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COMPOSANTS IN INDECOMPOSABLE INVERSE LIMITS OF UNIMODAL MAPS

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Abstract. We consider indecomposable inverse limits of certain unimodal maps on intervals, and we use the Axiom of Choice to assign sequences of zeros and ones to points of these spaces so that two points belong to the same composant if and only if their itineraries agree on their tails. This extends results long known to hold for any indecomposable continuum that arises as the inverse limit of a single tent core with a nonrecurrent or periodic critical point. For the context in which the inverse limit is generated by a single unimodal map, it is shown that sequences may be assigned in such a way that the shift on the resulting sequence space is semiconjugate to the shift homeomorphism on the inverse limit.

1. Introduction. Composant structure in indecomposable continua can often be described with sequences of zeros and ones. In inverse limits of tent cores with nonrecurrent critical points, backward itineraries of points in the inverse limit may be used to identify each point of the inverse limit with a sequence of zeros and ones with the effect that two points of the inverse limit belong to the same composant if and only if their backward itineraries agree on their tails [3]. Similar results exist for Knaster continua and, more generally, inverse limits of Markov maps [7, 8]. From the perspective of descriptive set theory, the composant equivalence relation of an indecomposable continuum is always Borel bireducible with one of two canonical forms, the simpler of which is the equivalence relation on $\{0,1\}^{\mathbb{N}}$ according to which two sequences of zeros and ones are equivalent if and only if they agree on their tails [9].

It is well known, however, that the equivalence relation obtained by equating points whose backward itineraries agree on their tails does not always correspond to the composant equivalence relation. For example, for some inverse limits of tent maps with recurrent critical points, the partition of the space thus induced is a proper refinement of the partition of the space into composants.

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In this paper, tails of sequences on two symbols are used to classify composants of indecomposable inverse limits generated by a sequence of unimodal maps. For some composants, backward itineraries suffice. Composants that contain a proper subcontinuum whose projections straddle the critical point for infinitely many positive integers are less easily described. The Axiom of Choice is used to extract a transversal to the collection of all such composants—that is, a set containing exactly one point from each such composant—and the transversal is used to assign a sequence of symbols to each point.

Transversals lay at the heart of a classic problem in the history of continuum theory: Does an indecomposable continuum admit a Borel transversal? When this problem was answered in the negative [9], the solution revealed two categories for the complexity of the composant equivalence relation of an indecomposable continuum. Specifically, the composant equivalence relation is always Borel bireducible with one of two canonical equivalence relations, $\mathbb{E}_0$ and $\mathbb{E}_1$. The simpler of the two, $\mathbb{E}_0$, is the equivalence relation on $\{0,1\}^{\mathbb{N}}$ according to which two sequences of zeros and ones are equivalent if and only if they agree on their tails. Associated with this category are all Knaster continua [9] and, more generally, all indecomposable inverse limits of Markov maps on intervals [8]. By contrast, all hereditarily indecomposable continua are associated with $\mathbb{E}_1$, the equivalence relation on $(\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$ according to which two sequences of sequences are equivalent if and only if they agree on their tails [9].

The composant equivalence relations of the spaces studied here are shown to have kinship with the $\mathbb{E}_0$-type continua. Certainly, the identification of each point in the space with a sequence of zeros and ones in such a way that two points of the space belong to the same composant if and only if their corresponding sequences agree on their tails does reduce the composant equivalence relation to $\mathbb{E}_0$. But is the function that makes these identifications Borel? This function, which is presented in Section 1.2, is defined in terms of a transversal selected by the Axiom of Choice. Hence its Borelness or lack thereof is in a sense subject to the whimsy of the Axiom of Choice. For some unimodal maps, those which are also Markov maps for example, it is known [8] that the inverse limit has the simpler $\mathbb{E}_0$-type, but whether or not indecomposable inverse limits of unimodal maps are always of the $\mathbb{E}_0$-type remains open.

In Section 2, it is shown that two points of the inverse limit belong to the same composant if and only if their corresponding sequences agree on their tails. This generalizes a well known result of Brucks and Diamond [3]. In Section 3, the focus is narrowed to indecomposable inverse limits generated by a single unimodal bonding map, and it is shown that if the transversal to the unruly composants is shift invariant, then the shift homeomorphism is semiconjugate to the shift on the corresponding symbolic sequence space. In Section 4, attention is further restricted to indecomposable inverse limits generated by a single tent core, and the Axiom of Choice is used to define a shift invariant transversal to the desired composants.

This paper does not address the question of which inverse limits of unimodal maps are indecomposable. However indecomposability has been characterized for inverse limits in general [6], as well as for inverse limits generated by a single bonding map in a variety of settings: organic maps on intervals [8], tent cores [3], cores of logistic maps [1], and piecewise linear unimodal maps [4]. Many of these results are summarized in Ingram and Mahavier's book [5] on inverse limits.

1.1. Definitions. A *continuum* is a compact connected metrizable space. A continuum is said to be *indecomposable* if it is not the union of two of its proper subcontinua; otherwise it is *decomposable*. A continuum is *irreducible* between two points if none of its proper subcontinua contain both points. The *composant* of a point p of a continuum X is the union of all proper subcontinua of X that contain p. The composants of an indecomposable continuum partition it into $2^{\aleph_0}$ sets. A *transversal* to the composants of an indecomposable continuum X is a set that contains exactly one point from each composant of X.

