z) = \sum_{j=0}^{m} \theta_j(r(z))\varphi_j(z), \qquad z \in U.$$

We see that $\tilde{S} = \{\rho = 0\}$ and $\nabla\rho(z) = \sum_{j=0}^{m} \theta_j(r(z))\nabla\varphi_j(z)$ for all $z \in U$, thus we further have

$$\mathcal{L}_{(p)}(\rho; \lambda) = \sum_{j=0}^{m} \theta_j(p)\mathcal{L}_{(p)}(\varphi_j; \lambda), \qquad p \in \tilde{S},\ \lambda \in T_p(U).$$

After shrinking U we deduce that $\{\nabla\rho = 0\} = \tilde{S}$ and ρ is strictly plurisubharmonic away from the points $p_1, \dots, p_m$.

It remains to show that $\Omega_\epsilon = \{\rho < \epsilon\}$ are Stein domains. Since ρ coincides with φ_j near p_j for every $j \in \{1, \dots, m\}$, each Ω_ϵ is strongly Levi pseudoconvex near p_j. For a given ϵ we can, much as in the proof of Theorem 4.1 (see (4.1)), choose a positive constant C such that $\tilde{\rho}(z) = \big(\rho(z) - \epsilon\big)e^{C(\rho(z)-\epsilon)}$ is a defining function for Ω_ϵ, and $\tilde{\rho}$ is strictly plurisubharmonic on $b\Omega_\epsilon$. The function $\tilde{\rho}$ might not be strictly plurisubharmonic only near $p_1, \dots, p_m$. Since X is Stein, we globally have a strictly plurisubharmonic function, and by standard cutting and patching techniques (see e.g. [14]) we obtain a strictly plurisubharmonic exhaustion function for Ω_ϵ. By Grauert's theorem [11, Theorem 2], Ω_ϵ is then Stein. ∎

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Tadej Starčič
Faculty of Education
University of Ljubljana
Kardeljeva Ploščad 16
1000 Ljubljana, Slovenia
and
Institute of Mathematics, Physics and Mechanics
Jadranska 19
1000 Ljubljana, Slovenia
E-mail: tadej.starcic@pef.uni-lj.si