A *map* is a continuous function. A *unimodal map* is a continuous function f from an interval $[a, b]$ onto itself for which there is a point $c \in (a, b)$ called a *critical point* such that f is monotone on both $[a, c]$ and $[c, b]$, $f(c) = b$, and $f(b) = a$. Note that this is more restrictive than the usual definition of unimodal map. Noticeably excluded are maps obtained by replacing "$f(b) = a$" with "$f(a) = a$" in the definition just given. This omission is mitigated somewhat in noting that, at least in the restricted setting of inverse limits generated by a single bonding map, the maps we are excluding do not generate indecomposable inverse limits.

A map $f : X \to X$ is said to be *semiconjugate* to a map $g : Y \to Y$ if there is a surjective map $m : X \to Y$ such that $m \circ f = g \circ m$. In this case, m is said to be a *semiconjugacy*, and f is said to be semiconjugate to g via m.

Suppose $\{X_n\}_{n\in\mathbb{Z}}$ is a sequence of topological spaces and $\{f_n\}_{n\in\mathbb{Z}}$ a sequence of maps such that f_n maps X_{n+1} into X_n for each $n \in \mathbb{Z}$. Then $\{X_n, f_n\}$ is an *inverse sequence*, and the *inverse limit* of $\{X_n, f_n\}$ is the subspace of $\prod_{\mathbb{Z}} X_n$ to which a point $(\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots)$ belongs if and only if $f(x_{n+1}) = x_n$ for each $n \in \mathbb{Z}$. The spaces X_n are called *factor spaces*, and the maps f_n are called *bonding maps*. The projection of the inverse limit into the factor spaces X_n is denoted for each $n \in \mathbb{Z}$ by π_n. It is well known that an inverse limit is a continuum if each of its factor spaces is a continuum.

If each factor space is an interval, the topology on the inverse limit is generated by the metric defined as follows:

$$d(x,y) = \sum_{n=0}^{\infty} \frac{|x_n - y_n|}{2^n}.$$

The continuity of the bonding maps guarantees that it is sufficient to sum over the nonnegative integers.

Suppose X is the inverse limit of an inverse sequence of the form $\{Y, f\}$ for some space Y and some map f. The *shift map* is the function $\hat{f} : X \to X$ defined coordinatewise by $\pi_n \circ \hat{f}(x) = f(x_n) = x_{n-1}$. It is well known that the shift map is a homeomorphism of the inverse limit onto itself.

1.2. Notation. The following notation will be used throughout without further reference. X denotes an indecomposable continuum that arises as the inverse limit of an inverse sequence $\{I, f_n\}$ where, for each $n \in \mathbb{Z}$, f_n is a unimodal map with critical point c_n.

For any collection of sets $\mathcal{G}$, the symbol $\mathcal{G}^*$ denotes the union of the members of $\mathcal{G}$. Denote by $\mathcal{C}$ the collection of composants of X that contain a proper subcontinuum K of X such that $\pi_n[K]$ contains c_n for infinitely many positive integers n. Denote by T a transversal to $\mathcal{C}$. For each x in $\mathcal{C}^*$, define $T(x)$ to be $T \cap C_x$, where C_x denotes the composant of x in X.

For each $n \in \mathbb{Z}$, define collections $\mathcal{A}_n^0$ and $\mathcal{A}_n^1$ as follows:

$$\mathcal{A}_n^0 = \left\{ \begin{array}{l} K : K \text{ is a subcontinuum of } X, \\ \qquad c_k \in \pi_k[K] \text{ for infinitely many } k \in \mathbb{N}, \\ \qquad \pi_k[K] \subset [c_k, 1] \text{ for some } k \leq n, \\ \qquad T[K] \in K, \\ \qquad \pi_n \circ T[K] \in [0, c_n) \end{array} \right\},$$

$$\mathcal{A}_n^1 = \left\{ \begin{array}{l} K : K \text{ is a subcontinuum of } X, \\ \qquad c_k \in \pi_k[K] \text{ for infinitely many } k \in \mathbb{N}, \\ \qquad \pi_k[K] \subset [c_k, 1] \text{ for some } k \leq n, \\ \qquad T[K] \in K, \\ \qquad \pi_n \circ T[K] \in [c_n, 1] \end{array} \right\}.$$

For each $n \in \mathbb{Z}$, set $B_n^0 = (\pi_n^{-1}[0, c_n) \cup \mathcal{A}_n^{0^*}) - \mathcal{A}_n^{1^*}$ and $B_n^1 = (\pi_n^{-1}[c_n, 1] \cup \mathcal{A}_n^{1^*}) - \mathcal{A}_n^{0^*}$, and let $\mathcal{B}_n = \{B_n^0, B_n^1\}$. By Lemma 1.1 below, each $\mathcal{B}_n$ is a partition of X. Denote the projection of X onto $\mathcal{B}_n$ by β_n for each $n \in \mathbb{Z}$, i.e., $\beta_n(x) = B_n^j$ if $x \in B_n^j$, $j \in \{0, 1\}$.

For any distinct pair of real numbers, a and b, the notation $[a, b)$ will be used independent of whether $a < b$ or $b < a$ to denote the interval that contains a, fails to contain b, and has endpoints a and b. If $a = b$, then $[a, b)$ refers to the empty set. Similar conventions will be used for each of (a, b), $(a, b]$, and $[a, b]$.

LEMMA 1.1. *The collections $\mathcal{A}_n^0$, $\mathcal{A}_n^1$, and $\mathcal{B}_n$ have the following properties for each $n \in \mathbb{Z}$:*

(1) $\mathcal{A}_n^0 \cup \mathcal{A}_n^1 \subset \mathcal{A}_{n+1}^0 \cup \mathcal{A}_{n+1}^1$.
(2) $\mathcal{A}_n^{0^*} \cup \mathcal{A}_n^{1^*} \subset \mathcal{A}_{n+1}^{0^*} \cup \mathcal{A}_{n+1}^{1^*}$.
(3) $\mathcal{A}_n^{0^*} \cup \mathcal{A}_n^{1^*} \subset \mathcal{C}^*$.
(4) *For any x in $\mathcal{A}_n^{0^*} \cup \mathcal{A}_n^{1^*}$, x is in $\mathcal{A}_n^{0^*}$ if $\pi_n \circ T(x) \in [0, c_n)$, and x is in $\mathcal{A}_n^{1^*}$ if $\pi_n \circ T(x) \in [c_n, 1]$.*
(5) $\mathcal{A}_n^0 \cap \mathcal{A}_n^1 = \emptyset$.
(6) $\mathcal{A}_n^{0^*} \cap \mathcal{A}_n^{1^*} = \emptyset$.
(7) *$\mathcal{B}_n$ is a partition of X.*

2. Composant classification. The goal of this section is to classify the composants of X. Roughly speaking, each point of X is identified with an element of the sequence space $\prod_{\mathbb{Z}}\{B_n^0, B_n^1\}$, and the composant of the point is determined by the tail of the sequence. More specifically, the main result of the section, Theorem 2.11, states that two points x and y of X belong to the same composant if and only if $\beta_n(x) = \beta_n(y)$ for cofinitely many positive integers n. Along the way the result is established for $x, y \in X - \mathcal{C}^*$ (Lemma 2.2), $x, y \in T$ (Lemma 2.4), $x, y \in \mathcal{C}^*$ (Lemma 2.8), and $x \in X$, $y \in \mathcal{C}^*$ (Lemma 2.10).

LEMMA 2.1. *If x and y are points of X such that $c_n \notin (x_n, y_n)$ for cofinitely many $n \in \mathbb{N}$, then x and y belong to the same composant of X.*

Proof. The restriction of f_n to $[x_n, y_n]$ is monotone for cofinitely many $n \in \mathbb{N}$, so $f_n[x_{n+1}, y_{n+1}] = [x_n, y_n]$ for cofinitely many $n \in \mathbb{N}$. It follows that x and y are in the same composant of X. ∎

LEMMA 2.2. *If x and y are points of $X - \mathcal{C}^*$, then the following are equivalent:*

(1) *$\beta_n(x) = \beta_n(y)$ for cofinitely many $n \in \mathbb{N}$.*
(2) *$c_n \notin (x_n, y_n)$ for cofinitely many $n \in \mathbb{N}$.*
(3) *x and y belong to the same composant of X.*

Proof. Since x and y are not in $\mathcal{C}^*$, it follows that $x_n, y_n \neq c_n$ for cofinitely many $n \in \mathbb{N}$, and $x_n, y_n \notin \mathcal{A}_n^{0^*} \cup \mathcal{A}_n^{1^*}$ for $n \in \mathbb{Z}$. Consequently, $\beta_n(x) = B_n^0$ if and only if $x_n \in [0, c_n)$, and $\beta_n(x) = B_n^1$ if and only if $x_n \in [c_n, 1]$, and similarly for y. Thus (1) implies (2), and, as $x_n, y_n \neq c_n$ for cofinitely many $n \in \mathbb{N}$, (2) implies (1). By Lemma 2.1, (2) implies (3). Conversely, if (3) holds, there is a proper subcontinuum of X that contains both x and y and fails to belong to $\mathcal{C}$. Then (2) follows from the definition of $\mathcal{C}$. ∎

LEMMA 2.3. *For each $x \in T$ and each $n \in \mathbb{Z}$,*

$$\beta_n(x) = \begin{cases} B_n^0 & \text{if } x_n \in [0, c_n), \\ B_n^1 & \text{if } x_n \in [c_n, 1]. \end{cases}$$

Proof. First suppose $x_n \in [0, c_n)$. If $x \notin \mathcal{A}_n^{0^*} \cup \mathcal{A}_n^{1^*}$, then $x \in B_n^0$. Hence $\beta_n(x) = B_n^0$. Suppose $x \in \mathcal{A}_n^{0^*} \cup \mathcal{A}_n^{1^*}$. Then there is a proper subcontinuum K of X containing x that belongs to one of $\mathcal{A}_n^0$ and $\mathcal{A}_n^1$. Since $\pi_n \circ T[K] = x_n$ and $x_n \in [0, c_n)$, we see that K belongs to $\mathcal{A}_n^0$. Then $x \in \mathcal{A}_n^{0^*}$, so $x \in B_n^0$ and $\beta_n(x) = B_n^0$. Similarly, if $x_n \in [c_n, 1]$, then $\beta_n(x) = B_n^1$. ∎

LEMMA 2.4. *If x and y are points of T, the following are equivalent:*

(1) *$\beta_n(x) = \beta_n(y)$ for cofinitely many $n \in \mathbb{N}$.*
(2) *$c_n \notin (x_n, y_n)$ for cofinitely many $n \in \mathbb{N}$.*
(3) *x and y belong to the same composant of X.*
(4) *$x = y$.*

Proof. The proof goes (1)⇒(2)⇒(3)⇒(4)⇒(1). The first two implications follow from Lemmas 2.3 and 2.1 respectively. The third follows from the fact that T contains at most one point from each composant of X, and the fourth is trivial. ∎

LEMMA 2.5. *If K is a proper subcontinuum of X, then $\pi_n[K] \subset [0, c_n) \cup (c_n, 1]$ for infinitely many $n \in \mathbb{N}$. Furthermore, if $c_n \in \pi_n[K]$ for infinitely many $n \in \mathbb{N}$, then $\pi_n[K] \subset (c_n, 1]$ for infinitely many $n \in \mathbb{N}$.*

Proof. If $c_n \in \pi_n[K]$ for cofinitely many $n \in \mathbb{N}$, then each of the following also holds for cofinitely many $n \in \mathbb{N}$: $1 \in \pi_n[K]$, $[c_n, 1] \subset \pi_n[K]$, and $[0, 1] \subset \pi_n[K]$. Thus $K = X$, contrary to hypothesis. Hence $\pi_n[K] \subset [0, c_n) \cup (c_n, 1]$ for infinitely many $n \in \mathbb{N}$. Suppose further that $c_n \in \pi_n[K]$ for infinitely many $n \in \mathbb{N}$. Then $1 \in \pi_n[K]$ for infinitely many $n \in \mathbb{N}$. Since it is not true that $[c_n, 1] \subset \pi_n[K]$ for infinitely many $n \in \mathbb{N}$, it follows that $\pi_n[K] \subset (c_n, 1]$ for infinitely many $n \in \mathbb{N}$. ∎

LEMMA 2.6. *If $x \in T$ and K is a proper subcontinuum of X containing x, then there are cofinitely many $n \in \mathbb{N}$ for which K is contained by one of B_n^0 and B_n^1.*

Proof. There is a proper subcontinuum L of X containing K such that $c_n \in \pi_n[L]$ for infinitely many $n \in \mathbb{N}$. By Lemma 2.5, there is a positive integer N such that $\pi_N[L] \subset (c_N, 1]$. Suppose $n \geq N$. Notice that $T[L] = x$, and therefore $T[L] \in L$. Then $L \in \mathcal{A}_n^0$ if $x_n \in [0, c_n)$, and $L \in \mathcal{A}_n^1$ if $x_n \in [c_n, 1]$. Consequently, $L \subset B_n^0$ if $x_n \in [0, c_n)$, and $L \subset B_n^1$ if $x_n \in [c_n, 1]$. The conclusion of the lemma follows. ∎

LEMMA 2.7. *If x is a point of T and y is a point of X from the composant of x, then $\beta_n(x) = \beta_n(y)$ for cofinitely many $n \in \mathbb{N}$.*

Proof. Denote by $\overline{xy}$ the continuum irreducible between x and y. By Lemma 2.6, there are cofinitely many $n \in \mathbb{N}$ such that $\overline{xy}$ is contained by one of B_n^0 and B_n^1. Consequently, $\beta_n(x) = \beta_n(y)$ for cofinitely many $n \in \mathbb{N}$. ∎

LEMMA 2.8. *If x and y are points of $\mathcal{C}^*$, then the following are equivalent:*

(1) *$\beta_n(x) = \beta_n(y)$ for cofinitely many $n \in \mathbb{N}$.*
(2) *x and y belong to the same composant of X.*

Proof. Denote $T(x)$ and $T(y)$ by a and b respectively. By Lemma 2.7, $\beta_n(x) = \beta_n(a)$ and $\beta_n(y) = \beta_n(b)$ both hold for cofinitely many $n \in \mathbb{N}$. Hence (1) is equivalent to the statement: $\beta_n(a) = \beta_n(b)$ for cofinitely many $n \in \mathbb{N}$. Note that (2) holds if and only if $a = b$. Thus the result follows from Lemma 2.4. ∎

LEMMA 2.9. *If y is a point of $\mathcal{C}^*$ and x is a point of X such that $\beta_n(x) = \beta_n(y)$ for cofinitely many $n \in \mathbb{N}$, then $x \in \mathcal{C}^*$.*

Proof. Denote $T(y)$ by a. It follows from Lemma 2.7 and the hypothesis of the present lemma that $\beta_n(a) = \beta_n(x)$ for cofinitely many $n \in \mathbb{N}$. By Lemma 2.3, for each $n \in \mathbb{N}$, $\beta_n(a) = B_n^0$ if $a_n \in [0, c_n)$, and $\beta_n(a) = B_n^1$ if $a_n \in [c_n, 1]$. Suppose, contrary to the lemma, that $x \notin \mathcal{C}^*$. Then, for each $n \in \mathbb{N}$, $x \notin \mathcal{A}_n^{0^*} \cup \mathcal{A}_n^{1^*}$, so $\beta_n(x) = B_n^0$ if $x_n \in [0, c_n)$, and $\beta_n(x) = B_n^1$ if $x_n \in [c_n, 1]$. It follows that $c_n \notin (x_n, a_n)$ for cofinitely many $n \in \mathbb{N}$. By Lemma 2.1, x and a belong to the same composant of X, which contradicts the assumption that $x \notin \mathcal{C}^*$. Consequently, $x \in \mathcal{C}^*$. ∎

LEMMA 2.10. *If y is a point of $\mathcal{C}^*$ and x is a point of X, then the following are equivalent:*

(1) *$\beta_n(x) = \beta_n(y)$ for cofinitely many $n \in \mathbb{N}$.*
(2) *x and y belong to the same composant of X.*

Proof. Both conditions imply that $x \in \mathcal{C}^*$: (1) via Lemma 2.9, and (2) trivially. Thus the result follows from Lemma 2.8. ∎

THEOREM 2.11. *Two points x and y of X belong to the same composant if and only if $\beta_n(x) = \beta_n(y)$ for cofinitely many $n \in \mathbb{N}$.*

Proof. Either x and y are both in $X - \mathcal{C}^*$, or at least one of them is in $\mathcal{C}^*$. Thus the theorem follows from Lemmas 2.2 and 2.10. ∎

3. Single bonding map. In this section, we will assume that there is a single unimodal map f such that $f_n = f$ for each positive integer n. Prior to this section, the sequences of symbols with which points of X were identified were points of the space $\prod_{\mathbb{Z}}\{B_n^0, B_n^1\}$. One naturally regards such points as sequences of zeros and ones, and that connection is formally established in this section. In Theorem 3.5, the resulting set Ω of all points of $\prod_{\mathbb{Z}}\{0, 1\}$

that correspond to points of X is shown to be invariant under the shift on $\prod_{\mathbb{Z}}\{0,1\}$, and the restriction of that shift to Ω is shown to be semiconjugate to the shift homeomorphism on X.

Notation. Define $\tilde{\beta}_n$ for each $n \in \mathbb{Z}$ by

$$\tilde{\beta}_n(x) = \begin{cases} 0 & \text{if } \beta_n(x) = B_n^0, \\ 1 & \text{if } \beta_n(x) = B_n^1, \end{cases}$$

and define $\tilde{\beta} : X \to \prod_{\mathbb{Z}}\{0,1\}$ coordinatewise by $\pi_n \circ \tilde{\beta}(x) = \tilde{\beta}_n(x)$ for each $n \in \mathbb{Z}$. Denote the range of $\tilde{\beta}$ by Ω.

Denote the quotient topology on $\{0,1\}$ induced by $\tilde{\beta}_n$ by $\mathcal{Q}_n$ for each n. ($\mathcal{Q}_n$ is nontrivial if and only if one of B_n^0 and B_n^1 is open.) Note that $\tilde{\beta}$ is a continuous function from X into $\prod_{\mathbb{Z}}(\{0,1\},\mathcal{Q}_n)$.

Define σ to be the shift on $\prod_{\mathbb{Z}}(\{0,1\},\mathcal{Q}_n)$ given by $\pi_n \circ \sigma(x) = \pi_{n-1}(x)$ for each $n \in \mathbb{Z}$. Since each coordinate of σ is a projection map, it appears that σ is continuous as the rising sun. However, π_{n-1}, although continuous as a projection onto $(\{0,1\},\mathcal{Q}_{n-1})$, is not necessarily continuous when regarded as a function into $(\{0,1\},\mathcal{Q}_n)$. That is, the continuity of $\pi_n \circ \sigma : \prod_{\mathbb{Z}}(\{0,1\},\mathcal{Q}_n) \to (\{0,1\},\mathcal{Q}_n)$ does not follow from the continuity of $\pi_{n-1} : \prod_{\mathbb{Z}}(\{0,1\},\mathcal{Q}_n) \to (\{0,1\},\mathcal{Q}_{n-1})$. If $\mathcal{Q}_n = \mathcal{Q}_{n-1}$ for each n, then $\pi_n \circ \sigma$ is continuous for each n, and σ is continuous. It is shown in Theorem 3.4 that this is the case if T is shift invariant.

Lemma 3.1. *If $f_n = f$ for each $n \in \mathbb{N}$, and T is a shift invariant transversal to the composants in $\mathcal{C}$, then $\hat{f}|_{\mathcal{C}^*} \circ T = T \circ \hat{f}|_{\mathcal{C}^*}$.*

Proof. It suffices to show that $\hat{f} \circ T(x) = T \circ \hat{f} \circ T(x) = T \circ \hat{f}(x)$ for each $x \in \mathcal{C}^*$. The first equality holds since $\hat{f} \circ T(x)$ belongs to the transversal T, and the second holds since $\hat{f}(x)$ and $\hat{f} \circ T(x)$ belong to the same composant. ∎

Lemma 3.2. *If $f_n = f$ for each $n \in \mathbb{N}$, and T is a shift invariant transversal to the composants in $\mathcal{C}$, then, for each $i \in \{0,1\}$, a subcontinuum K of X belongs to the collection $\mathcal{A}_n^i$ if and only if $\hat{f}[K]$ belongs to the collection $\mathcal{A}_{n+1}^i$.*

Proof. The proof is similar for $i = 0$ and $i = 1$, so only the case in which $i = 0$ will be considered. Suppose K is a subcontinuum of X. Since $f_n = f$ and hence $c_n = c$ for each $n \in \mathbb{Z}$, the definition of $\mathcal{A}_n^0$ is as follows, and that of $\mathcal{A}_{n+1}^0$ is similar:

$$\mathcal{A}_n^0 = \left\{ \begin{array}{ll} K: & 1)\ K \text{ is a subcontinuum of } X, \\ & 2)\ c \in \pi_k[K] \text{ for infinitely many } k \in \mathbb{N}, \\ & 3)\ \pi_k[K] \subset [c,1] \text{ for some } k \leq n, \\ & 4)\ T[K] \in K, \\ & 5)\ \pi_n \circ T[K] \in [0,c) \end{array} \right\}.$$

It is straightforward to verify that K satisfies the first three properties of $\mathcal{A}_n^0$ if and only if $\hat{f}[K]$ satisfies the first three properties of $\mathcal{A}_{n+1}^0$. Notice that $T[K] \in K$ if and only if $\hat{f} \circ T[K] \in \hat{f}[K]$. Applying Lemma 3.1 shows that $T[K] \in K$ if and only if $T \circ \hat{f}[K] \in \hat{f}[K]$. Hence K satisfies the fourth property of $\mathcal{A}_n^0$ if and only if $\hat{f}[K]$ satisfies the fourth property of $\mathcal{A}_{n+1}^0$. Finally, notice that $\pi_n \circ T = \pi_{n+1} \circ \hat{f} \circ T = \pi_{n+1} \circ T \circ \hat{f}|_{\mathcal{C}^*}$. Hence $\pi_n \circ T[K] = \pi_{n+1} \circ T \circ \hat{f}[K]$ for each subcontinuum K of X lying in $\mathcal{C}^*$ of X, and K satisfies the fifth property of $\mathcal{A}_n^0$ if and only if $\hat{f}[K]$ satisfies the fifth property of $\mathcal{A}_{n+1}^0$. Consequently, $K \in \mathcal{A}_n^0$ if and only if $\hat{f}[K] \in \mathcal{A}_{n+1}^0$. ∎

Lemma 3.3. *If $f_n = f$ for each $n \in \mathbb{Z}$, and T is a shift invariant transversal to the composants in $\mathcal{C}$, then $\hat{f}(B_n^0) = B_{n+1}^0$ and $\hat{f}(B_n^1) = B_{n+1}^1$ for each $n \in \mathbb{Z}$.*

Proof. Since $\hat{f}$ is one-to-one and $\{B_n^0, B_n^1\}$ is a partition of X for each $n \in \mathbb{Z}$, it suffices to show that $\hat{f}(B_n^0) = B_{n+1}^0$ for each $n \in \mathbb{Z}$. It follows from Lemma 3.2 that, for each $i \in \{0,1\}$, a point x of X belongs to $\mathcal{A}_n^{i^*}$ if and only if $\hat{f}(x)$ belongs to $\mathcal{A}_{n+1}^{i^*}$. Since a point x belongs to $\pi_n^{-1}[0,c)$ if and only if $\hat{f}(x)$ belongs to $\pi_{n+1}^{-1}[0,c)$, it follows that a point x belongs to $B_n^0 = (\pi_n^{-1}[0,c) \cup \mathcal{A}_n^{0^*}) - \mathcal{A}_n^{1^*}$ if and only if $\hat{f}(x)$ belongs to $B_{n+1}^0 = (\pi_{n+1}^{-1}[0,c) \cup \mathcal{A}_{n+1}^{0^*}) - \mathcal{A}_{n+1}^{1^*}$. ∎

Theorem 3.4. *If $f_n = f$ for each $n \in \mathbb{Z}$, and T is a shift invariant transversal to the composants in $\mathcal{C}$, then σ is continuous.*

Proof. It was noted in the remarks prior to Lemma 3.1 that σ is continuous if $\mathcal{Q}_n = \mathcal{Q}_{n-1}$ for each n. Suppose $n \in \mathbb{Z}$. Since $\mathcal{Q}_n$ and $\mathcal{Q}_{n-1}$ both contain the empty set and $\{0,1\}$, it suffices to show that $\mathcal{Q}_n$ contains $\{0\}$ if and only if $\mathcal{Q}_{n-1}$ does, and similarly for $\{1\}$. Note that $\mathcal{Q}_n$ contains $\{0\}$ if and only if $\tilde{\beta}_n^{-1}(0)$, which is equal to B_n^0, is open. Similarly $\mathcal{Q}_{n-1}$ contains $\{0\}$ if and only if B_{n-1}^0 is open. By Lemma 3.3, B_n^0 and B_{n-1}^0 are homeomorphic. Consequently, $\mathcal{Q}_n$ contains $\{0\}$ if and only if $\mathcal{Q}_{n-1}$ does. Similarly $\mathcal{Q}_n$ contains $\{1\}$ if and only if $\mathcal{Q}_{n-1}$ does. It follows that $\mathcal{Q}_n = \mathcal{Q}_{n-1}$. Hence σ is continuous. ∎

Theorem 3.5. *If $f_n = f$ for each $n \in \mathbb{Z}$, and T is a shift invariant transversal to the composants in $\mathcal{C}$, then Ω is a shift invariant subset of $\prod_{\mathbb{Z}}\{0,1\}$, and $\sigma \circ \tilde{\beta} = \tilde{\beta} \circ \hat{f}$.*

Proof. The shift invariance of Ω follows from $\sigma \circ \tilde{\beta} = \tilde{\beta} \circ \hat{f}$, which we establish by showing that $\pi_n \circ \sigma \circ \tilde{\beta} = \pi_{n-1} \circ \tilde{\beta} = \pi_n \circ \tilde{\beta} \circ \hat{f}$ for each $n \in \mathbb{Z}$. The former equality follows from the definition of σ. To see the latter, suppose

$x \in X$ and $n \in \mathbb{Z}$. The definition of $\tilde{\beta}$ and Lemma 3.3 yield

$$\begin{aligned}\tilde{\beta}_n \circ \hat{f}(x) &= \begin{cases} 0 & \text{if } \hat{f}(x) \in B_n^0, \\ 1 & \text{if } \hat{f}(x) \in B_n^1 \end{cases} \\ &= \begin{cases} 0 & \text{if } x \in B_{n-1}^0, \\ 1 & \text{if } x \in B_{n-1}^1 \end{cases} \\ &= \tilde{\beta}_{n-1}(x). \ \blacksquare\end{aligned}$$

4. The tent family. In this section it is shown that there is a shift invariant transversal to the composants of an indecomposable inverse limit of tent cores (Theorem 4.3). The proof makes use of the fact that every composant that is periodic under the shift map contains a periodic point with the same period. This result about periodic composants was first proved by Block, Jakimovik, and Keesling [2]. The proof given in Theorem 4.2 below for the sake of completeness and consistency of notation is similar in spirit.

The *tent map* with slope s, for any $s \in [0,2]$, is given by

$$T_s(x) = \begin{cases} sx & \text{if } x \in [0,1/2], \\ s(1-x) & \text{if } x \in [1/2,1]. \end{cases}$$

A *tent core* is a map of the form $T_s|_{[T_s^2(c),T_s(c)]}$, where $c = 1/2$. Notice that if f is a tent core, then f is unimodal with critical point $c = 1/2$, and the domain and range of f are both equal to $[f^2(c), f(c)]$.

LEMMA 4.1. *If f is a tent core, and I is an interval such that $f^2[I] \neq [f^2(c), f(c)]$, and $r = \sqrt{2}/s$, then* $\operatorname{diam}(I) \le r^2 \operatorname{diam}(f^2[I])$.

Proof. First note that if J is a subinterval of $[f^2(c), f(c)]$ such that $f[J] \neq [f^2(c), f(c)]$, then

$$\operatorname{diam}(f[J]) \begin{cases} = s \operatorname{diam}(J) & \text{if } c \notin J, \\ \ge (s/2) \operatorname{diam}(J) & \text{if } c \in J. \end{cases}$$

Applying this to $f[I]$ and I, at least one of which fails to contain c, shows that $\operatorname{diam}(f^2[I]) \ge (s^2/2) \operatorname{diam}(I)$. The conclusion follows. $\blacksquare$

THEOREM 4.2 (Block, Jakimovik, Keesling; [2]). *Suppose $f_n = f$ for each $n \in \mathbb{Z}$ where f is a tent core, and suppose $s > \sqrt{2}$. Then every periodic composant of $\hat{f}$ contains a periodic point of $\hat{f}$ with the same period.*

Proof. Suppose C is a periodic composant of $\hat{f}$, and choose a point $x \in C$. Denote the period of C under $\hat{f}$ by p. Then x and $\hat{f}^{-p}(x)$ belong to the same composant, so there is a proper subcontinuum K of X that contains both x and $\hat{f}^{-p}(x)$. There is a positive integer N such that $\pi_n[K]$ is a proper subset of $[f^2(c), f(c)]$ for each $n \ge N$. Set $z = \hat{f}^{-Np}(x)$ and $L = \hat{f}^{-Np}[K]$. Then z belongs to the composant C, z and $\hat{f}^{-p}(z)$ belong to L, and $\pi_n[L]$ is a proper subset of $[0,1]$ for each nonnegative integer n.

We wish to show that $\{\hat{f}^{-np}(z)\}$ converges, and that its limit satisfies the conclusion of the theorem. To that end, we show $\{\hat{f}^{-np}(z)\}$ is Cauchy.

Set $r = \sqrt{2}/s$ and $M = \max\{\operatorname{diam}(\pi_0[L]), r^{-1}\operatorname{diam}(\pi_1[L])\}$, and note that $|r| < 1$. By Lemma 4.1, $\operatorname{diam}(\pi_{n+2}[L]) \le r^2 \operatorname{diam}(\pi_n[L])$ for each nonnegative integer n. Then $\operatorname{diam}(\pi_n[L]) \le Mr^n$ for each nonnegative integer n. It follows that $\operatorname{diam}(\pi_n[\hat{f}^{-kp}[L]]) \le Mr^{n+kp}$ for all nonnegative integers k and n. Since $\hat{f}^{-kp}(z)$ and $\hat{f}^{-(k+1)p}(z)$ belong to $\hat{f}^{-kp}[L]$ for each nonnegative integer k, it follows that $d(\hat{f}^{-kp}(z), \hat{f}^{-(k+1)p}(z)) \le \sum_{n=0}^{\infty} Mr^{n+kp} = Mr^{kp}/(1-r)$ for each nonnegative integer k.

Then, for $j \ge k$,

$$\begin{aligned} d(\hat{f}^{-kp}(z), \hat{f}^{-jp}(z)) &\le \sum_{i=k}^{j-1} d(\hat{f}^{-ip}(z), \hat{f}^{-(i+1)p}(z)) \\ &\le \sum_{i=k}^{\infty} M\frac{1}{1-r}r^{ip} = \frac{M}{1-r}\,\frac{r^{kp}}{1-r^p} = o(1) \quad \text{ as } j,k \to \infty. \end{aligned}$$

Hence $\{\hat{f}^{-np}(z)\}$ is a Cauchy sequence, and therefore converges to some point $\zeta \in X$. Furthermore, $\hat{f}^{-p}(\zeta) = \zeta$. Thus ζ is periodic under $\hat{f}$ with period at most p. To see that the period of ζ is at least p, it suffices to show that $\zeta \in C$, or equivalently, that ζ belongs to the composant of z.

For each $n = 0, 1, \ldots, p-1$,

$$\begin{aligned} \zeta_n = \pi_n(\zeta) = \pi_n\Big(\lim_{k\to\infty} \hat{f}^{-kp}(z)\Big) &= \lim_{k\to\infty} \pi_n \circ \hat{f}^{-kp}(z) \\ = \lim_{k\to\infty} \pi_{n+kp}(z) &= \lim_{k\to\infty} z_{n+kp}. \end{aligned}$$

Thus, for each $n = 0, 1, \ldots, p-1$, ζ_n and z_{n+kp} fail to straddle c for cofinitely many nonnegative integers k. But $\zeta_{n+kp} = \zeta_n$ for each nonnegative integer k since $\hat{f}^p(\zeta) = \zeta$. Consequently, for each $n = 0, 1, \ldots, p-1$, ζ_{n+kp} and z_{n+kp} fail to straddle c for cofinitely many nonnegative integers k. Equivalently, ζ_k and z_k fail to straddle c for cofinitely many nonnegative integers k, and ζ and z belong to the same composant by Lemma 2.1. ∎

THEOREM 4.3. *If $f_n = f$ for each $n \in \mathbb{Z}$ where f is a tent core, then there is a shift invariant transversal to the composants in $\mathcal{C}$.*

Proof. Consider the collections $\mathcal{P}$ and $\mathcal{N}$ defined as follows:

$$\mathcal{P} = \Big\{\bigcup_{n\in\mathbb{Z}} \hat{f}^n(C) : C \in \mathcal{C} \text{ and } \hat{f}^k(C) = C \text{ for some } k \in \mathbb{Z}\Big\},$$

$$\mathcal{N} = \Big\{\bigcup_{n\in\mathbb{Z}} \hat{f}^n(C) : C \in \mathcal{C} \text{ and } \hat{f}^k(C) \ne C \text{ for each } k \in \mathbb{Z}\Big\}.$$

By the Axiom of Choice, there are a transversal $S_{\mathcal{N}}$ to $\mathcal{N}$ and, via Theorem 4.2, a transversal $S_{\mathcal{P}}$ to $\mathcal{P}$ such that each point of $S_{\mathcal{P}}$ is periodic under $\hat{f}$ with period equal to that of the composant in which it resides. Set $S = \bigcup_{\mathbb{Z}} \hat{f}^n[S_{\mathcal{P}} \cup S_{\mathcal{N}}]$, and note that $\hat{f}[S] = S$.

It remains to show that S is a transversal to $\mathcal{C}$. Suppose $C \in \mathcal{C}$. Then $\bigcup_{\mathbb{Z}} \hat{f}^n[C]$ contains a unique point x of $S_{\mathcal{P}} \cup S_{\mathcal{N}}$. Hence $x \in \hat{f}^k[C]$ for some $k \in \mathbb{Z}$. Consequently, $\hat{f}^{-k}(x) \in C$, and C contains a point of S.

To see that C contains only one point of S, suppose $y \in S \cap C$. We wish to show that $y = \hat{f}^{-k}(x)$. Since $y \in S$, $y = f^m(x')$ for some point $x' \in S_{\mathcal{P}} \cup S_{\mathcal{N}}$ and some integer $m \in \mathbb{Z}$. Since $y \in C$, $\hat{f}^{-m}[C]$ contains x'. But x is the unique point of $S_{\mathcal{P}} \cup S_{\mathcal{N}}$ from $\bigcup_{\mathbb{Z}} \hat{f}^n[C]$, so $x = x'$. Thus we have $\hat{f}^m(x) = y \in C$ and $\hat{f}^{-k}(x) \in C$. It follows that $\hat{f}^{k+m}[C] = C$. If C fails to be periodic under $\hat{f}$, then $m = -k$, which implies that $y = f^{-k}(x)$. If C is periodic under $\hat{f}$, then $m + k$ is a multiple of p where p is the period of C, and hence of x. Thus $\hat{f}^{m+k}(x) = x$. Applying $\hat{f}^{-k}$ yields $y = \hat{f}^{-k}(x)$. ∎

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